THE p-SMOOTH LOCUS OF SCHUBERT VARIETIES

GEORDIE WILLIAMSON

ABSTRACT. These are notes from talks given at Jussieu (seminaire Chevalley), Newcastle and Aberdeen (ARTIN meeting). They are intended as a gentle introduction to the papers [FW] and [JW] which give two descriptions of the p-smooth locus of Schubert varieties.

1. p-SMOOTHNESS

Let k be a ring and X be an n-dimensional variety over $\mathbb{C}$ equipped with the classical topology.

Definition 1. *A point $x \in X$ is* k-smooth *if we have an isomorphism of graded rings*

$$H^\bullet(X, X \setminus \{x\}; k) \xrightarrow{\sim} H^\bullet(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}; k).$$

The k-smooth locus *is the largest open subset $U \subset X$ consisting of k-smooth points.*

Let U be a contractible neighbourhood of x. Then excision and the long exact sequence of cohomology shows that $x \in X$ is k-smooth if and only if we have an isomorphism of graded rings

$$H^\bullet(U \setminus \{x\}; k) \cong H^\bullet(S^{2n-1}; k).$$

This is clearly true at smooth points of X.

The slogan is: $x \in X$ is k-smooth if k cannot distinguish x from a smooth point.

Remark 1.
(1) X is k-smooth if and only if $\underline{k}_X$ is Verdier self-dual (up to a shift of $2n$). Hence if X is k-smooth then we have Poincaré duality between $H^\bullet(X; k)$ and $H^\bullet_c(X; k)$.
(2) The universal coefficient theorem shows that the k-smooth locus only depends on the characteristic of k. We call the $\mathbb{F}_p$-smooth locus the *p-smooth locus*, and the $\mathbb{Q}$-smooth locus the *rationally smooth locus*.

(3) The universal coefficient theorem also shows that we always have inclusions:

$$\left\{\begin{array}{c}\text{smooth}\\ \text{locus}\end{array}\right\} \subset \left\{\begin{array}{c}\mathbb{Z}\text{-smooth}\\ \text{locus}\end{array}\right\} \subset \left\{\begin{array}{c}p\text{-smooth}\\ \text{locus}\end{array}\right\} \subset \left\{\begin{array}{c}\text{rationally}\\ \text{smooth locus}\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\mathbb{Z}\text{-smooth}\\ \text{locus}\end{array}\right\} = \bigcap_p \left\{\begin{array}{c}p\text{-smooth}\\ \text{locus}\end{array}\right\}$$

Here are some examples:

(1) If $X = \mathbb{C}^2/\pm 1 = \operatorname{Spec} \mathbb{C}[X^2, Y^2, XY] = \operatorname{Spec} \mathbb{C}[U, V, W]/(UV = W^2)$ is an affine quadric cone then a punctured regular neighbourhood of $0 \in X$ is homotopic to $S^3/\pm 1 = \mathbb{RP}^3$. Now

$$H^i(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & i = 3 \\ \mathbb{Z}/2\mathbb{Z} & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0. \end{cases}$$

Hence $0 \in X$ is p-smooth if and only if $p \neq 2$.

(2) Let Φ be a simply laced root system and X the corresponding simple surface singularity. Let $Q \subset P \subset V$ denote the root and weight lattice respectively. Then, if U denotes a regular neighbourhood of the only singular point $0 \in X$ then (see [Jut09]):

$$H^i(U \setminus \{0\}) = H^i(X \setminus \{0\}) = \begin{cases} \mathbb{Z} & i = 3 \\ Q/P & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0. \end{cases}$$

In particular, if Φ is of type E_8 then $Q = P$ and X is $\mathbb{Z}$-smooth.

Also note that the previous case is a special case of this example if we take $\Phi = A_1$.

