Vanishing Cycles (S. Schneider)

(VUDVR (2004) PAPER MEBKHOUT'S BOOK)

1) Vanishing cycles for $D^b_{const}(X,\mathbb{Q})$

X a complex mfld, $f : X \to \Delta = \{ |z| < 1, z \in \mathbb{C} \}$.

$X^* = f^{-1}(\Delta^*)$

$$\begin{tikzcd}
\tilde{X} \arrow[r, "k"] \arrow[d] & X \arrow[d, "f"] & X_0 \arrow[l, hook', "i"'] \arrow[d] \\
\mathbb{H} \arrow[r] & \Delta & 0 \arrow[l]
\end{tikzcd}$$

$z \longrightarrow e^{2\pi i z}$

$K \in D^b_c(X,\mathbb{Q})$

$\Psi_f(K) = i^* R k_* k^* K$

"Complex of nearby cycles".

$\Phi_f(K) = \text{cone}(i^{-1}K \xrightarrow{can} \Psi_f(K))$ "complex of vanishing cycles.

Have a monodromy operator $T : \Psi_f(K) \longrightarrow \Psi_f(K)$

$T : \Phi_f(K) \longrightarrow \Phi_f(K)$

$can : \Psi_f(K) \to \Phi_f(K)$ but also $var : \Phi_f(K) \to \Psi_f(K)(-1)$

$$\begin{tikzcd}
i^{-1}K \arrow[r] \arrow[d] & \psi_f(K) \arrow[r] \arrow[d, "\frac{1}{2\pi i}\log T"] & \Phi_f(K) \arrow[d] \\
0 \arrow[r] & \psi_f(K)(-1) \arrow[r, "id"] & \psi_f(K)(-1)
\end{tikzcd}$$

this defines Var.

Rmk Sometimes, instead of $\frac{1}{2\pi i}\log T$ people use $1-T$

$\log T = (1-T) + \frac{1}{2}(1-T)^2 + \dots$ so $\log T$ is well defined if $(1-T)^m = 0 \; \forall m >> 0$. In general one has to make choices on the branch of the log.

Gabber If $K \in Perv(X, \mathbb{C}) \Rightarrow {}^p\psi_f(K) = \psi_f(K)[-1]$ and ${}^p\Phi_f(K) = \Phi_h(h)[-1]$ are perverse

$\Rightarrow {}^p\psi_f(K) = \bigoplus_{\lambda \in \mathbb{C}^*} {}^p\psi_{f,\lambda}(K)$, $\psi_{f,\lambda}(K) = Ker(T - \lambda Id)^m$, $m >> 0$

Perv$(X, \mathbb{C})$ is a $\mathbb{C}$-lin cat of finite length

Similarly ${}^p\Phi_f(K) = \bigoplus_{\lambda \in \mathbb{C}^*} {}^p\Phi_{f,\lambda}(K)$

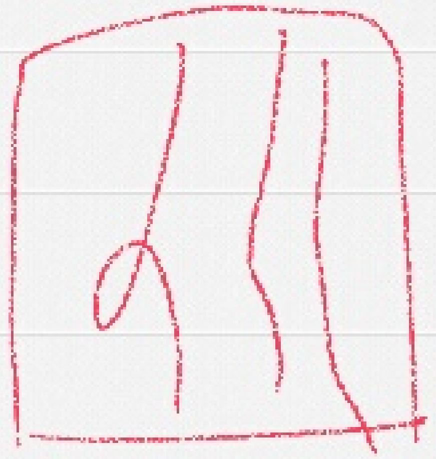
If $\lambda \neq 1 \Rightarrow \psi_{f,\lambda}(K) \xrightarrow{\sim} \Phi_{f,\lambda}(K)$ because T acts trivially on $i^{-1}h$

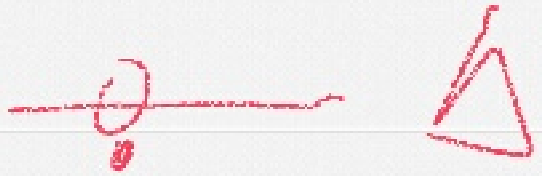
T comes from an automorphism of $\mathbb{H}$ which doesn't affect X_0

The cohomology of semistable families

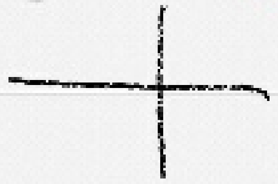
Def A degeneration is a proper flat hol map $\pi : \mathcal{X} \to \Delta$ which is smooth above Δ^* st

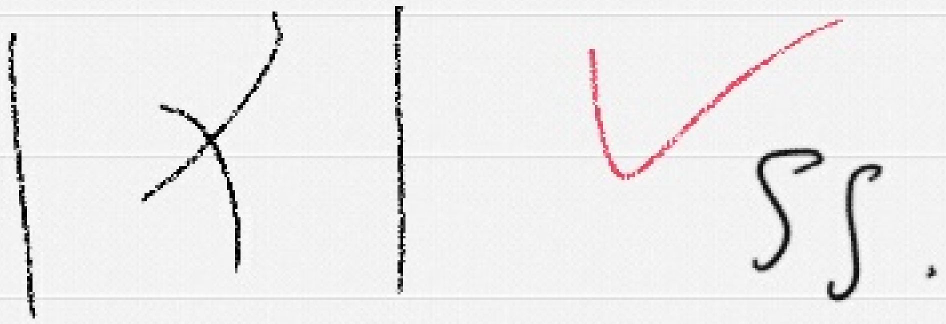
- $X_t = \pi^{-1}(t)$ is a smooth complex proj variety $\forall t \neq 0$
- $\mathcal{X}$ is Kähler





2) A degeneration is semistable if $X_0 \in \mathcal{X}$ is a reduced nc, i.e.
$X_0 = \overset{n}{\underset{i=1}{\bigcup}} X_{0_i}$, X_{0_i} = smooth & reduced & locally $X_0 = \{z_1 \dots z_m = 0\}$





Semistable red thm (MUMFORD)

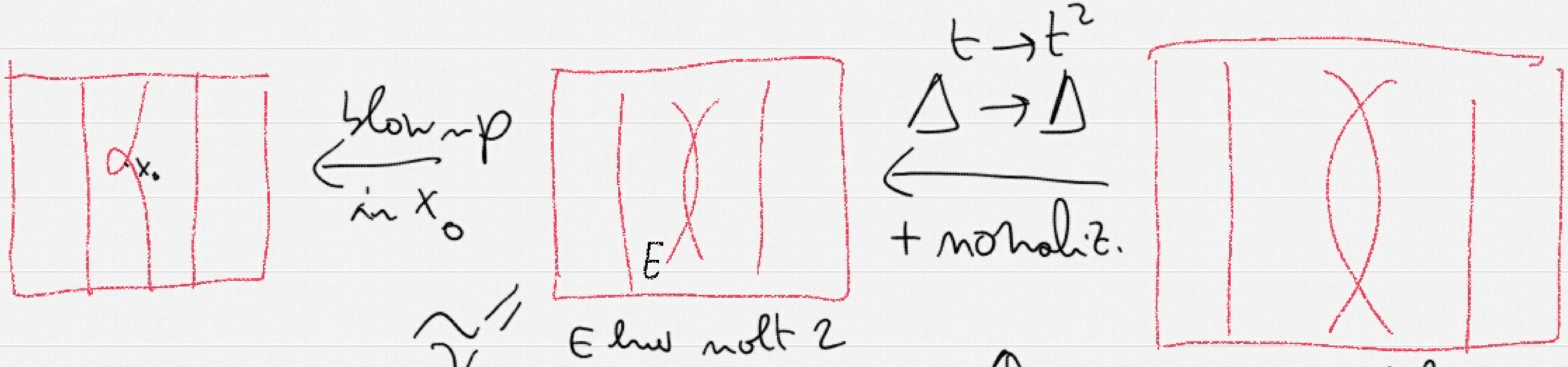
Given a degeneration $\mathcal{X} \to \Delta$, there is a finite $b : \Delta \to \Delta^k$ $(t \mapsto t^k)$ s.t

$$\begin{tikzcd} \text{semistable} \arrow[r] & \mathcal{X}^{ss} \arrow[r] \arrow[dr] & \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & & \Delta \arrow[r, "b"] & \Delta \end{tikzcd}$$

