1. Examples of algebraic varieties and maps

Exercise 1.1
Let C be a smooth curve and $f : C \to \mathbb{P}^1$ a degree two map ramified at n points. (C is called a *hyperelliptic curve*.) Use the n ramification points as the 0-simplices of a triangulation of $\mathbb{P}^1$ and argue that one obtains an induced triangulation of C. By relating the 0, 1 and 2-simplices and applying Euler's formula

$$\chi(C) = v - e + f$$

deduce that n is even and that the genus of C is $n/2 - 1$. (*Hint:* If $z_i \in \mathbb{P}^1$ denotes a ramification point, then there exists neighbourhoods of z_i and $f^{-1}(z_i)$ such that f is isomorphic to a neighbourhood of zero of the map $z \mapsto z^2 : \mathbb{C} \to \mathbb{C}$.)

Exercise 1.2
Consider Whitney's umbrella W, which is the affine surface given by the equation

$$x^2 = zy^2.$$

(1) Draw a picture of the zero set of W in the real numbers $\mathbb{R}^3$, (and convince yourself that it looks something like an umbrella!).
(2) Show that the singular locus of W is the line $x = y = 0$.
(3) Show that the map $f : (u, v) \mapsto (x, y, z) = (uv, v, u^2)$ gives a resolution of singularities of W. (In fact f is the normalization of W which you might like to prove.)

From now on we consider the complex points of W with its metric topology.

(4) Use f to describe the topology of a neighbourhood of $(0, 0, z)$ in W for any $z \in \mathbb{C}$. Deduce that the topology of W at $(0, 0, 0)$ is different to that at $(0, 0, z)$ for any $z \neq 0$.
(5) Consider the stratification $W = W_{\text{reg}} \cup \{(0, 0, *)\}$, where W_{reg} denotes the non-singular locus. Show that the Whitney conditions are satisfied at any point $(0, 0, z)$ with $z \neq 0$, but fail at $(0, 0, 0)$.

Exercise 1.3
(The Springer resolution) Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the Lie algebra of traceless $n \times n$ complex matrices.

(1) Consider the characteristic polynomial as a function

$$c : \mathfrak{g} \to \mathbb{C}[t].$$

Define $\mathcal{N} := c^{-1}(t^n) \subset \mathfrak{g}$. Show that $\mathcal{N}$ is the set of nilpotent matrices.

(2) $G = SL_n(\mathbb{C})$ acts on $\mathfrak{g}$ and hence $\mathcal{N}$ by conjugation. Show that G has finitely many orbits on $\mathcal{N}$, and that these orbits are classified by partitions of n. (*Hint:* Jordan normal form.)

(3) Calculate the singular locus of $\mathcal{N}$ for $n = 2, 3$. Guess the answer in general.

(4) A *flag* is a sequence $F_\bullet := (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n)$ where $\dim F_i = i$ for all i. Convince yourself that the set of flags has the structure of a smooth projective algebraic variety. What is its dimension? This variety is called the *flag variety* and will be denoted $\mathcal{F}\ell_n$.

(5) Consider the set

$$\widetilde{\mathcal{N}} := \{(x, F_\bullet) \in \mathcal{N} \times \mathcal{F}\ell_n \mid xF_i \subset F_{i-1} \text{ for } 1 \leq i \leq n\}.$$

Show that the second projection $\widetilde{\mathcal{N}} \to \mathcal{F}\ell_n$ identifies $\widetilde{\mathcal{N}}$ as a vector bundle over $\mathcal{F}\ell_n$. In particular, $\widetilde{\mathcal{N}}$ is smooth. (Challenge: in fact $\widetilde{\mathcal{N}} \cong T^*\mathcal{F}\ell_n$.)

(6) Show that the first projection

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$$

is a (G-equivariant) resolution of singularities. (π is the famous *Springer resolution.*)

(7) Calculate the fibres of π over all G-orbits for $n = 2, 3$.

Exercise 1.4

(The Grothendieck-Springer alteration) We keep the notation of the previous section. Consider

$$\widetilde{\mathfrak{g}} := \{(x, F_\bullet) \in \mathfrak{g} \times \mathcal{F}\ell_n \mid xF_i \subset F_i \text{ for } 1 \leq i \leq n\}.$$

(1) Show that $\widetilde{\mathfrak{g}}$ is a smooth variety.

(2) Show that the first projection

$$\pi_{GS} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$$

is a projective map, which is generically finite of degree $n!$. (Hence π_{GS} is not a *resolution*, but rather an *alteration.*)

(3) Show that the restriction of π_{GS} to $\pi_{GS}^{-1}(\mathcal{N})$ is the Springer resolution.

(4) For $n = 2, 3$ find a stratification of $\mathfrak{g}$ into locally closed subvarieties, so that π_{GS} is a fibration in (possibly singular) varieties over each stratum. What is the least number of strata you need in each case?

Exercise 1.5

(The Gauß map) Let X be a smooth degree d hypersurface in $\mathbb{P}^N$.