(3) (Probably will omit). More generally, let $Y \subset \mathbb{P}^n$ be a (smooth) projective variety and $C \subset \mathbb{A}^{n+1}$ the cone over X. Then we have a spectral sequence

$$H^q(\mathbb{C}^*) \otimes H^p(Y) \Rightarrow H^{p+q}(C \setminus 0)$$

with diffierential given by $c(\mathcal{O}(1))$. So for example, if we embed $\mathbb{P}^n$ in projective space via $\mathcal{O}(m)$ (a Veronese embedding) then C is p-smooth if and only if p doesn't divide m.

2. SCHUBERT VARIETIES

Let G be a complex reductive algebraic group and $T \subset B \subset P$ a maximal torus, Borel subgroup and parabolic subgroup. We have the Bruhat decomposition

$$G/P = \bigsqcup BxP/P.$$

The closures $X_x = \overline{BxP/P}$ are *Schubert varieties*.

Example 2.1. *Consider $\mathbb{C}^4 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$ equipped with the symplectic form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is the 2×2 identity matrix. Set $V_0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. Let $G = SP(4)$ be the symplectic automorphisms of $\mathbb{C}^4$ and let P denote the stabiliser in G of V_0. Then*

$$G/P = \textit{isotropic 2-spaces} \subset \textit{Gr}(2,4).$$

One may show that

$$X = \{V \in G/P \mid \dim(V \cap V_0) \geq 1\} \subset G/P$$

is a Schubert variety.

Let us examine the local structure of X around the point V_0. A chart around V_0 in Gr$(2,4)$ is given by sending $a, b, c, d \in \mathbb{C}$ to the subspace V given by the span of the column vectors of the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & c \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$. The condition that $\dim(V \cap V_0) \geq 1$ may be expressed as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 0.$$

The condition that V is isotropic gives $a = d$. Hence the above neighbourhood of the singularity is isomorphic to the subvariety $a^2 = bc$ in $\mathbb{C}^3$, which we have already seen.

The conclusion is that X is rationally smooth but not 2-smooth.

Many authors have worked on questions of smoothness and rational smoothness of Schubert varieties. Let us mention:

(1) Chevalley: all Schubert varieties are smooth!
(2) Lakshmibai-Seshadri (global smoothness, pattern avoiding, classical types).
(3) Ryan, Wolper (SL_n),

(4) Jantzen, Kazhdan-Lusztig, Carrell-Peterson, Kumar (rational smoothness in general),
(5) Kumar (smoothness in general).

In this talk I want to discuss the following extension of these results:

Theorem 2.2 (Fiebig-W (2009), Juteau-W (2010)). *Suppose G is simple and*

(1) $p \neq 2$ *in type* $BCFG$,
(2) $p \neq 3$ *in type* G

then the p-smooth locus and rationally smooth locus of Schubert varieties coincide.

The Fiebig-W proof uses parity sheaves, moment graphs and commutative algebra. It also generalises to give the above equality for Schubert varieties in general Kac-Moody groups, however it gives no information for "bad" primes.

For the rest of the talk I will discuss the Juteau-W proof. It uses the FW work, but also provides more detailed information it doesn't use the hard bit!

3. Equivariant cohomology and multiplicities

As always X is a variety, and k denotes a fied of coefficients. (If in doubt $k = \mathbb{Z}$).

Let $H_{BM,\bullet}(X)$ denote the Borel-Moore homology of X. It is the cohomology of the chain complex of locally finite chains.

Example 3.1. *We have*

$$H_{BM,i}(\mathbb{C}^*) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z} & i = 1 \\ 0 & i = 0 \end{cases}$$

[Draw the chains on the black-board.]

Throughout we use cohomological numbering. We set

$$\tilde{H}^n(X) = H_{BM,-n}(X)$$

Remark 2. Both are isomorphic to (and may be defined as) $H^n(X, \omega_X)$ where ω_X denotes the dualising sheaf on X).

Given any subvariety $Z \subset X$ we may triangulate it, and hence obtain a fundamental class $[Z]$ in the Borel-Moore homology of X.