& $\mathcal{X}^{ss}\big|_{\Delta^*} = \mathcal{X}'\big|_{\Delta^*}$

Example

$\mathcal{X} =$



$\widetilde{\mathcal{X}} =$ E has mult 2

semistable

this is just the branched 2:1 cover of $\widetilde{\mathcal{X}}$ branched along X_0

Monodromy Thm

$\pi : \mathcal{X} \to \Delta$ a degeneration, $t \in \Delta^*$, $T : H^k(X_t) \to H^k(X_t)$ $(K = R\pi_* \mathbb{Q})$

1) $(T^r - 1)^{k+1} = 0$ for some $r \geq 0$ (quasi-unipotent)

2) If π is semistable $\Rightarrow (T-1)^{k+1} = 0$

Retraction Thm $\mathcal{X} \to \Delta$ s.s. $\exists$ retraction $r : \mathcal{X} \to X_0$ s.t. $r^* : H^*(X_0) \to H^*(\mathcal{X})$

$r_* : H_*(\mathcal{X}) \to H_*(X_0)$

Def $N = \log T = (1-T) + \frac{1}{2}(1-T)^2 + \dots$

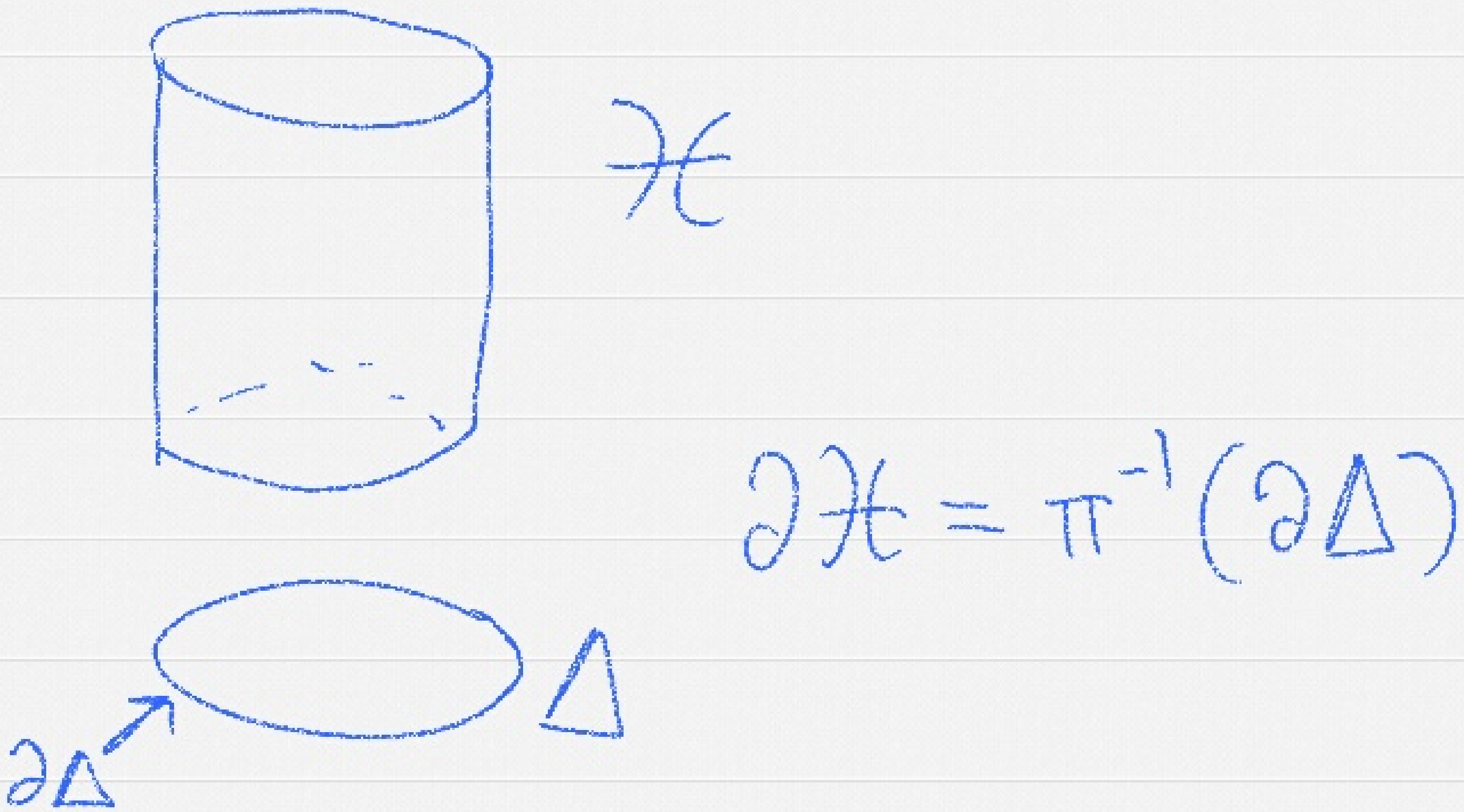
The Clemens-Schmid exact sequence

$\pi : \mathcal{X} \to \Delta$ s.s. $n = \dim X_t$, $t \neq 0$: have l.e.s.

$$\to H_{2n+2-k}(\mathcal{X}) \xrightarrow{\alpha} H^k(\mathcal{X}) \to H^k(X_t) \xrightarrow{N} H^k(X_t) \xrightarrow{\beta} H_{2n-k}(\mathcal{X}) \to \dots$$

Here $\alpha : H_{2n+2-k}(X) \overset{P.D.}{\cong} H^k(\mathcal{X}, \partial\mathcal{X}) \longrightarrow H^k(\mathcal{X})$

$\beta : H^k(X_t) \overset{PD}{\cong} H_{2n-k}(X_t) \longrightarrow H_{2n-k}(\mathcal{X})$



$K = R\pi_* \mathbb{Q}$, $\pi : \mathcal{X} \to \Delta$ s.s. $\rightsquigarrow H^k(\psi(K)) = H^k(X_t, \mathbb{Q})$ $t \in \Delta^*$

$$i^{-1}K \to \psi(K) \longrightarrow \Phi(K) \xrightarrow{+1}$$

$$\Longrightarrow H^k(\Phi(K)) \cong H^k(X_t)/\ker(N) \oplus \operatorname{Im}(H_{2n+2-k}(\mathcal{X}) \longrightarrow H^{k+1}(X_0))$$

$$H^k(\psi(K)) = H^k(X_t) \to H^k(\Phi(K)) \to H^{k+1}(X_0) \to H^{k+1}(X_t)$$

<u>Ex</u> $k=2$

$$\begin{tikzcd} H_{2m}(X_0) \arrow[r] \arrow[d, equal] & H^2(X_0) \\ \bigoplus [X_{0i}]\mathbb{Q} & \end{tikzcd}$$

$$[X_{0i}] \longmapsto [\mathcal{O}_X(X_{0i})|_{X_0}]$$

THE KASHIWARA-MALGRANGE FILTRATION

<u>Aim</u> Define nearby and vanishing cycles for D-modules

<u>Motivations</u> 1) $D^b_{rh}(\text{mod}-D_X) \longrightarrow D^b_c(X,\mathbb{C})$ $\quad = \Lambda^{top}\Omega_X$

$$M \longrightarrow DR_X(M) = R\mathcal{H}om(\omega_X, M)$$

2) A polarized Hodge module $M \in HM^p(X)$ is $(M, F, \mathcal{M}, K)$ where $(\mathcal{M}, F_\cdot, M)$ is a filtered r.h. D-module, $K \in Perv(X,\mathbb{Q})$ s.t. $DR_X(\mathcal{M}) \cong K \otimes_{\mathbb{Q}} \mathbb{C}$ + additional conditions...