(1) For any point $x \in X$ let T_x denote the tangent space to x (which we view as a hyperplane in $\mathbb{P}^N$). Show that the map $x \to T_x$ determines a morphism

$$\phi : X \to (\mathbb{P}^N)^\vee$$

to the dual projective space of hyperplanes in $\mathbb{P}^N$. (This map is the *Gauß map.*)

(2) Show that the degree of ϕ is given by $(d-1)^N$.

(3) Calculate the image of this map (the *dual variety*) in a few examples (e.g. a smooth quadric and cubic curve in $\mathbb{P}^2$). (Wikipedia has some nice pictures.)

Exercise 1.6

Let X denote the total space of $\mathcal{O}(m)$ on $\mathbb{P}^1$ (a complex surface). Show that there exists a map

$$f : X \to X'$$

which "contracts $\mathbb{P}^1$" (i.e. maps $\mathbb{P}^1 \subset X$ to a point, and is an isomorphism on the complement) if and only if $m < 0$.

The following exercises are "just for fun". They would become important if we had a few more lectures to explore the (still poorly understood) subject of perverse sheaves on surfaces.

Exercise 1.7

Consider $C \subset \mathbb{C}^2$, the plane curve singularity given by

$$x^n = y^m.$$

For any $\varepsilon > 0$ show that C intersects $S^3_\varepsilon = \{(x, y) \mid |x| + |y| = \varepsilon\} \subset \mathbb{C}^2$ in an (m, n) torus knot.

The goal of the following exercise is to calculate the fundamental group of the complement of a curve in $\mathbb{P}^2$. We follow an old method due to Zariski described in

Zariski, Oscar *On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve.* Amer. J. Math. 51 (1929), no. 2, 305–328.

which provides a lovely introduction to the "philosophy of hyperplane sections". This exercise is broader than the previous ones and you might find it harder.

Exercise 1.8

Let $C \subset \mathbb{P}^2$ denote a (not necessarily smooth) curve. Set $\infty := (0 : 0 : 1)$ and let

$$\mathrm{pr} : \mathbb{P}^2 \setminus \{\infty\} \to \mathbb{P}^1 : (x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$$

denote the projection. Let $\mathbb{P}^2_\infty$ denote the blow up of $\mathbb{P}^2$ at ∞, so that pr extends to a morphism $\mathrm{pr} : \mathbb{P}^2_\infty \to C$.

(1) Argue that by applying projective transformations one can assume that $\infty \notin C$ and that the induced map $\mathrm{pr} : C \to \mathbb{P}^1$ has at most one singularity in each fibre, which is either a singularity of C or an ordinary double point. (We assume this for the rest of the exercise.)

(2) Show that the projection induces an isomorphism

$$\pi_1(\mathbb{P}^2_\infty \setminus C) \xrightarrow{\sim} \pi_1(\mathbb{P}^2 \setminus C).$$

Let $\widetilde{\mathrm{pr}} : \mathbb{P}^2_\infty \setminus C \to \mathbb{P}^1$ denote the induced projection and let $U \subset \mathbb{P}^1$ denote the locus over which the restriction of pr to C is smooth. Fix a base point $u \in U$.

(3) Explain why every $\ell \in \pi_1(U, u)$ leads to an automorphism ϕ_ℓ of $\pi_1(\widetilde{\mathrm{pr}}^{-1}(u), \infty)$.

(4) Show that $\pi_1(\mathbb{P}^2_\infty \setminus C, \infty)$ has a presentation

$$\langle \gamma_1, \dots, \gamma_d \mid \gamma_1 \dots \gamma_d = 1, \gamma = \phi_\ell(\gamma) \text{ for all } \ell \in \pi_1(U, u) \text{ and } \gamma \in \langle \gamma_i \rangle \rangle.$$

for an appropriate choice of generators $\gamma_1, \dots, \gamma_d$ of $\pi_1(\widetilde{\mathrm{pr}}^{-1}(u), \infty)$.

(5) Make the presentation in (4) explicit in the case of a smooth degree d curve, and the curve $X^2 Z = Y^3$. See Zariski's article for more examples.

(6) Make sense of the following: Given a family C_t curves in $\mathbb{P}^2$ then $\pi_1(\mathbb{P}^2 \setminus C_t)$ gets "less complicated" (i.e. there are "more relations") as C_t becomes less singular.

(7) Use the Lefschetz hyperplane theorem to show that if $X \subset \mathbb{P}^n$ is a smooth subvariety, then $\pi_1(\mathbb{P}^n \setminus X)$ is abelian.

2. Cohomology and local systems

Exercise 2.1

In this question X is a connected complex algebraic variety.

(1) Show that the category of local systems on X is an abelian subcategory of $\mathrm{Sh}X$. (Compare with vector bundles!)
(2) Show that if X is contractible and if $\mathcal{L}$ is a local system on X then $\mathcal{L}$ is canonically isomorphic to the constant sheaf with values in $\mathcal{L}_x$ for any $x \in X$.
(3) Convince yourself that, after fixing a base-point $x \in X$, the category of local systems on X is equivalent to the category of finite dimensional representations of $\pi_1(X, x)$.