Under our normalisation:

$$[Z] \in \tilde{H}^{-2\dim_{\mathbb{C}} Z}(X).$$

The basic functoriality properties of Borel-Moore homology are the following:

(1) If $i : Z \to X$ is proper then we have a push-forward map
$$i_* : \tilde{H}(Z) \to \tilde{H}(X).$$

(2) If $j : U \to X$ is an open inclusion then we have a pull-back
$$j^* : \tilde{H}(X) \to \tilde{H}(U).$$

(3) If $X = Z \sqcup U$ (disjoint union) with Z closed and U open then the above maps fit into a long exact sequence:
$$\dots \to \tilde{H}^\bullet(Z) \to \tilde{H}^\bullet(X) \to \tilde{H}(U) \to \tilde{H}^{\bullet+1}(Z) \to \dots$$

(4) One also has Gysin like maps

Now let T denote an algebraic torus, $\mathcal{X}(T)$ the character lattice of T and $S = S(\mathcal{X}(T))$ the symmetric algebra on $\mathcal{X}(T)$.

If X is a T-variety then one may also consider the T-equivariant Borel-Moore homology $\tilde{H}^\bullet_T(X)$. After taking care with degrees, it may be defined as the Borel-Moore homology of the Borel construction of X.

One has all of the above properties: fundamental classes for T-stable subvarieties, push-forward (resp. pull-back) maps for proper morphims (resp. open inclusions) and the long exact sequence for a T-stable open-closed decomposition.

The crucial extra property is that the T-equivariant Borel-Moore homology $\tilde{H}^\bullet(X)$ is a module over $S = H^\bullet_T(\mathrm{pt})$ and one has the following fundamental fact:

Fact 3.2. *If $X^T = \emptyset$ then $\tilde{H}^\bullet_T(X)$ is torsion over S.*

Consider the following open-closed decomposition

$$X^T \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X \setminus X^T.$$

For simplicity assume that X^T is finite. We have a long exact sequence:

$$\dots \to \tilde{H}^\bullet_T(X^T) \to \tilde{H}^\bullet_T(X) \to \tilde{H}(X \setminus X^T) \to \tilde{H}^{\bullet+1}(X^T) \to \dots$$

But $\tilde{H}^\bullet_T(X^T) = \bigoplus_{x \in X^T} S[x]$ is free and $\tilde{H}^\bullet_T(X \setminus X^T)$ is torsion. We conclude that we have a short exact sequence

$$0 \to \bigoplus_{x \in X^T} S[x] \to \tilde{H}^\bullet_T(X) \to \tilde{H}(X \setminus X^T) \to 0$$

and that when we tensor with $Q = \mathrm{Frac} S$ we obtain an isomorphism

$$Q \otimes_S \tilde{H}_T^\bullet(X^T) = \bigoplus_{x \in X^T} Q[x] \xrightarrow{i_*} Q \otimes_S \tilde{H}_T^\bullet(X).$$

In particular, we can find an element $f = \sum_{x \in X^T} (e_x X)[x]$ which maps to $1 \otimes [X]$ under the above isomorphism (where $[X]$ denotes the equivariant fundamental class of X).

Definition 2. *For any $x \in X^T$, $e_x X \in Q$ is the* equivariant multiplicity *of $x \in X$.*

Example 3.3. (1) *If $x \in X$ is smooth then*

$$e_x X = \frac{1}{\chi_1 \cdots \chi_n}$$

where $\chi_1 \ldots \chi_n$ are the weights of T on $T_x X$.

(2) *In our standard example $X = \mathrm{Spec}\, \mathbb{C}[x,y,z]/(xy = z^2)$ then X becomes a $\mathbb{C}^*$-variety via $l \cdot (x,y,z) = (lx, ly, lz)$. In this case one has*

$$e_0 X = \frac{2}{\chi^2}.$$

where $\chi : \mathbb{C}^* \to \mathbb{C}^*$ is the identity character.