If M has strict support $Z \subseteq X$ then $\exists$ Zariski open $U \subseteq Z$ s.t. $M|_U = VHS$

Thm X cx. mfld. $f: X \to \mathbb{C}$ hol fctn non constant. $Z = f^{-1}(0)$.
Then $M \in HM^p(X)$ is uniquely det by $(M|_{X-Z}, \Phi_{f,1}(M), \text{can}, \text{var})$

$$\text{can}: \Psi_{f,1}(M) \to \Phi_{f,1}(M), \ \text{var}\ \Phi_{f,1}(M) \longrightarrow \Psi_{f,1}(M)(-1)$$

Rmk This hints that we can use induction on the dimension.

X cx. mfld., $t: X \to \mathbb{C}$ a smooth & hol. fctn., ∂_t a vector field with $[\partial_t, t] = 1$

Def The KM-filtration on $M \in \text{mod}-D_X$ is an exhaustive increasing filtration $V_\cdot M$, $\cdot \in \mathbb{Z}$, s.t.
1) $V_k M$ coherent over $V_0 D_X = \{P \in D_X \mid P \cdot I_{X_0} \subset I_{X_0}\}$ (I_{X_0} ideal sheaf of $X_0 \subseteq X$)
2) $V_k(M)t \subseteq V_{k-1}(M)$, $V_k(M) \cdot \partial_t \subseteq V_{k+1}(M)$
3) $V_k(M)t = V_{k-1}(M)$ $\forall k << 0$
4) each eigenvalue of $E = t\partial_t$ on $gr^V_k M$ has $Re(\lambda) \in (k-1, k]$

Thm (KASHIWARA)
- If M is holonomic $\Rightarrow$ the KM filtration exists & it's unique
- If M is (reg) hol. $\Rightarrow$ $gr^V_k M$ are (reg) hol with support $\subset t^{-1}(0)$
  $\forall k \leq -1$

PART 2 06/05

The V-filtration

Last time: X a cx. mfld, $f : X \to \Delta$ smooth above $\Delta^* = \Delta \setminus \{0\}$

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "k"] \arrow[d] & X \arrow[d] & X_0 \arrow[l, hook', "i"'] \arrow[d] \\
\mathbb{H} \arrow[r, "\exp"] & \Delta & \{0\} \arrow[l, hook']
\end{tikzcd}$$

$K \in Perv(X, \mathbb{Q})$

$\psi_f(K) = i^{-1} Rk_* k^{-1} K$

${}^p\psi_f(K) = \psi_f(K)[-1] \in Perv(X_0, \mathbb{Q})$

$$\begin{tikzcd}
i^{-1}K \arrow[r] \arrow[d] & \psi_f K \arrow[r, "can"] \arrow[d, "T-1"] & \phi_f K \arrow[d, dashed, "var"] \\
0 \arrow[r] & \psi_f K \arrow[r] & \psi_f K
\end{tikzcd}$$

(Var is unique for some reason...)

$$\left.\begin{array}{l} \Phi_f K = \bigoplus_{\lambda \in \mathbb{C}} \Phi_{f,\lambda} K \\ \Psi_f K = \bigoplus_{\lambda \in \mathbb{C}} \Psi_{f,\lambda} K \end{array}\right\} \text{dec. in generalized eigenspaces for } T$$

Var : $\Phi_{f,1}K \to \Psi_{f,1}K(-1)$ s.t. $\mathrm{Var} \circ \mathrm{Can} = \frac{1}{2\pi i} \log T$ $\quad (\log T = (1-T) - \frac{1}{2}(1-T)^2 + \dots$

$\Phi_{f,\lambda}K \cong \Psi_{f,\lambda}K \quad \forall \lambda \neq 1$

The KM-filtration

$t : X \to \mathbb{C}$ a small and holomorphic, ∂_t vector field s.t. $[\partial_t, t] = 1$

Def The KM-filt. on $M \in \text{mod-}D_X$ wrt to $X_0 = t^{-1}(0)$ is an increasing exhaustive filt. $V_\cdot M$, $\cdot \in \mathbb{Z}$ s.t.

1) $V_k M$ is coherent over $V_0 D = \{ P \in D_X \mid P I_{X_0} \subseteq I_{X_0} \} \overset{\text{locally}}{=} O_X < D_{X_0}, E = t\partial_t >$

(D_{X_0} = diff. op. acting tangent to X_0)

2) $V_k M \cdot t \subseteq V_{k-1} M$

$V_k M \cdot \partial_t \subseteq V_{k+1} M$

3) $V_k M t = V_{k-1} M \quad k << 0$

4) Each eigenvalue of E on $gr^V_k M$ has $Re(\lambda) \in (k-1, k]$

5) $gr^V_k M$ decomposes into a sum of gen. eigenspaces wrt to E

Thm If M is (reg.) holonomic, then the KM-filtration exists & is unique. Moreover $gr^V_k M$, $k\leq 0$ are (reg.) holonomic on X_0

EXERCISE
a) $\cdot t : V_k M \xrightarrow{\sim} V_{k-1} M$ $\forall k<0$ (and surj for $k=0$)
b) $\cdot\partial_t : gr^V_k M \xrightarrow{\sim} gr^V_{k+1} M$ $\forall k \neq -1$
c) M is generated as a mod-D_X by $V_0 M$

Prop $V_k M = V_0 M \cdot t^{-k}$ $\forall k$. Hence, if $k>0$ $V_0 M\cdot t^{-k} = \{ m\in M \mid m t^k \in V_0 M\}$

Proof If $k<0 \rightsquigarrow$ Ex a) ✓

If $k=0$ ✓

If $k>0$: Induction.
$$V_k M = V_{k-1} M\,\partial_t + V_{k-1} M \text{ (by ex. (b))} =$$
$$= V_0 M \underbrace{t^{-k+1}\partial_t}_{\text{use } \partial t^k = kt^{k-1} + t^k\partial_t} + V_0 M t^{-k+1} \subseteq V_0 M t^{-k}$$

Conversely, let $m \in V_0 M\cdot t^{-k}$, then $m t^k \in V_0 M \Longrightarrow m t^k \partial_t^k \in V_k M$

Use $t^k \partial_t^k = (E+k)(E+k-1)\cdots(E+1) \Longrightarrow m(E+k)\cdot(E+k-1)\cdots(E+1) \in V_k M$

$\exists k_0 \in \mathbb{Z}$ s.t. $m \in V_{k_0} M$, choose it minimal. Sup $k > k_0 \Rightarrow \overline{m} \neq 0$ in $gr^V_{k_0}$

But $\overline{m}(E+k) \cdots (E+1) \underset{\substack{\uparrow \\ \text{in } gr^V_{k_0} M}}{=} 0$ $\Big\}$ to (4) $\downarrow$

$\square$

EXAMPLES 1) $M = \omega_X \in$ mod-D_X. Then $V_0 \omega_X = \omega_X$, $V_{-k}\omega_X = \omega_X t^k$ $\forall k \geq 0$

$\rightsquigarrow gr^V_{-1} \omega_X = \omega_X t / \omega_X t^2 \cong \omega_{X_0}$

2) $\operatorname{supp} M \subseteq X_0$, $V_{-1} M = 0$, $V_k M = \{ m \in M \mid m t^{k+1} = 0 \}$

in fact $V_\cdot$ splits $V_k M = \bigoplus_{n=0}^{k} M_n$ where $M_n = \{ m \in M \mid mE = n \cdot m \}$

Prop The KM-filtr. is unique if it exists.

