THE GEOMETRY OF REFLECTION GROUPS

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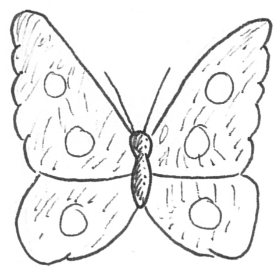
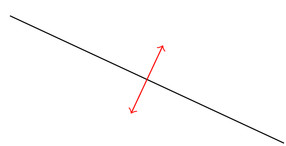
1. Reflection groups

Our first encounter with symmetry might be an encounter with a butterfly



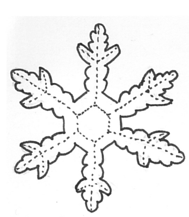
or perhaps with the face of our mother or father. We quickly learn to identify the axis of symmetry and know intuitively that an object is symmetric if it "the same" on both sides of this axis.

In mathematics symmetry is abundant and takes many forms. Symmetry like that of the butterfly or a face is referred to as *reflexive symmetry*. Any line in the plane determines a unique symmetry which reflects the plane about this line:

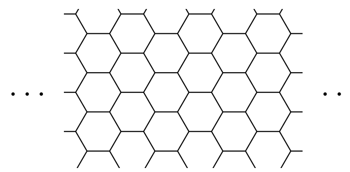


A figure in the plane has reflexive symmetry if the reflection about a given axis of symmetry yields an identical figure in the plane.

As children we were struck by the beauty of objects with many reflexive symmetries. For example, a snowflake has six axes of symmetry:

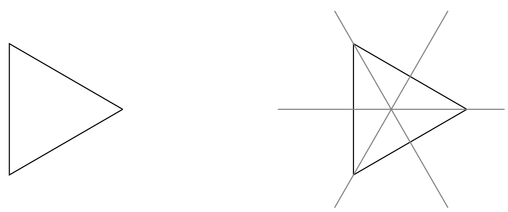


An infinite beehive has infinitely many symmetries:



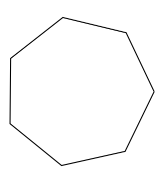
A fascinating area of current mathematical study is that of reflection groups. These are collections, or *groups*, of symmetries in which every symmetry can be expressed as compositions of reflexive symmetries. The symmetry group of a butterfly is the set $\{id, s\}$ where s is the reflexive symmetry. We write $ss = s^2 = id$ to express the fact that if we perform s twice we "do nothing", referred to as the identity transformation in the theory of groups.

The next simplest example of a reflection group is the symmetries of an equilateral triangle:



The reflections in the marked axes of symmetries give three reflexive symmetries. The reader can check that performing two reflections in two different axes of symmetry yields a rotation. This gives a complete description of the group of symmetries of the triangle: it has three reflexive symmetries, two rotational symmetries and the identity transformation.

Similarly, the symmetries of a regular polygon with n faces

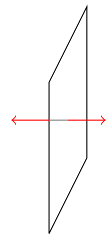


$n = 7$

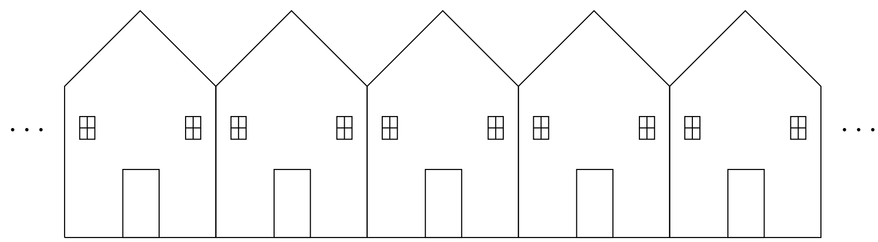
yields a group with n reflections, $n - 1$ rotations and the identity transformation, giving a total of $2n$ symmetries. If $n = 6$ we recover the symmetries of the snowflake!

The notion of reflexive symmetry makes sense in any dimension. In one dimension the "axis of symmetry" is a point:

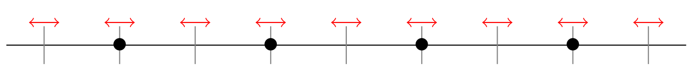
In three dimensions reflections take place about a plane:



How many reflection groups are there? In one dimension there are only two. The first is the symmetries of the butterfly, which is really a one-dimensional example (perhaps the reader can see why?). The second can be described as the symmetries of an infinite row of symmetrical houses:

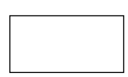


Or as the symmetries of the whole numbers amongst all real numbers:



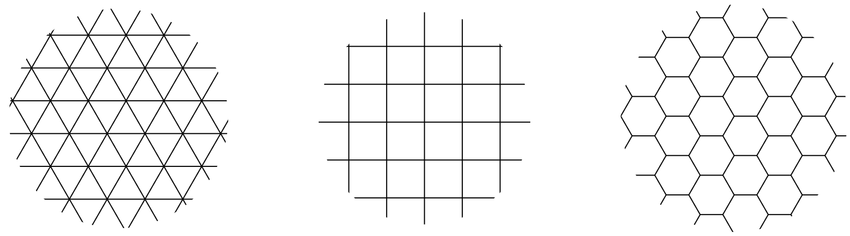
Here there are infinitely many axes of symmetry. This is an example of an infinite reflection group.