Exercise 2.2

Let L be a local system on S^1 corresponding to a (finite dimensional) vector space V with monodromy ϕ. Show that one has canonical isomorphisms:

$$H^0(S^1, L) = V^\phi \qquad \text{(invariants)},$$
$$H^1(S^1, L) = V_\phi \qquad \text{(coinvariants)}.$$

Exercise 2.3

Let X be an algebraic variety over $\mathbb{C}$.

(1) Assume that X has a filtration by closed subvarieties

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_n = X$$

such that $Z_i \backslash Z_{i-1} \cong \mathbb{C}^{n_i}$ for all $1 \leq i \leq n$. (In this situation we say that X admits an *affine paving.*) Show that $H^!_{odd}(X, \mathbb{Z}) = 0$ and

$$H^!_{2j}(X, \mathbb{Z}) = \mathbb{Z}^{d_j}$$

where $d_j = |\{1 \leq j \leq n \mid n_j = j\}|$. (*Hint:* Use the long exact sequence in homology with closed supports, and the calculation of $H^!_*(\mathbb{C}^n)$.)

(2) Find an example to show that the conclusion of (1) does not hold under the weaker assumption that X admits a decomposition

$$X = \bigsqcup X_\lambda$$

into locally closed subsets such that each X_λ is isomorphic to $\mathbb{C}^{d_\lambda}$ for some d_λ. (*Hint:* The coordinate hyperplanes $xyz = 0$ in $\mathbb{P}^2\mathbb{C}$ provides an example.)

Exercise 2.4
Calculate the homology (with closed support) of all fibres of the Springer resolution π for $n = 2, 3$ (see Question 1.3). (*Hint:* All fibres admit affine pavings.)

Exercise 2.5
Consider $U = \mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ for distinct points $z_i \in \mathbb{P}^1$. Let $\mathcal{L}$ denote the local system on U with stalk $\mathbb{Q}$ and monodromy -1 around each z_i. Show that

$$H^0(U, \mathcal{L}) = H^2(U, \mathcal{L}) = 0 \quad \text{and} \quad H^1(U, \mathcal{L}) = \mathbb{Q}^2.$$

(*Hint:* One possibility is to triangulate $\mathbb{P}^1$ as a tetrahedron with vertices z_1, z_2, z_3, z_4 and then use the edges and faces of the triangulation to build a Cech complex.)

Exercise 2.6
Suppose that $\mathcal{L}_\lambda$ is a family of local systems on a space X depending continuously on a parameter λ. Find a heuristic argument for the fact that the Euler characteristic

$$\chi(H^\bullet(X, \mathcal{L}_\lambda))$$

does not change as we vary λ. (On the other hand the individual cohomology groups may jump, as we have seen.)

Exercise 2.7
Give all details for the calculation of the monodromy in the family of elliptic curves ramified at the points $(a, b, c) \in \mathbb{C}^3$ which was outlined in lectures.

Exercise 2.8
We continue the notation of Question 2.2.

(1) Let L_λ denote the local system with $V = \mathbb{C}$ and monodromy given by multiplication by $\lambda \in \mathbb{C}^*$. Calculate $H^0(S^1, L \otimes L_\lambda)$ and $H^1(S^1, L \otimes L_\lambda)$.
(2) Let J_n denote the local system with stalk $\mathbb{C}^n$ and ϕ given by the sum of $\mathrm{id}_{\mathbb{C}^n}$ and a nilpotent Jordan block of size n. We has inclusions $\dots \hookrightarrow J_n \hookrightarrow J_{n+1} \hookrightarrow \dots$. Show that one has isomorphisms

$$\lim_{\to} H^0(S^1, L \otimes J_n) = V^{(1)},$$
$$\lim_{\to} H^1(S^1, L \otimes J_n) = 0$$

where $V^{(1)} \subset V$ denotes the generalized 1-eigenspace of ϕ. (*Hint:* It might help to decompose V into Jordan blocks for ϕ.)

(The point of this question is that the stalk of a local system L on S^1 can be recovered by taking global cohomology, if we allow ourselves to tensor with "standard" local systems and take limits. This is important in algebraic approaches to vanishing cycles.)

The following exercise is "just for fun".

Exercise 2.9

Give an explicit injective resolution of the constant sheaf on S^1 with coefficients in a field, and hence calculate $H^*(S^1)$ "directly from the definitions". (Once you have done this exercise, you can disagree with anyone that tells you that "no-one ever works out sheaf cohomology by actually taking an injective resolution"!)

3. Constructible sheaves

Exercise 3.1
Let L be a local system on $\mathbb{C}^*$ corresponding to a finite-dimensional vector space V and monodromy μ. Let $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ denote the inclusion. Show that $(j_* L)_0 = V^\mu$ (invariants). (Here j_* denotes the non-derived functor.)

Exercise 3.2
Let $f : X \to Y$ be a morphism of algebraic varieties and consider the sheaf $\mathcal{F}$ on Y associated to the presheaf $\mathcal{F}'$:

$$U \mapsto H^i(f^{-1}(U), k)$$

(1) Calculate $\mathcal{F}$ for a few interesting maps. Try to find some interesting constructible sheaves in this way.
(2) Give examples of maps to show that $\mathcal{F}'$ need not satisfy either of the sheaf conditions in general (i.e. sheafification is really necessary). (*Hint:* It might be easier to find non-algebraic examples first, e.g. think about the Hopf fibration.)
(3) Show that $\mathcal{F}$ is a local system if f is proper and smooth.