Pf Sup. $V_\cdot, W_\cdot$ satisfy (1)-(5). Then $V_\cdot \cap W_\cdot$ satisfy (1)-(5)

wlog $V_\cdot \subseteq W_\cdot$

1) $\Rightarrow W_0 M \subset V_N M \quad N \geq 0$

Prop $\Rightarrow W_k M \subseteq V_{k+N} M \quad \forall k$

Choose N minimal $\Longrightarrow \exists m \in W_k M$ with $m \notin V_{k+N-1} M$
(sup $N \geq 1$)

5) $\Longrightarrow m E = \lambda m + m'$, $m' \in W_{k-1} M \quad \mathrm{Re}\,\lambda \in (k-1, k]$

Contradiction if we go to $gr^V_{k+N} M \quad \bar{m} E = \lambda \underbrace{\bar{m}}_{\neq 0} \Rightarrow \mathrm{Re}\,\lambda \in (k+N-1, k+N]$

GENERAL CASE $f : X \to \mathbb{C}$ non const. hol. fctn.

$(id, f) : X \hookrightarrow X \times \mathbb{C}$. $\quad M_f := (id, f)_* M = M[\partial_t]$

Let t be the coord. on the 2nd factor of $X \times \mathbb{C}$, $\partial_t = \frac{\partial}{\partial t}$, $[\partial_t, t] = 1$

So KM-filtr. for M_f exists if M is reg. hol.

Also, $gr^V_k M$ lives on $X \times \{0\} \cap \mathrm{supp}\, M_f = f^{-1}(0) \subseteq X \quad (k \leq 0)$

For $\alpha \in \mathbb{C}$, set $M_{f,\alpha} = \ker (E - \alpha \mathrm{Id})^m \subseteq \mathrm{gr}^V_k M_f$, $k = \lceil \alpha \rceil$, $m \gg 0$

FACTS (Recall $DR(M) = \mathbb{R}\mathcal{H}om(\omega_X, M)$)

put $\lambda = e^{2\pi i \alpha}$, then $DR(M_{f,\alpha}) \simeq {}^p\psi_{f,\lambda}(DR(M))$ for $-1 \leq Re(\alpha) < 0$

$DR(M_{f,\alpha}) \cong {}^p\phi_{f,\lambda}(DR(M))$ for $-1 < Re(\alpha) \leq 0$

T corresponds to $e^{2\pi i E}$

can: $\Psi_{f,1} K \to \Phi_{f,1} K$ & Var $\Phi_{f,1} K \to \Psi_{f,1} K(-1)$

correspond to $\partial_t : M_{f,-1} \to M_{f,0}$ & $\cdot t : M_{f,0} \to M_{f,-1}$

$$\mathrm{Var} \circ \mathrm{Can} \cong t\partial_t = E = \frac{1}{2\pi i} \log T$$

The V-filtration

Motivation $M = (\underbrace{\mathcal{M}, F_\cdot \mathcal{M}}_{\text{filt. reg. hol. mod-}\mathcal{D}_X}, \overset{\in \mathrm{Perv}(X,\mathbb{Q})}{K})$ $\mathrm{DR}(\mathcal{M}) = K \otimes_{\mathbb{Q}} \mathbb{C}$

$\rightsquigarrow$ need to take care of F.

1) The monodromy T of $\mathbb{Z}$-VHS on Δ^* is quasi-unipotent, i.e. $\lambda = \sqrt{1}$
$\Rightarrow \alpha \in \mathbb{Q}$

2) The Hodge filtr. on the nearby cycles should be the limit MHS, it should be compatible with gen. eigenspaces of ψ_f

Def 1) The rational V-filtration on $\mathcal{M}_f$ is an increasing filtration $V_\cdot \mathcal{M}_f$, $\cdot \in \mathbb{Q}$, s.t. $\forall \alpha \in \mathbb{Q}$, $[\alpha] = k$: $V_\alpha \mathcal{M}_f \subset \underset{\text{kM-filt.}}{V_k} \mathcal{M}_t$ is the preimage of $\bigoplus_{-k-1<\beta\leq\alpha} \mathcal{M}_{f,\beta} \subseteq \mathrm{gr}_k V$

2) If M is filtered w $F_\cdot M$, we define $F_p \, gr^V_\alpha M = \dfrac{F_p M_f \cap V_\alpha M_f}{F_p M_f \cap V_{<\alpha} M_f}$

where $F_p M_f = \bigoplus_{i=0}^{\infty} F_{p-i} M \partial_t^i$ $\qquad (M_f = M[\partial_t])$

MEROMORPHIC CONNECTIONS & HOLONOMIC REGULAR $\mathcal{D}$-MODULES IN DIM 1

20/05
BRUNO
BARBE

X complex manifold, $Y \subseteq X$ hypersurface
$\mathcal{O}_X(*Y)$ sheaf of meromorphic functions on X with poles in Y.

Locally, if $Y = \{f=0\}$, $\mathcal{O}_X(*Y) = \mathcal{O}_X\left[\frac{1}{f}\right] \rightsquigarrow \mathcal{O}_X(*Y)$ is a coh. sheaf of ring.

Def An A-mod. M is coherent if it is fin. gen. + if $0 \to \ker \to A^n \to M \to 0$
$\ker$ is fin. gen.

De Rham complex $\quad \Omega^p(*Y) = \mathcal{O}_X(*Y) \otimes_{\mathcal{O}_X} \Omega^p_X.$

Def A meromorphic connection is given by: - $\mathcal{V}$ a coherent $\mathcal{O}_X(*Y)$-module
- $\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X(*Y)} \Omega^1_X(*Y)$ $\mathbb{C}$-linear map satisfying Leibniz's rule + integrability.

$\rightsquigarrow$ De Rham complex: $\Omega^p(\mathcal{V}) := \mathcal{V} \otimes_{\mathcal{O}_X(*Y)} \Omega^p(*Y)$

$(\mathcal{V}, \nabla)$ Meromorphic connection $\rightsquigarrow$ $\mathcal{D}_X$-module structure on $\mathcal{V}$

<u>Thm</u> This is a holonomic $\mathcal{D}_X$-module.

<u>Ex</u> $(\mathcal{O}_X(*Y), d)$.

LOCALIZATION

$Y \subseteq X$, $M \in Hol(X)$

<u>Def</u> The localization of M along Y is defined by $M \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y)$

<u>Thm</u> Localization preserves holonomicity and regularity.

<u>Prop</u> M coherent $\mathcal{D}_X$-module on X s.t.
- $\mathrm{Sing}(M) \subseteq Y$
- M is localized at Y $\left(\text{i.e. } M \xrightarrow{\sim} M(*Y)\right)$

Then M is a meromorphic connection on (X, Y).

$M \in \mathrm{Hol}(X)$. Find Y a hypersurface containing $\mathrm{Sing}(M)$

$0 \to K \to M \to M(*Y) \to N \to 0$, K and N are supported on Y

Proof Locally there exists a good filtration $\{M_p\}$ of M

$\mathrm{Sing}(M) \subseteq Y \Rightarrow \forall \ell \geqslant \ell_0$, $\mathrm{supp}(M/M_{\ell_0}) \subset Y$

$\forall \ell \geq \ell_0$ $\quad M_{\ell_0} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y) \xrightarrow{\sim} M_\ell \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y) \xrightarrow{\sim} M(*Y)$

But $M = M(*Y) \Rightarrow M = M_{\ell_0} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y)$ $\square$

We used the Lemma No nonzero coh. $\mathcal{O}_X(*Y)$ module is supp. on Y.

SIMPLE NORMAL CROSSING CASE

Assume D is n.c. divisor, $D = \bigcup D_i$, D_i smooth.