In two dimensions the situation is more complicated. One might start with the symmetries of a rectangle:



However mathematicians regard this as being simply two copies of the symmetries of the butterfly. (The horizontal and vertical symmetries do not interract. Hence the symmetry group of the rectangle is simply a "product" of the symmetries in the horizonal and verticle directions.)

Ignoring examples that "come from one dimension" it turns out that all finite reflection groups are given by the symmetries of a regular n-gon, which we discussed above. There are infinite examples of two types. The first type consists of the symmetries of crystal structures in the plane:



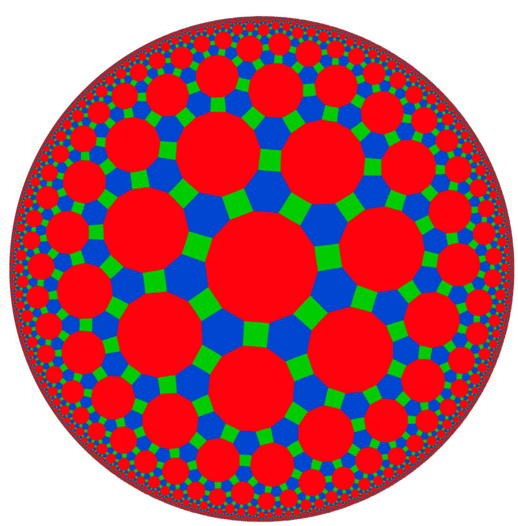
(The last example is the infinite beehive.)

The second class of infinite reflection groups consists of symmetries of the hyperbolic plane, an example of a non-euclidean geometry. In school we learn that sum of the angles of a triangle is always equal to π. However this is only true in the plane. On the surface of a sphere the angle sum of a triangle lies between π and 3π, depending on how big the triangle is. In the hyperbolic plane all triangles have angle sums between 0 (big triangles) and π (small triangles).

It turns out that for any positive integers $p, q, r \geq 2$ such that

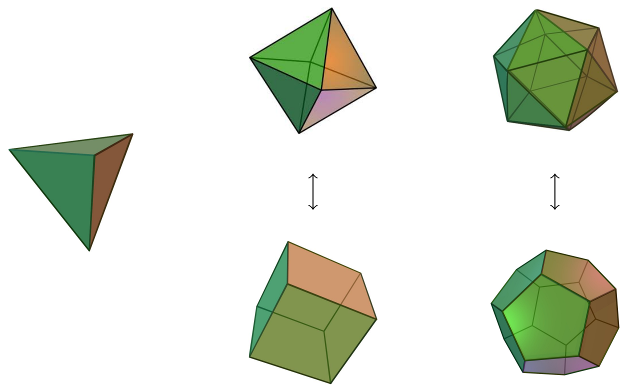
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

there exists a two dimensional reflection group, acting as symmetries on the hyperbolic plane. For example, the symmetries of the following configuration in the hyperbolic plane



corresponds to $p = 2$, $r = 3$ and $q = 7$. Because almost all triples (p, q, r) of positive integers satisfy the above inequality, almost all two dimensional reflection groups are hyperbolic.

In three dimensions there are many infinite reflection groups. However, it turns out that the only finite reflection groups are those that "come from dimension 2" together with the symmetries of the platonic solids:

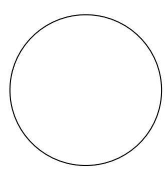


The reader might remember that the cube and octahedron as well as the icosahedron and dodecahedron are "dual", and hence their symmetry groups are equal.

2. CONTINUOUS AND DISCRETE SYMMETRY

Reflection groups are *discrete*: one cannot move a symmetry a small amount and obtain a new symmetry. For example in any of the examples above, moving an axis of symmetry a small amount never results in a new axis of symmetry.

An example of a group which does not have this property is the group of symmetries of a circle:



Here *any* line through the origin serves as an axis of symmetry. Similarly any rotation about the origin is a symmetry. Here one speaks of *continuous* symmetry. Other examples of continuous symmetry include the symmetries of a sphere, or the group of all rigid motions of space.

Groups of continous symmetry were first investigated in depth by a Norwegian mathematician Sophus Lie, and today are called Lie groups. Lie noticed that many equations governing the world around us have large degrees of continuous symmetry. He hoped that the presence of this symmetry could be used to constrain solutions. This simple idea plays a fundamental role in modern mathematics and physics. (Fourier analysis, a basic and pervasive tool in modern science, gives one of the simplest examples of this phenomenon.)