Exercise 3.3
Let C be a smooth connected curve and suppose that $\mathcal{F}$ is a constructible sheaf on C, constructible with respect to the stratification

$$C = U \sqcup \bigsqcup_{i=1}^{n} \{x_i\}.$$

For each $x_i \in C$ fix nearby points x_i' and let ℓ_i denote a small loop at x_i' encircling x_i. Show that $\mathcal{F}$ is uniquely determined by the following data:

(1) the local system $\mathcal{F}_{|U}$;
(2) for each $1 \leq i \leq n$ a map

$$\mathcal{F}_{x_i} \to \mathcal{F}_{x_i'}^{\mu_i}$$

where μ_i denotes the monodromy of $\mathcal{F}_{x_i}$ around the loop ℓ_i and $\mathcal{F}_{x_i'}^{\mu_i}$ denotes invariants under μ_i.

Exercise 3.4
Let E be an elliptic curve and $p : E \to \mathbb{P}^1$ a degree 2 map ramified at 4 points (see Question 1.1).

(1) Show that we have a decomposition $p_*\mathbb{Q}_E = \mathbb{Q}_{\mathbb{P}^1} \oplus \mathcal{L}'$ where $\mathcal{L}' = j_!\mathcal{L} = j_*\mathcal{L}$ ($\mathcal{L}$ is the local system from Question 2.5 and j denotes the appropriate inclusion).
(2) Use your knowledge of the cohomology of $\mathbb{P}^1$ and the calculation in Question 2.5 to (re)calculate the cohomology of E.

Exercise 3.5

(More work) Let X be a simplicial complex. Find a reasonable notion of "constructible sheaf" on X, and find a quiver description of the resulting category.

4. Constructible derived category

Exercise 4.1
Do Question 3.1 with j_* replaced by Rj_*. (That is, show that $\mathcal{H}^m(Rj_*L) = 0$ for all $m \neq 0, 1$ and $\mathcal{H}^1$ is a skyscraper at zero with stalk V_μ.) Deduce that the canonical map $j_!L \to Rj_*L$ is an isomorphism if and only if 1 does not belong to the spectrum of μ.

Exercise 4.2
Let $H : S^3 \to S^2$ denote the Hopf fibration. Show that the "decomposition theorem fails" for $RH_*\mathbb{Q}_{S^3}$. That is, it is not true that one has a decomposition in $D^b(S^2)$

$$RH_*\mathbb{Q}_{S^3} \cong \mathcal{H}^0(RH_*\mathbb{Q}_{S^3}) \oplus \mathcal{H}^1(RH_*\mathbb{Q}_{S^3})[-1].$$

(*Hint:* Think about the implications of such a decomposition when taking global sections.)

Exercise 4.3
Calculate the stalks of the dualizing complex on the following spaces:

(1) m copies of $\mathbb{R}_{\geq 0}$ all joined at $0 \in \mathbb{R}_{\geq 0}$.
(2) m copies of $\mathbb{C}$ all joined at $0 \in \mathbb{C}$.

Exercise 4.4

(1) Let $x \in X$, and let $d_X = \dim_{\mathbb{C}} X$. Show that the stalk of the dualizing complex at x agrees with that of $\mathbb{Q}_X[2d_X]$ iff the following condition is satisfied: there exists an embedding $j : U \hookrightarrow \mathbb{C}^n$ of a neighbourhood of U such that $j(x) = 0$ and so that, for all small enough ε we have

$$H^*((\mathbb{C}^n \setminus U) \cap B(0,\varepsilon)) \cong H^*((\mathbb{C}^n \setminus \mathbb{C}^{d_X}) \cap B(0,\varepsilon))$$

where $B(0,\varepsilon) \subset \mathbb{C}^n$ denotes an open ε-ball. (In other words, the dualizing complex looks like the constant sheaf at x iff X cohomology can't distinguish U from $\mathbb{C}^{d_X}$.)

(2) Suppose that the condition of the previous exercise is met for all $x \in X$. Show that $\omega_X \cong \mathbb{Q}_X[2d_X]$. (*Hint:* This exercise becomes easier (at least for me) once we have discussed perverse sheaves and intersection cohomology complexes. Don't worry if you can't solve it immediately.)

Exercise 4.5
Let W be the Whitney umbrella (see Question 1.2), let $W_{\mathrm{reg}} \subset W$ (resp. Z) denote the non-singular (resp. singular) locus, and $j : W_{\mathrm{reg}} \hookrightarrow W$ (resp. $i : Z \hookrightarrow W$) the inclusion. Show that the cohomology sheaves of

$$i^* Rj_* \underline{\mathbb{Q}}_{W_{\mathrm{reg}}} \in D^b_c(Z, \mathbb{Q})$$

is not a local system on Z. Describe this complex as explicitly as you can.