$\Omega^1_X(\log D) = \left\{ \begin{array}{l} \text{locally generated as a } \mathcal{O}_X\text{-module} \\ \text{by } \quad \frac{dz_1}{z_1}, \dots, \frac{dz_p}{z_p}, dz_{p+1}, \dots, dz_n \end{array} \right\}$

<u>Def</u> A logarithmic connection $(\mathcal{V}, \nabla)$ on (X, D) is

- a free $\mathcal{O}_X$-mod. $\mathcal{V}$
- $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$

<u>Rem</u>: from a logarithmic connection is canonically associated to a meromorphic connection.

<u>Def</u> A meromorphic connection (X, D) is regular if it comes from a log. connection

<u>RESIDUE</u> $D_i \subseteq D$ smooth irr. component.

$\mathrm{res}_{D_i} : \Omega^1_X(\log D) \longrightarrow \mathcal{O}_{D_i}$ $\quad \mathcal{O}_X$-linear

locally $\sum_{i=1}^{p} b_i \frac{dz_i}{z_i} + \sum_{p+1}^{n} b_j dz_j \longmapsto b_i(0, z_2, \dots, z_n)$

For $(\mathcal{V},\nabla)$ a log. connection on (X,D)

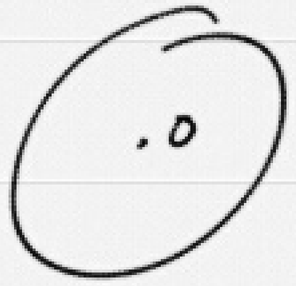
$$\mathcal{O} \xrightarrow{\nabla} \mathcal{V}\otimes_{\mathcal{O}_X}\Omega^1_X(\log D) \xrightarrow{1\otimes \mathrm{res}} \mathcal{V}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_i} = \mathcal{V}_{|D_i}$$

$$\mathrm{res}_{D_i}(\nabla) \in \mathrm{End}(\mathcal{V}_{|D_i}).$$

Locally $\nabla = d + \sum_{i=1}^{n} N_i(z)\frac{dz_i}{z_i} + \sum_{p+1}^{m} N_j(z)\,dz_j$

$$\mathrm{res}_{D_i}(\nabla) = N_i(0, z_2, \dots, z_m)$$

<u>Ex</u> $(\mathcal{V},\nabla)$ on $(\Delta, 0)$



$\exists$ $T \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{V})$ commuting with ∇ s.t. — $T|_{\Delta^*}$ is the monodromy endomorphism

— $T_0 = \exp(-2\pi i\, \mathrm{Res}_0(\nabla)) \in \mathrm{End}(\mathcal{V}_0)$

Thm $(\mathcal{V},\nabla)$ reg. meromorphic connection on (X,D)

Let τ be a section of $\mathbb{C}\to\mathbb{C}/\mathbb{Z}$

Then there exists a unique loc. free $\mathcal{O}_X$-submodule V^τ of V such that
- ∇ has log poles with respect to V^τ
- the eigenvalues of the the residues of ∇ wrt. V^τ are contained in $\mathrm{Im}(\tau)$

Def The lattice corresponding to the section $[0,1[$ is called Deligne's lattice.

DELIGNE'S RIEMANN-HILBERT

ANALYTIC CASE $Y\subseteq X$ hypersurface on X

$$\left(\begin{array}{c}\text{Regular meromorphic}\\ \text{connections on } (X,Y)\end{array}\right) \xleftrightarrow[\text{equivalence of cat.}]{\sim} \left(\begin{array}{c}\text{local system}\\ \text{on } X-Y\end{array}\right)$$

COMPARAISON THEOREM

$M = (\mathcal{V}, \nabla)$ reg. meromorphic conection on (X, D), $U = X \setminus D$

$$DR(M) \xrightarrow{\sim} Rj_* j^{-1} DR(M) \xrightarrow{\sim} Rj_* \gamma$$

$$\gamma = \ker(\nabla_{|U})$$

ALGEBRAIC CASE

X smooth complex alg. variety

$$\left\{ \begin{array}{c} \text{flat alg. connection on } X \\ \text{"regular at infinity"} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{local system} \\ \text{on } X \end{array} \right\}$$

(take a compactification, then we require its analytization to be regular) ← does not depend on the compactification!

Comparaison thm $M = (\mathcal{V}, \nabla)$ flat connection regular at infinity, $X \overset{j}{\hookrightarrow} \overline{X}$

$$\gamma = (M^{an})^{\nabla}$$

$$j_* DR(M) \simeq j_* \gamma$$

$$\mathbb{H}^i(X, DR(M)) = H^i(X, \gamma)$$

QUIVER DESCRIPTION

27/05 YOHAN

Regular meromorphic connection on the disc

Thm Let M is a regular meromorphic connection on Δ with sing $M \subset \{0\}$ (= regular $\mathcal{D}$-mod M on the disc with sing $M \subset \{0\}$ satisfying $M \xrightarrow{\sim} \underset{M \otimes_{\mathcal{O}_D} \mathcal{O}_D(*0)}{\underset{\shortparallel}{M(*0)}}$ -

For any section τ of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ there exists a unique lattice $V^\tau \subseteq M$ (coh. sub $\mathcal{O}_\Delta$-mod of M s.t. $V^\tau \otimes_{\mathcal{O}_D} \mathcal{O}_D(*0) = M$)

s.t. 1) V^τ is stable under $t\partial_t$

2) the eigenvalues of $t\partial_t$ acting on $V^\tau/(t)V^\tau$ are in τ

Ex If you take τ s.t $\mathrm{Im}(\tau) = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) < 1\}$ then you get the Deligne's lattice

Cor Let $\nabla = d + N(z)\frac{dz}{z}$ on $\mathcal{O}_\Delta^m$ (N is a $m \times m$ matrix of holom. fcts. on Δ)

$(\mathcal{O}_\Delta^m, \nabla)$ is equivalent to $(\mathcal{O}_\Delta^m, \nabla' = d + N(z)\frac{dz}{z})$ as soon as the eigenvalues of $N(0)$ does not differ by a non-zero integer.

In particular the monodromy $T = \exp(-2\pi i N(0))$

Counterexample: $\nabla = d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dz}{z}$ on $\mathcal{O}_\Delta^2 = (e_1, e_2)$

$n \geq 1$, $F_1 = z^{-N} e_1$, $F_2 = e_2$, $\nabla' = d + \begin{pmatrix} n & t^n \\ 0 & 0 \end{pmatrix} \frac{dz}{z}$

$T = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}$ $\quad T' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

cannot use this matrix since $\underbrace{n - 0}_{\text{eigenvalues}} \in \mathbb{Z}$

Regular hol. D-mod on the disc

$X = \Delta \qquad I = (z)$ ideal of $\{0\}$

$V^k(\mathcal{D}_X) = \{P \in \mathcal{D}_X \mid \forall i \;\; P(I^i) \subseteq I^{k+i}\}$ $\searrow$ exhaustive $\mathbb{Z}$-filtration of $\mathcal{O}_D$-coh. submodules

It is compatible with the ring structure. In particular $V^0 \mathcal{D}_X$ is a ring and $gr_V^0 \mathcal{D}_X = \mathbb{C}(E)$