(3) *The T-fixed points in Schubert varieties $X = X_z = \overline{BzB/B} \subset G/B$ are those x which are less than z in the Bruhat order. In this case there exists a combinatorial formula for $e_x X$ in terms of the action of W on the root system of G.*

In general:

$$e_x X = \frac{f_x}{\chi_1 \cdots \chi_n}$$

where the characters $\chi_1, \ldots \chi_m$ occur in $T_x X$ and $f \in S$. It is also known that $e_x X$ is homogeneous of degree $-\dim_{\mathbb{C}} X$.

(Possibly mention link with the K-theory of the tangent cone.)

It is perhaps amazing that the above theorem has a converse:

Theorem 3.4 (Kumar's Criterion)**.** *Suppose $x \in X$ is an attractive isolated fixed poin of X. Then*

$$x \in X \text{ is smooth} \Leftrightarrow f_x = 1.$$

If in addition $X \setminus \{x\}$ is rationally smooth. Then

$$X \text{ is rationally smooth} \Leftrightarrow f_x \in \mathbb{Z}.$$

(This was proved by Kumar around 1987 for Schubert varieties and was generalised by Arabia and Brion.)

Theorem 3.5 (Juteau-W)**.** *Suppose that X is affine and and that $x \in X^T$ is an isolated attractive fixed point. Suppose in addition that $U := X \setminus \{x\}$ is p-smooth and that $H_T^\bullet(U;\mathbb{Z})$ is free of p-torsion. Then*

$$X \text{ is } p\text{-smooth} \Leftrightarrow f_x \in \mathbb{Z} \text{ and } p \text{ does not divide } f_x.$$

The condition that $H_T^\bullet(U;\mathbb{Z})$ is torsion free (obviously) has no analogue in the Kumar criterion but is certainly necessary, as is seen by considering Dynkin singularities of types D and E. They admit attractive $\mathbb{C}^*$-actions but the equivariant multiplicity tells us nothing about p-smoothness.

The link with Schubert varieties is obtained via the following theorem:

Theorem 3.6 (FW + JW)**.** *Let $x \in X_z \subset G/B$ be a T-fixed point and let $N \subset X_z$ be an affine T-stable transverse slice to $x \in X_z$. If $N \setminus \{x\}$ is p-smooth then $H_T^\bullet(N \setminus \{x\};\mathbb{Z})$ is p-torsion free.*

Corollary 3.7.

(1) *This gives a purely combinatorial criterion for p-smoothness of Schubert varieties:*

$$x \in X_z \text{ is } p\text{-smooth} \Leftrightarrow \begin{array}{c} \text{for all } x \le y \le z \text{, if we write:} \\ e_y X_z = \frac{f_y}{\chi_1 \cdots \chi_n} \\ \text{then } f_y \in \mathbb{Z} \text{ and } p \text{ does not divide } f_y. \end{array}$$

(2) *This shows that for Schubert varieties:*

$$x \in X_z \text{ is smoooth} \Leftrightarrow x \in X_z \text{ if } \mathbb{Z}\text{-smooth}.$$

Fiebig-W show that for Schubert varieties in flag varieties of simply lace groups, the rationally smooth locus is p-smooth for all p. Combining this results with the above, gives a new (and quite bizarre) proof of an old result of Peterson: in simply laced types, the rationally smooth locus of Schubert varieties is smooth.

Lastly, the theorem stated earlier follows from the following theorem of Dyer:

Theorem 3.8 (Dyer)**.**
Write the numerator of the equivariant multiplicity $e_x X_z$ as $f_{x,z}$. Then, if $f_{x,z} \in \mathbb{Z}$ then

(1) $f_{x,z} = 1$ *in types ADE,*
(2) $f_{x,z} = 2^m$ *in types BCF,*
(3) $f_{x,z} = 2^m 3^n$ *in type G.*

References

[FW] P. Fiebig and G. Williamson. Parity sheaves, moment graphs and the p-smooth locus of schubert varieties. Preprint, arXiv:1008.0719v1.

[Jut09] D. Juteau. Decomposition numbers for perverse sheaves. *Ann. Inst. Fourier (Grenoble)*, 59(3):1177–1229, 2009.

[JW] D. Juteau and G. Williamson. Kumar's criterion modulo p. In preparation.