Exercise 4.6
Let X be an affine "quasi-homogenous cone", in other words, X is an affine variety admitting an action of $\mathbb{C}^*$ such that

$$\lim_{z \to 0} z \cdot x = x_0$$

for some fixed $x_0 \in X$ and all $x \in X$. Denote the inclusions of $\{x_0\}$ and $U := X \setminus \{x_0\}$ into X by i and j. Show that

$$H^m(i^* j_* \mathbb{Q}_U) = H^m(U; \mathbb{Q}).$$

Exercise 4.7

(1) Let $f : Z \hookrightarrow X$ be the inclusion of a locally closed subset and $\mathcal{F} \in \mathrm{Sh}_X$. For any open set $V \subset Z$ choose an open set U in X such that $X \cap Z = V$ and set

$$\mathcal{F}^!_Z(V) := \{ s \in \mathcal{F}(U) \mid \operatorname{supp} s \subset V \}.$$

Show that $\mathcal{F}^!_Z(V)$ is independent of the choice of U and defines a sheaf on Z. This sheaf $\mathcal{F}^!_Z$ is called the *sections of $\mathcal{F}$ with support in Z*.

(2) Show that the assignment $\mathcal{F} \mapsto \mathcal{F}^!_Z$ extends to a functor $\mathrm{Sh}_X \to Sh_Z$ and that the resulting functor is right adjoint to $f_!$.

(3) Deduce that (with f as above) $f^!$ is the derived functor of $\mathcal{F} \mapsto \mathcal{F}^!_Z$.

(4) Show that $f^* = f^!$ if f is the inclusion of an open subset.

(5) Give an example of a map $f : X \to Y$ between algebraic varieties such that $f_! : \mathrm{Sh}_X \to \mathrm{Sh}_Y$ does not have a right adjoint. (Hence the passage to the derived category to construct $f^!$ is essential.)

(6) Suppose that we have a contravariant duality $\mathbb{D} : D^b_c(X; k) \to D^b_c(X, k)$ and functorial isomorphisms

$$H^*(X, \mathcal{F}) \to H^*_!(X, \mathbb{D}\mathcal{F})^*$$

(more precisely, one would like a functorial isomorphism in the derived category) for all sheaves $\mathcal{F}$ and varieties X. Show that

$\mathbb{D}\mathcal{F}$ is isomorphic to the functor

$$\mathbb{D}\mathcal{F} := \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}, (X \to \mathrm{pt})^! \underline{k}_{\mathrm{pt}}).$$

(In part this motivates the search for a right adjoint to $f_!$.)

Hint: Fix X, then swapping $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ the canonical isomorphism $H^*(X, \mathbb{D}\mathcal{F}) = H^*_!(X, \mathcal{F})^*$ leads to (for all $U \subset X$ open)

$$\begin{aligned} \mathbb{D}\mathcal{F}(U) &= R\,\mathrm{Hom}(p_{U!}\mathcal{F}_U, \underline{k}_{\mathrm{pt}}) = R\,\mathrm{Hom}(\mathcal{F}_U, p_U^! \underline{k}_{\mathrm{pt}}) = \\ &= R\,\mathrm{Hom}(\mathcal{F}_U, i_U^! p_X^! \underline{k}_{\mathrm{pt}}) = \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}, p_X^! \underline{k}_{\mathrm{pt}})(U) \end{aligned}$$

where p_U, p_X denote the projections to a point, $i_U : U \hookrightarrow X$ denotes the inclusion and we have used that $i_U^! = i_U^*$ because i_U is an open inclusion.

Exercise 5.1
Prove Beilinson's lemma: Consider a triangulated functor

$$F : D \to D'$$

and suppose that we have a set of objects $\{M_\lambda\}$ such that

$$F : \mathrm{Hom}_D(M_\lambda, M_\mu[i]) \xrightarrow{\sim} \mathrm{Hom}_{D'}(F(M_\lambda), F(M_\mu[i]))$$

for all λ, μ and i. Then F induces an equivalence between the triangulate subcategories of D and D' generated by $\{M_\lambda\}$ and $\{F(M_\lambda)\}$. In particular, F is an equivalence if D and D' are generated by $\{M_\lambda\}$ and $\{F(M_\lambda)\}$ respectively.

Exercise 5.2

(1) Let $\mathbb{Z}_{(p)}$ denote the local ring of $(p) \subset \mathbb{Z}$ (the subring of $\mathbb{Q}$ consisting of a/b where $gcd(p, b) = 1$). Classify all t-structures on the derived category of finite generated $\mathbb{Z}_{(p)}$-modules.
(2) Classify all t-structures on the derived category of finitely generated $\mathbb{Z}$-modules.

Exercise 5.3
Suppose that we are in the glueing situation axiomatized in lectures. Prove that

$$D^{\leq 0} := \{X \in D \mid i^*X \in D_Z^{\leq 0} \text{ and } j^*X \in D_U^{\leq 0}\}$$
$$D^{\geq 0} := \{X \in D \mid i^!X \in D_Z^{\geq 0} \text{ and } j^!X \in D_U^{\geq 0}\}$$

defines a t-structure on D. (*Hint:* If you get stuck, you can look at the proof of Théorème 1.4.10 in [BBD].)

Exercise 5.4
Use the previous question to convince yourself that perverse sheaves do indeed form an abelian category.