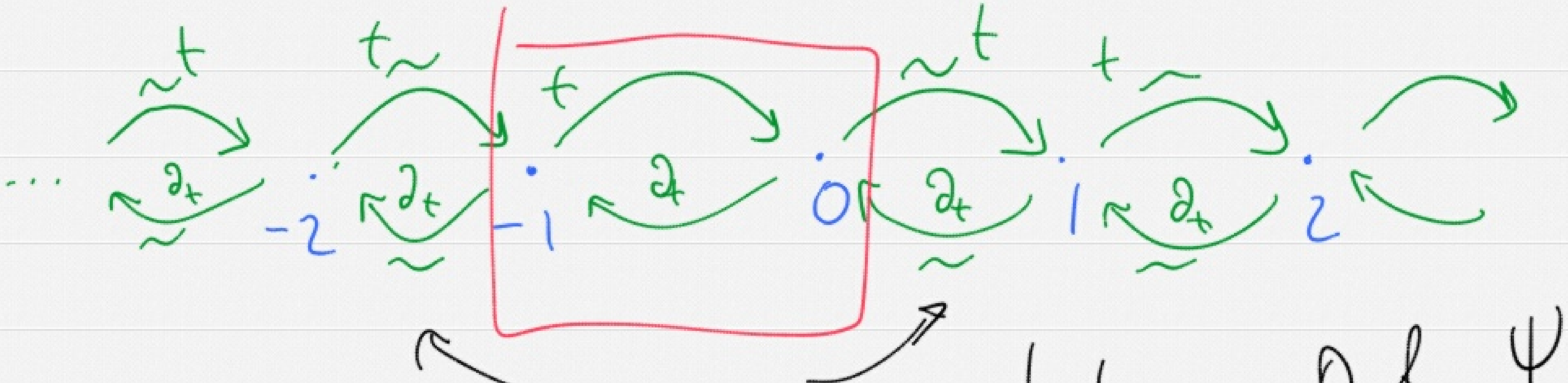
(E is the image of $t\partial_t$)

Def-Thm Let M be a holonomic $\mathcal{D}_\Delta$-mod. There exists a unique exhaustive filtration $V^k M$ s.t.

1) $(V^k \mathcal{D}_X)(V^\ell M) \subset V^{k+\ell} M$ $\qquad$ 2) in "1" we have $=$ for $\begin{cases} k \geq 0 & \ell >> 0 \\ k \leq 0 & \ell << 0 \end{cases}$

3) The eigenvalues of $E = t\partial$ acting on $Gr_V^k M$ have real part in $[k, k+1)$

Rmk τ section with $\tau(0)=0$. Replace (3) with eigenvalues on Gr^k of $E \subset \mathrm{Im}(\tau)+k$



t, ∂_t are iso. outside the box.

Def $\Psi M = Gr^0_V M$

$\varphi M = Gr^{-1}_V M$

Example M be a regular meromorphic connection. $V^{Del} \subseteq M$ $\forall k$ $V^k M = (z^k) V^{Del}$

Rmk From uniqueness it follows that any morphism of V-filtration is strict

($f(V^k M) = \mathrm{Im}(f) \cap V^k N$ would be two V-filt. on $\mathrm{Im}\, f$)

Def Let $\mathcal{C}$ be the abelian cat. of $\mathcal{D}_X$-module ($X = \Delta$) holon. regular with $\mathrm{Sing}(M) \subset \{0\}$

Thm There exists an eq. of categories

$$\mathcal{C} \longleftrightarrow \left\{ E \overset{u}{\underset{v}{\rightleftarrows}} V \right\} \text{ E,V finite } \mathbb{C}\text{-vs.} \quad : \quad M \mapsto \left(\Psi M \overset{\partial}{\underset{t}{\rightleftarrows}} \varphi M \right)$$

where the eigenvalues of $u \circ v$, $v \circ u$ have real parts in $[0,1)$

Fix z a coord. and ∂ s.t $[\partial, z] = 1$. Set $E = z\partial$

$\overline{M} = \{m \in M \mid \exists P(x) \in \mathbb{C}[x]\setminus\{0\},\ P(E)\cdot m = 0\} = \bigcup$ fin. dim. $\mathbb{C}$-subspace of M stable by E

$\overline{M}$ is a $\mathbb{C}[z,\partial]$-sheaf of modules (but not a $\mathcal{D}$-module, we can not multiply by $\exp(z)$...)

KEY LEMMA 1) $M = \mathcal{O}_X \otimes_{\mathbb{C}[z]} \overline{M}$

$\overline{M}$ is the "algebraization" of M

2) $\forall \lambda \in \mathbb{C}$, $\overline{M} \supseteq \overline{M}_\lambda = \bigcup_N \ker(E - \lambda\,\mathrm{Id})^N$ is a finite dim. $\mathbb{C}$-vector space.

$$\Rightarrow \overline{M} = \bigoplus_{\lambda\in\mathbb{C}} \overline{M_\lambda}, \quad V^k\overline{M} = \bigoplus_{Re(\lambda)\geq k} \overline{M}_\lambda$$

INVERSE FUNCTOR $M \in \mathcal{C}$, $E = \bigoplus_{0\leq Re(\lambda)<1} \overline{M}_\lambda$, $F = \bigoplus_{-1\leq Re(\lambda)<0} \overline{M}_\lambda$

$$0 \to (\mathcal{D}\otimes_{\mathbb{C}} E)\oplus(\mathcal{D}\otimes_{\mathbb{C}} F) \xrightarrow{P} (\mathcal{D}\otimes_{\mathbb{C}} E)\oplus(\mathcal{D}\otimes_{\mathbb{C}} F) \longrightarrow M \longrightarrow 0$$

$$P = \begin{pmatrix} \partial\otimes\mathrm{Id}_E & -(1\otimes v) \\ -(1\otimes v) & z\otimes\mathrm{Id}_F \end{pmatrix}$$

$\overline{M}$ is the $\mathbb{C}[z,\partial]$-mod. generated by E and F with relations $u(x) = \partial x \quad x \in E$
$v(x) = t x \quad x \in F$

Thm Let $M \in \mathcal{C}$

$$\begin{tikzcd}
\Psi_z(DR(M)) \arrow[d, bend right, "can"'] & = Gr^0_V M & = E \arrow[d, bend right, "u"'] \\
\Phi_z(DR(M)) \arrow[u, bend right, "var"'] & = Gr^{-1}_V M & = F \arrow[u, bend right, "v"']
\end{tikzcd}$$

$can = u$

$var = \varphi(vu)v$ $\qquad \varphi(z) = \dfrac{e^{2\pi i z} - 1}{z}$

Rm $u\,\varphi(vu)v = (uv)\varphi(uv) = \varphi(uv)uv$

$T - Id|_{\Psi} = var \circ can = \varphi(uv)uv = e^{2\pi i uv} - Id$

$$T|_{\Phi(DR(M))} = \exp(-2\pi i\, uv)$$

Prop Let $M \in \mathcal{C}$. 1) u is onto $\iff$ M has no quotient supported in 0

2) v is injective $\iff$ M " " submodule " " 0

3) $F = Im(u) \oplus Ker(v) \iff M$ is support decomposable, i.e.

$M = M' \oplus M''$
- M'' supp. in 0
- M' has no quotient nor submodule supported in 0

$\mathcal{C}$ is an abelian category of finite length

$\mathcal{C}^* = \mathcal{D}$-mod on Δ^* with no singularities.

Simple objects

$$\mathcal{C}^* : \left\{ \begin{array}{l} \mathcal{D}_U / \mathcal{D}_U(t\partial - \lambda) \quad \text{on } U = \Delta^* \\ \quad \big\downarrow \text{ the cor. loc. syst.} \\ \mathbb{C} \circlearrowleft_{e^{-2\pi i \lambda}} \text{ (monodromy) } =: \mathbb{C}_\lambda \end{array} \right.$$

(so λ and $\lambda+1$ give the same $\mathcal{D}^*$-mod)

$$\mathcal{C} : \left\{ \begin{array}{l} \forall \lambda \notin \mathbb{Z} \quad \mathcal{D}_X / \mathcal{D}_X(\partial - \lambda) \longleftrightarrow j_! \mathbb{C}_\lambda \underset{j_{!*}\mathbb{C}_\lambda}{\overset{\shortparallel \qquad \shortparallel}{=}} j_* \mathbb{C}_\lambda \\ \mathcal{D}_X / \mathcal{D}_X t \longleftrightarrow i_* \underline{\mathbb{C}}_0 \\ \mathcal{D}_X / \mathcal{D}_X \partial \longleftrightarrow \underline{\mathbb{C}}_\Delta \end{array} \right.$$

BLOCKS

Equivalence relation on simple objects generated by $M \sim N \iff \mathrm{Ext}^1(M,N)$ or $\mathrm{Ext}^1(N,M) \neq 0$

$R = \{\text{eq. classes}\}$ $\quad \mathcal{C}_r \subset \mathcal{C}$ full subcat. gen. by simple elements in r.

