Equivariant cohomology, localisation and moment graphs

University of Melbourne
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Monday Problem Sheet

1. Let G be a topological group and E a free G-space. Show that the diagonal G action on $E \times X$ is free for any G-space X.

2. Describe classifying spaces for the following groups: *a)* $\mathbb{Z}$, *b)* $\mathbb{Z}^n$, *c)* $\pi_1(\Sigma)$ where Σ is a compact Riemann surface of genus $g \geq 1$, *d)** $G = \pi_1(M)$ where M is a compact hyperbolic manifold.

3. Denote by $\mathbb{C}^\infty$ the direct sum $\bigoplus_{i=1}^\infty \mathbb{C}e_i$. Show that $\mathbb{C}^\infty \setminus \{0\}$ is contractible first using Whitehead's theorem and then without it. Similarly show that $S^\infty = \{v \in \mathbb{C}^\infty \mid |v| = 1\}$ is contractible.

4. Let $\mathbb{P}^\infty\mathbb{C}$ denote the quotient of $\mathbb{C}^\infty \setminus \{0\}$ by the diagonal $\mathbb{C}^*$-action.

a) Show that
$$H^i(\mathbb{P}^\infty\mathbb{C}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i \geq 0 \text{ is even,} \\ 0 & \text{if } i \geq 0 \text{ is odd.} \end{cases}$$

b) Apply the Leray-Serre spectral sequence to the fibration $\pi : \mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty\mathbb{C}$ to conclude that one has an isomorphism of rings
$$H^\bullet(\mathbb{P}^\infty\mathbb{C}, \mathbb{Z}) = \mathbb{Z}[X]$$
where X denotes the first Chern class of π.

(You might like to start by proving an analagous statement for $H^\bullet(\mathbb{P}^{n-1}\mathbb{C}, \mathbb{Z})$ using the fibration $\pi_n : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}\mathbb{C}$.)

5. Let G be a finite group. Show that $H^\bullet_G(pt; \mathbb{Q}) = \mathbb{Q}$. Give an example to show that $H^\bullet_G(pt; k)$ can be non-trivial if k is of positive characteristic.

6. Using the Question 3 to describe BG for $G = S^1$ and a finite cyclic group. Show that BG cannot be taken to be a finite CW complex in these cases. (*Hint:* Use that $H^\bullet_G(pt) = H^\bullet(BG)$ does not depend on the choice of BG.)

7. Let B be a simplicial complex, and $\pi : E \to B$ a principal $\mathbb{Z}$-bundle.

a) Show that π is trivial over each simplex of B.

b) Show that there exists a $\mathbb{Z}$-equivariant map $f : E \to \mathbb{R}$ and conclude that $\pi = f^*\pi_\mathbb{Z}$ where $\pi_\mathbb{Z} : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ denotes the quotient map.

8. (This exercise is an attempt to convince the reader that much EG can also be constructed as a direct limit of algebraic varieties. This is useful to define equivariant étale cohomology, equivariant intersection theory etc.) Let G be an algebraic group over $\mathbb{C}$ and $\rho : G \to GL(V)$ be a faithful (algebraic) representation on a finite dimensional complex vector space.

a) Show that the set
$$Z_n := \{v = (v_1, \dots, v_n) \mid \mathrm{Stab}_G v \neq \{e\}\}$$
is a closed subvariety.

b) Show that
$$\lim_{\to} \mathrm{codim}(Z_n, V^n) = \infty.$$
and hence conclude that
$$\lim_{\to} V^n \setminus Z_n$$
can be taken as a model for EG.

c) Explore what this construction gives if one takes V to be the natural representation of $G = (\mathbb{C}^*)^n$ and $G = GL_n(\mathbb{C})$.

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