Exercise 5.5
Consider $X = \mathbb{P}^1$ with the stratification Λ given by $X = \{0\} \cup \mathbb{C}$. Let $D = D^b_\Lambda(X)$ and let $D_Z := D^b(\mathrm{pt})$ and $D_U := D^b_{\mathrm{const}}(\mathbb{C})$ (complexes of sheaves on $\mathbb{C}$ with locally constant (= constant in this case) cohomology). Fix $m \in \mathbb{Z}$ and consider the t-structures on D_Z and D_U with hearts

$$\mathrm{Loc}_{\mathrm{pt}} \quad \text{and} \quad \mathrm{Loc}_{\mathbb{C}}[m].$$

respectively. Let $\mathcal{A}_m$ denote the heart of the abelian category obtained by glueing these two t-structures.

(1) Show that $\mathcal{A}_0$ is simply the abelian category of Λ-constructible sheaves on $\mathbb{P}^1$, and hence is equivalent to representations of the quiver $0 \to 1$.
(2) Use Verdier duality to deduce that the $\mathcal{A}_2$ is equivalent to representations of the quiver $0 \leftarrow 1$.
(3) Show that $\mathcal{A}_m$ for $m \notin \{2, 1, 0\}$ is semi-simple with two simple objects. (*Hint:* It might be helpful to do the exercise following this one first.)
(4) For general reasons (explained in [BBD], §3.1) there is always a functor
$$\text{real} : D^b(\mathcal{A}_m) \to D^b_\Lambda(\mathbb{P}^1).$$
Show that real cannot be an equivalence if $m \neq 1$. (*Hint:* The rough idea is that the categories $\mathcal{A}_m$ for $m \neq 1$ are "too homologically simple" to capture $D^b_\Lambda(\mathbb{P}^1)$.)

We will see in lectures that $\mathcal{A}_1$ is the category of perverse sheaves, and that it is equivalent to representations of the quiver

$$0 \underset{e}{\overset{f}{\rightleftarrows}} 1$$

subject to the relation $fe = 0$.

(5) Show that real : $D^b(\mathcal{A}_1) \to D^b_\Lambda(\mathbb{P}^1)$ is an equivalence. (This is another exercise which might be tricky now, but gets much easier when we learn a little more about perverse sheaves.)

The following question is simply Proposition 1.4.14 of [BBD] made into an exercise. It is worth going through carefully.

Exercise 5.6

Let X be variety with a fixed (Whitney) stratification Λ. Suppose that $Z \subset X$ is a smooth closed stratum and let $U := X \setminus Z$.

Given $\mathcal{F} \in D_U$, an *extension* $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ is a pair $(\widetilde{\mathcal{F}}, \alpha)$ where $\widetilde{\mathcal{F}} \in D^b_\Lambda(X)$ and $\alpha : j^*\widetilde{\mathcal{F}} \to \mathcal{F}$ is an isomorphism .

Let $D^{\leq 0}_Z, D^{\geq 0}_Z$ on $D_\Lambda(Z)$ denote the standard t-structure with heart local systems on Z and $\tau^Z_{\leq 0}, \tau^Z_{\geq 0}$ the truncation functors (as explained in lectures, or see [BBD, 1.4.10]).

(1) Fix $p \in \mathbb{Z}$ and $\mathcal{F} \in D_U$. Consider extensions $\widetilde{\mathcal{F}}$ satisfying
$$(*) \quad i^!\widetilde{\mathcal{F}} \in D^{\geq p+1}_Z \quad \text{and} \quad i^*\widetilde{\mathcal{F}} \in D^{\leq p-1}_Z.$$

Show that:

(a) $\tau^F_{\leq p-1} j_* \widetilde{\mathcal{F}}$ satisfies (*);

(b) any $\widetilde{\mathcal{F}}$ satisfying (*) is unique (up to unique isomorphism);

(c) $\tau^F_{\leq p-1} j_* \widetilde{\mathcal{F}} = \tau^F_{\geq p+1} j_! \mathcal{F}$.

(2) If $\mathcal{F}$ is a perverse sheaf, deduce that we have

$$ {}^p j_! \mathcal{F} = \tau^F_{\leq -d_Z - 2} j_* \mathcal{F}, j_{!*} \mathcal{F} = \tau^F_{\leq -d_Z - 1} j_* \mathcal{F} \quad \text{and} \quad {}^p j_* \mathcal{F} = \tau^F_{\leq -d_Z} j_* \mathcal{F}. $$

(3) Let $i : Z \hookrightarrow X$ be the inclusion. Let $\mathcal{F} \in \mathcal{M}_X$ denote a perverse sheaf. Show that the adjunction morphism $i_!({}^p i^! \mathcal{F}) \to \mathcal{F}$ may be identified with the inclusion of the largest subobject supported on Z. What is the corresponding statement for ${}^p i^*$?

Exercise 5.7

Let C be a smooth curve and Λ a stratification of C (given by $U \subset C$ and points $z_1, \dots, z_d \in C$ consisting of the complement of U in C).

(1) Suppose that the stratification is trivial (i.e. $U = C$). Find conditions on C for the natural functor

$$ D^b(\mathrm{Sh}_\Lambda) \to D^b_\Lambda(C) $$

to be an equivalence. (Here $\mathrm{Sh}_\Lambda(C)$ denote the abelian category of Λ-constructible sheaves.)