Claim Blocks for $\mathcal{C}$: $\mathcal{D}_X/\mathcal{D}_X\partial \sim \mathcal{D}_X/\mathcal{D}_X\partial$ is the only nontrivial relation.

$0 \to \mathcal{O}_\Delta \to \mathcal{O}_\Delta(*0) \to \delta \to 0$ is non trivial

BERNSTEIN-SATO POLYNOMIAL & V-FILTRATION

27/05
AZILATA

HISTORY Gelfand's question from '54

$f(x)$ polynomial

$$f_+(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & o/w \end{cases}$$

If $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$s \longmapsto \int_{\mathbb{R}^n} f_+^s(x)\varphi(x)\,dx$$ is analytic in s where $Re(s) > 0$

Q: Does it extend to other values of s?

BS-POLYNOMIAL

For any polynomial f : Let s be a formal parameter.

For some P, diff. operator depending polynomially in s and some polynomial $b(s)$ in s

$$P \cdot f^{s+1} = b(s) f^s$$

Use this equation to continue

$$s \longmapsto \int f_+^s(x)\varphi(x)\,dx = \int \frac{P f_+^{s+1}}{b(s)}\varphi(x)\,dx$$

By integrating by parts: extends meromorph. (in s) to $\mathrm{Re}(s) > -1$. By induction extends anywh.

Motivation #2 The existence of $b(s)$ gives us the existence of V-filtrations

Proof of existence of P, b

1) $X = \mathbb{A}^n$ ✓

2) More gen. case

Recall that D_X has an order filtration, but $D_X = D_{\mathbb{A}^n}$ has another one → Bernstein filt.

$$D_{\mathbb{A}^n} = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / \sim \qquad |x_i| = |\partial_i| = 1$$

Graded pieces are finite dim.

FACT If $X = \mathbb{A}^n$, then $M \in D_X$-mod is holonomic iff it is holonomic wrt. Bernstein filtration

$f \in \mathbb{C}[x]$. Consider $D_X(s) \cong D_X \otimes_{\mathbb{C}} \mathbb{C}(s)$

$M \subseteq \mathcal{O}_X(s)[f^{-1}] \cdot f^s$ generated by the symbol f^s.

If $g \in \mathcal{O}_X$ $g f^s = g f^s$. If $\xi \in T_X$, $\xi \cdot f^s = s \cdot f^{-1}(\xi f) f^s$. M is a $D_X(s)$-module.

Claim M is holonomic as a $D_X(s)$-module; D_X has a filtration.

Consequence of claim Holonomic module $\Longrightarrow$ finite length

So M has fin. length

$$Mf \supseteq Mf^2 \supseteq \dots \supseteq Mf^n \supseteq \dots$$

this stabilizes $\Longrightarrow$ $f^{s+k} \in Mf^{k+1} = \mathcal{D}_X(s)$-submod. gen. by f^{s+k+1}

Then there is some $L \in \mathcal{D}_X(s)$ s.t. $L \cdot s^{k+s+1} = f^{s+k}$ $\Rightarrow$ clear denominators in s

$$P f^{s+k+1} = b'(s) f^{s+k} \xrightarrow{\text{shifting } s+k \to s}$$

$$P f^{s+1} = b(s) f^s$$

Claim M is holonomic, in fact $N := \mathcal{O}_X(s)[f^{-1}] f^s$ is also hol. as $\mathcal{D}_X(s)$-modules

Consider a filtration on N: $F_i N = \{ P \cdot f^{s-i} \mid p \in \mathbb{C}(s)[X], \deg p \le i(1+\deg f)\}$

Compare with Bernstein filtration $x_j \cdot p f^{s-i} = x_j p \cdot f \cdot f^{s-i-1}$

Count degrees: $\deg(x_j p f) \le (i+1)(1+\deg f)$

Rmk Graded piece of $F_i N$ are fin. dim. In fact $Gr_i N$ has dim = dim of polynomials of degree $i(1+\deg f)$

LEMMA If $\exists$ a filtration on N such that $\dim \mathrm{Gr}_i N \leq \frac{i^{m-1} e}{(m-1)!} + O(i^{m-2})$, then N is holonomic

In our case we have $\binom{i(1+\deg f)+m-1}{m-1} = \frac{i^{m-1}(1+\deg f)^{m-1}}{(m-1)!} + O(i^{m-2})$

$\Rightarrow N$ is holonomic $\Rightarrow$ So is M since $M \subseteq N$

V-FILTRATION

t-coord.

$i : X \hookrightarrow X \times \mathbb{C}$

$x \mapsto (x, f(x))$

$$\begin{tikzcd} \mathcal{O}_X \arrow[d, squiggly] \\ i_+\mathcal{O}_X \end{tikzcd}$$

$i_+\mathcal{O}_X \cong \mathcal{O}_X \otimes \mathbb{C}[\partial_t]$ as $\mathcal{O}_X$-mod.

As a $\mathcal{D}$-mod, action depends on the embedding

$i_+\mathcal{O}_X = \mathcal{O}_{X\times\mathbb{C}}$-module generated by "$\delta_{t-f}$" i.e.

$(t-f)\delta_{t-f} = 0$; freely gen. as an $\mathcal{O}_X$-mod by $\delta_{t-f}, \partial_t \delta_{t-f}, \partial^2 \delta_{t-f}, \ldots$

$N = i_+ \mathcal{O}_X$

$\cup|$

$M = \mathcal{D}_X[\partial t]\,\delta_{t-f}$ a $\mathcal{D}_X$-submod.

$\wr$

$\mathcal{D}_X(s)$-mod. gen. by the symbol f^s

$$\begin{tikzcd}
\delta_{t-f} \arrow[d, leftrightarrow] & t\cdot\delta_{t-f} = f\cdot\delta_{t-f} \arrow[d, leftrightarrow] & -\partial_t t\cdot\delta_{t-f} \arrow[d, leftrightarrow] \\
f^s & f\cdot f^s & s\cdot f^s
\end{tikzcd}$$

$b(s)$ can be thought as the minimal polynomial of s acting on the quotient module $\mathcal{D}_X[s]\cdot f^s \big/ \mathcal{D}_X[s]\cdot f^{s+1}$

$\forall m \in A$

$P(m f^{s+1}) = b(s)(m f^s)$

On the other side $b(s)$ is the min. pol. of $-\partial_t t$ on M/tM

Let A any hol. $\mathcal{D}$-mod. on sing X. We can construct $\mathcal{M} = A \otimes_{\mathcal{D}_X} \mathcal{D}_X(s) f$

It is enough to show that $\mathcal{M}$ is hol.

<u>Idea</u>: Find some $\mathcal{M}' \subseteq \mathcal{M}$ holonomic s.t. $\mathcal{M}'|_U = \mathcal{M}|_U$ where $U = X \setminus \{f=0\}$

Need Gabber's filtration + a ham. filtration on M.

VARIATIONS OF HODGE STRUCTURES ON THE PUNCTURED DISK

10/06
YOHAN

Reminder on PVHS.