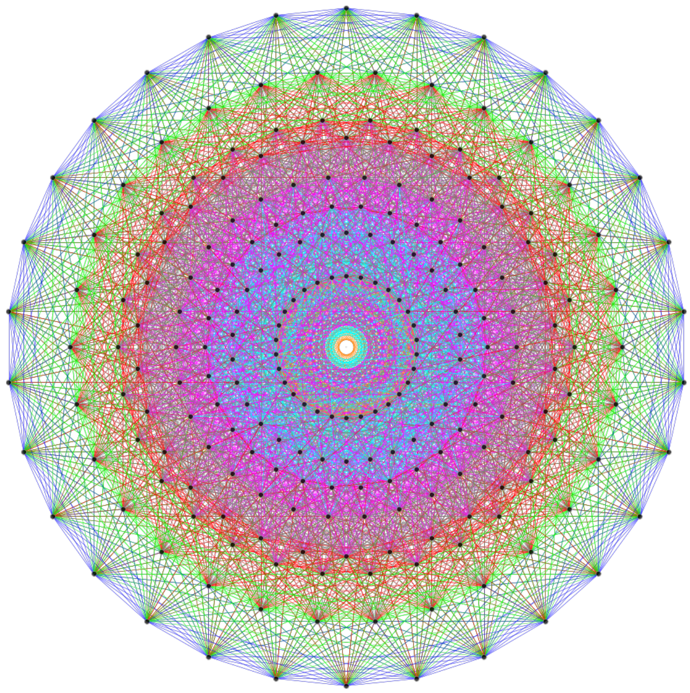
Lie also initiated a program, which continues to this day, to understand the structure of groups of continous symmetry. Here an amazing fact emerges: one can associated to any Lie group (usually a rather complicated high-dimensional object) a finite reflection group! This finite reflection group is called Weyl group after the German mathematician Hermann Weyl who emphasised its importance in understanding the structure of Lie groups.

The way in which one associates the Weyl group to a Lie group is a somewhat tricky process, and we will have to be content with discussing the first interesting example. If we consider the Lie gorup of symmetries of the sphere, then its Weyl group is the symmetries of the butterfly!

(For the interested reader, one obtains the Weyl group in this example as follows: first, fix a line ℓ through the origin and consider all rotations of the sphere which preserve this line. Such rotations are of two types: those rotations in the axis ℓ (which fix ℓ pointwise), and those rotations through π in an axis perpendicular to the chosen line (which act as a reflection on the line ℓ). The residual action of these rotations on ℓ is the simplest example of a reflection group!)

These ideas allowed Lie, Cartan and Killing to obtain a complete classification of the so-called connected compact Lie groups. A startling feature of this classification is that one has a number of families in all dimensions (for example the symmetries of the spheres in all dimensions), together with certain exceptional groups, which occur only in “small” dimensions. The largest of these examples if a 248 dimensional Lie group which mathematicians call E_8.

The Weyl group of the Lie group E_8 is a fascinating, but complicated 8-dimensional reflection group which is the symmetries of a certain collection of vectors in 8 dimenionsal space. Mathematicians have drawn the following picture of these vectors:



The group Lie group E_8 been the subject of much investigation and speculation by mathematicians and physicists alike, with some even positing it as giving the elusive theory of everything!

3. Current directions

Fourier series and the Fourier transform emerges naturally when on does calculus on the two simplest Lie groups: the groups of rotations of the circle, and the group of translations of the real line. Doing calculus on the Lie group of rotations of the 2-sphere may be used to determine the spectrum of the hydrogen atom, which was the first breakthrough in quantum mechanics.

Today the subject of calculus on Lie groups is an active and fascinating subject. In modern language one wishes to determine the *unitary dual* of a Lie group. This remains an unsolved problem. A major breakthrough was obtained by Kazhdan and Lusztig in 1979 who defined certain polynomials for every pair of elements in the Weyl group of a Lie group. Work of a number of mathematicians has highlighted the importance of these polynomials in many situations where discrete or continuous symmetry is present. In particular, this led to a major breakthrough (though not a complete solution) to the problem of determining the unitary dual.

One curious feature of Kazhdan and Lusztig's construction is the following: if one starts with a Lie group, its Weyl group is necessarily a finite reflection group of a special type. For example the only two dimensional reflection groups which occur are the symmetries of the triangle, square and hexagon. Similarly, in dimension three the symmetry group of the a tetrahedron and cube occurs, whereas the symmetry group of the icosahedron does not. However Kazhdan and Lusztig's constructure still makes sense, and produces polynomials with remarkable properties. Only time will tell what these polynomials are trying to tell us!

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