(2) (Harder, and not yet done by me) Find conditions on the stratification for

$$ D^b(\mathrm{Sh}_\Lambda) \to D^b_\Lambda(C) $$

to be an equivalence in general.

6. Perverse sheaves

Exercise 6.1

Let C denote a smooth curve and $\mathcal{L}$ a local system on a Zariski open subset of C. Calculate the stalks of $\mathbf{IC}(C,\mathcal{L})$ in terms of $\mathcal{L}$. (*Hint:* Use the Deligne construction and Question 4.1.)

Exercise 6.2

Let $j : U \hookrightarrow X$ denote an open inclusion. Show that $j_{!*} : \mathcal{M}_U \to \mathcal{M}_X$ preserves injections and surjections.

Exercise 6.3

Let $\mathcal{L}$ denote the local system on $\mathbb{C}^*$ with stalk $\mathbb{Q}^n$ and monodromy given by a single Jordan block, all of whose eigenvalues are 1. Let $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ denote the inclusion. Show that $j_{!*}\mathcal{L}$ is uniserial and calculate its composition series. Deduce that the functor $j_{!*}$ is *not* exact. (*Hint:* Use the quiver description of $\mathcal{M}_{C,\Lambda}$ given in lectures. Can you do this exercise without using the quiver description?)

Exercise 6.4

Let X denote a complex algebraic variety and $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ the perverse t-structure on $D := D^b_c(X)$. Show that

(1) ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$ are exchanged by $\mathbb{D}$;
(2) $\mathcal{F} \in D^{\leq 0}$ if and only if

$$\dim_{\mathbb{C}} \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i$$

for all i.
(3) If $\mathcal{F}$ is perverse then $\mathcal{H}^i(\mathcal{F}) = 0$ for $i < -\dim_{\mathbb{C}} X$.

Exercise 6.5

In the setup of Question 6.1, assume further that C is complete, and that $\mathcal{L}$ is simple and not the trivial local system. Show that $H^i(C, \mathbf{IC}(C,\mathcal{L})) = 0$ unless $i = 0$ and find a formula for the dimension of $H^0(C, \mathbf{IC}(C,\mathcal{L})) = 0$ in terms of data attached to $\mathcal{L}$.

Exercise 6.6

(1) Let X be an affine variety. Show that $H^i(X,\mathcal{F})$ and $H_!^{-i}(X,\mathcal{F})$ are zero for $i > 0$ and any perverse sheaf $\mathcal{F}$ on X. (This is the famous "weak Lefschetz theorem for perverse sheaves".)
(2) What goes wrong in the previous exercise if we assume instead that $\mathcal{F}[-\dim_{\mathbb{C}} X]$ is constructible?

(3) (Maybe harder, and not yet done by me.) Can perverse sheaves on an arbitrary projective variety be charaterised by the vanishing conditions in (1) on the complement of all hyperplane sections?

Exercise 6.7

Let $X \subset \mathbb{P}^n$ be a smooth projective variety, and $C_X \subset \mathbb{C}^{n+1}$ be the affine cone over X. Calculate the stalk of $\mathbf{IC}(C_X, \mathbb{Q})$ at the unique singular point in terms of the cohomology of X and the action of the Lefschetz operator. (*Hint:* Question 4.6 is the key here. Note that $C_X \setminus \{0\}$ is a $\mathbb{C}^*$-bundle over X. Calculate its cohomology via the Gysin sequence.)

Exercise 6.8

(Continuing the previous question.)

(1) Let $PC_X \subset \mathbb{P}^{n+1}$ denote the projective cone over X (so if X is given by equations $\{f_i\} \in k[x_0, \ldots, x_n]$ then $PC_X \subset \mathbb{P}^{n+1}$ is given $\{f_i\} \in k[x_0, \ldots, x_n, x_{n+1}]$, in particular PC_X has a decomposition $X \sqcup C_X$). Calculate the Betti numbers of $IH^*(PC_X)$ in terms of those of $H^*(X)$.
(2) Calculate $H^*(PC_X)$ and compare it to $IH^*(PC_X)$.
(3) (Amusing!) Let $PC_X^m \subset \mathbb{P}^{n+m}$ denote (the projective cone over)^m X. Calculate the Betti numbers of $IH^*(PC_X^m)$ in terms of $H^*(X)$.

Exercise 6.9

Suppose that X is an irreducible projective variety of dimension d_X with only isolated singularities. Show that one has natural isomorphisms

$$IH^m(X) = \begin{cases} H^{m+d_X}(U) & \text{for } m < 0, \\ H_!^{m+d_X}(U) & \text{for } m > 0 \end{cases}$$

where $U \subset X$ denotes the non-singular locus. Can you describe $IH^0(X)$ in terms of $H^*(U)$ and the singularities of X?

Exercise 6.10

Let C be a smooth complete curve with a fixed stratification Λ and let $U \subset C$ denote the open stratum. Fix two points $z_1, z_2 \in U$ and let

$C_i := C - \{z_i\}$, $C_{12} := C - \{z_1, z_2\}$. Consider the inclusions:

$$\begin{tikzcd} C_{12} \arrow[r, hook, "j_2"] \arrow[d, hook, "j_1"'] \arrow[dr, hook, "j_{12}"] & C_1 \arrow[d, hook, "j_1"] \\ C_2 \arrow[r, hook, "j_2"] & C \end{tikzcd}$$

In the following $\mathcal{F}$ is a perverse sheaf on C constructible with respect to Λ. (All that really matters is that z_1, z_2 are "non-singular" points of $\mathcal{F}$.)

(1) Show that we have a canonical isomorphisms

$$j_{1*}j_{2!}j_{12}^*\mathcal{F} = j_{2!}j_{1*}j_{12}^*\mathcal{F} = j_{2!}j_2^!j_{1*}j_1^*\mathcal{F} = j_{1*}j_1^*j_{2!}j_2^!\mathcal{F}.$$

We call the resulting sheaf $\mathcal{F}_{!*}$.

(2) Show that $H^m(C, \mathcal{F}_{!*}) = 0$ for $m \neq 0$.

(3) If you haven't already done so, deduce (2) from the weak Lefschetz theorem for perverse sheaves.

(4) Show that one has an exact sequence of perverse sheaves

$$0 \to i_{2*}i_2^*\mathcal{F}[-1] \to \mathcal{F}_{!*} \to i_{1!}i_1^!\mathcal{F}[1] \to 0.$$

and that all sheaves in this sequence satisfy $H^m(-) = 0$ for $m \neq 0$.

(5) Deduce that

$$0 \to H^{-1}(i_2^*\mathcal{F}) \to H^0(\mathcal{F}_{!*}) \to H^1(i^!\mathcal{F}) \to 0$$

is a complex computing $H^*(C, \mathcal{F})$.

For more details on this beautiful complex (which exists more generally for any perverse sheaf on any projective variety) was discovered and is explored in last few pages of:

Beilinson, A. A. *On the derived category of perverse sheaves.* K-theory, arithmetic and geometry (Moscow, 1984–1986), 27–41, LNM, **1289**, Springer, Berlin, 1987.

7. The decomposition theorem

Exercise 7.1

What does the decomposition theorem say in the case where $f : Bl_Z X \to X$ is the blow-up of a smooth variety X in a smooth closed subvariety $Z \subset X$?

Exercise 7.2

Suppose that D is a triangulated category with a non-degenerate t-structure $(D^{\leq 0}, D^{\geq 0})$. Let $\mathcal{A} \in D$ and suppose that we are given a morphism

$$\eta : \mathcal{A} \to \mathcal{A}[2]$$

such that η^i induces an isomorphism

$$\eta^i : \mathcal{H}^{-i}(\mathcal{A}) \to \mathcal{H}^i(\mathcal{A})$$

for all $i \geq 0$. (Here $\mathcal{H}^i$ denotes the cohomology functors with respect to the t-structure $(D^{\leq 0}, D^{\geq 0})$. Show that we have an isomorphism

$$\mathcal{A} \cong \bigoplus \mathcal{H}^i(\mathcal{A})[-i].$$

(*Hint:* If $\mathcal{H}^i(\mathcal{A}) = 0$ for $i < -m \leq 0$ then we have maps

$$\mathcal{H}^{-m}(\mathcal{A})[m] = \tau_{\leq -m}\mathcal{A} \hookrightarrow \mathcal{A} \xrightarrow{\eta} \mathcal{A}[2] \xrightarrow{\eta} \cdots \xrightarrow{\eta} \mathcal{A}[2m] \to \tau_{\geq m}\mathcal{A}[2m] = \mathcal{H}^m(\mathcal{A})[m].$$

such that the composition $\mathcal{H}^{-m}(\mathcal{A})[m] \to \mathcal{H}^m(\mathcal{A})[m]$ is an isomorphism. Hence $\mathcal{H}^{-m}(\mathcal{A})[m]$ is a summand of $\mathcal{A}$, $\mathcal{A}[2]$, ..., $\mathcal{A}[2m]$. Now extract these summands and repeat...)

What is the relevance of this result for the decomposition theorem?

Exercise 7.3

(1) Make the decomposition thoerem explicit (i.e. determine all simple summands and their stalks) in the case of the Weierstraß family of elliptic curves

$$y^2 = x(x-a)(x-1)$$

over $a \in \mathbb{C}$. We calculated the monodromies of the local systems which occur in lectures; check that they are semi-simple.

(2) Use the hard Lefschetz theorem applied to the locus of curves over $a \in \mathbb{C} - \{0,1\}$ to give a direct proof of the decomposition theorem using Question 7.2 in this case.

Exercise 7.4

Make the decomposition theorem explicit in the case of the Springer resolution for $n = 2, 3$. (See Questions 1.3 and 2.4.)

Exercise 7.5

(Harder) Let X be a smooth degree d hypersurface in $\mathbb{P}^3$ and let $f : \widetilde{X} \to \mathbb{P}^1$ be the corresponding Lefschetz fibration (so $\widetilde{X}$ is the blow up of X in m points). Make the decomposition theorem explicit for f. (Here Question 1.5 is relevant.)