Let $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$

Def A R-Hodge structure of weight k consists of

1) A fin. gen. R-module

2) a bigrading $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$ satisfying $V^{p,q} = \overline{V^{q,p}}$

2)' a finite decreasing filtration $F^{\cdot}$ of $V_{\mathbb{C}}$ satisfying $V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$ $\forall p$

A polarization of a R-HS $(V_R, F^{\cdot})$ of weight k is a $(-1)^k$-symmetric bilinear form $S : V_R \otimes_R V_R \longrightarrow R$ satisfying

1) $S_{\mathbb{C}}(V^{p,q}, V^{r,s}) = 0$ unless $p = s$ & $q = r$

2) $\forall\, v \in V^{p,q} \setminus \{0\}$, $i^{p-q} S_{\mathbb{C}}(v, \bar{v}) > 0$

Ex $X^n/\mathbb{C}$ smooth proj. variety, L a ample bundle on X

$S : H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ $\quad (u, v) \mapsto u \wedge v$

Define $PH^m(X,\mathbb{Z}) = \ker(\wedge c_1(L) : H^m \to H^{m+2})$

if we restrict to PH^m (primitive coh.) then S is a polarization. (HRBR)

<u>Def</u> Let (V, F, S) a $\mathbb{R}$-HS

i) the Weil operator: linear op. on $V_{\mathbb{C}}$ s.t. $C|_{V^{p,q}}$ = multiplication by i^{p-q}

ii) $(x,y) \longmapsto S(Cx, \bar{y})$ defines a hermitean symmetric positive definite form on $V_{\mathbb{C}}$

("the Hodge norm")

PVHS

<u>Def</u> A $\mathbb{R}$-VHS of weight k over a complex mfld is

1) $\mathcal{L}$ a $\mathbb{R}$-local system

2) $\mathcal{F}^{\cdot}$ a finite decreasing filtration on $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_X$ by holomorphic subbundles such that

a) $\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_X$

b) $\forall x \in X$, $(\mathcal{V}_x, \mathcal{F}^{\cdot})$ is a $\mathbb{R}$-HS of weight k

A polarization is a $(-1)^k$-symmetric map $\mathcal{L} \otimes_{\mathbb{R}} \mathcal{L} \to \underline{\mathbb{R}}_X$ that induces a polarization on each fiber

$\rightsquigarrow$ from the polarization one gets a canonical hermitean $\mathcal{C}^\infty$-metric ("the Hodge metric")

DEGENERATIONS OF PVHS

X complex manifold, $D \subseteq X$ a NCD

$\mathbb{V} = R$-PVHS on $X \setminus D$

<u>Thm</u> (Landman, Borel) the eigenvalues of the local monodromies around D have mod 1

In particular, if $\alpha_{\mathbb{R}}$ has an integral structure $(\alpha_{\mathbb{R}} \cong \alpha_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R})$

then the local monodromies are quasi-unipotent

(enough to prove on the punctured disk)

NILPOTENT ORBIT

(X, D) $\mathbb{V}$ a $\mathbb{R}$-PVHS on $X \setminus D$ with quasi-unipotent monodromies around D

Let $\mathcal{V}^{Del}$ be Deligne's canonical extensions of $(\mathcal{V}, \nabla)$ to X

(i.e. $\nabla : \mathcal{O}^{Del} \to \mathcal{O}^{Del} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$ + eigenvalues of the residue belongs to $[0,1)$

Thm (Schmid) The Hodge filtration $\mathcal{F}^\cdot$ extends to X as a filtration $\tilde{\mathcal{F}}^\cdot$ of $\mathcal{O}^{Del}$ by coherent sub $\mathcal{O}_X$-mod. When the local monodromies are unipotent $\tilde{\mathcal{F}}^\cdot$ is a filtration of $\mathcal{V}^{Del}$ by subbundles (i.e. locally direct factor)

$$\tilde{\mathcal{F}}^p = (j_*\mathcal{F}^p) \cap \mathcal{V}^{Del}.$$

EXAMPLE $X = \Delta$ disk, $D = \{0\}$. $\mathbb{V}$ on Δ^* with unipotent monodromy

$$\mathcal{V}^{Del} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_\Delta$$

(the idea is to show that we can find the limit of the filtrations looking at is as to a point in a Grassmanian)

$X = \Delta$, $D = \{0\}$, z the coordinate

1) Let $(\mathcal{V}, \nabla)$ on Δ^* with unipotent monodromy. Let $T \in End(\mathcal{V}, \nabla)$ the monodromy

$N := \log T$ (there finite term because T is unipotent)

Claim $\mathcal{V} \to \Delta^*$ has a privileged trivialization

(trivialization on $\underset{V\otimes\mathcal{O}_X}{\underset{\shortparallel}{\mathcal{V}}}$ $\Longleftrightarrow$ flat connection on $\mathcal{V}$ with trivial monodromy)

Define $\nabla^c := \nabla - N(z)\frac{dz}{z}$. It is a connection on $\mathcal{V}\to\Delta^*$

__Prop__ ∇^c has trivial monodromy

__Proof__

$$\begin{pmatrix}\text{flat (multi)-section}\\ \text{of } \nabla\end{pmatrix} \longrightarrow \begin{pmatrix}\text{flat (multi)-sections}\\ \text{of } \nabla^c\end{pmatrix}$$

$$h(z) \longmapsto \exp\left(-\frac{1}{2i\pi}\log(z)\cdot N\right)h(z)$$

__Cor__ ∇^c determines a trivialization of $\mathcal{V}$. $\mathcal{V} = V\otimes_{\mathbb{C}}\mathcal{O}_{\Delta^*}$, hence an extension $\mathcal{V}'$ on Δ

__Claim__ $\nabla^c N = 0$ (clearly $\nabla N = 0$)

$$\underset{\mathrm{End}\,V}{\underset{\cap}{(\nabla^c N)}}(v) = \nabla^c(N(v)) - N(\nabla^c(v))$$

__Prop__ $\mathcal{V}'$ is Deligne's canonical extensions of $(\mathcal{V},\nabla)\to\Delta^*$

(because in the new trivialization we have $\nabla = d + \frac{1}{2i\pi}N(0)\frac{dz}{z}$

Let $\mathbb{V}=(\alpha_{\mathbb{R}}, \mathcal{V}, \nabla, \mathcal{F}^{\cdot}, S)$ a $\mathbb{R}$-PVHS with integral structure on Δ^* with unipotent monod.

By Schmid $\mathcal{F}^{\cdot}$ extends to Δ as a filtration by submodules of $\mathcal{V}^{Del}$

Let $\mathcal{F}^{\cdot}_{nil}$ be the filtration of $\mathcal{V}^{Del}$ which satisfies 1) $\mathcal{F}^{\cdot}_{nil}(0) = \mathcal{F}^{\cdot}(0)$

2) $\mathcal{F}_{nil}$ is parallel with respect to ∇ (∇ gives a canonical isomorphism b/w different fibers because there is no monodromy

<u>Thm</u> (Schmid) The data $(\alpha_{\mathbb{Z}}, \mathcal{V}, \nabla, \mathcal{F}^{\cdot}_{nil}, S)$ define a $\mathbb{R}$-PVHS with integral structure in a nbhd of 0 in Δ^*

<u>SPOILER</u>: In 0 it will be a MHS.

17/06

Let $\mathbb{V} = (V_{\mathbb{R}}, \mathcal{V}, \nabla, F^{\cdot}, S)$ be a $\mathbb{R}$-PVHS of weight k with unipotent monodromy on $\Delta_t^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let $T \in \mathrm{End}(\mathcal{V}, \nabla)$ be the monodromy corresponding to the conterclockwise generator loop of Δ^* and set $N = \log(T) \in \mathrm{End}(\mathcal{V}, \nabla)$.

Consider the connection $\nabla^c = \nabla - \frac{1}{2i\pi} N(z) \frac{dz}{z}$ on $\mathcal{V} \to \Delta^*$.

Its monodromy is trivial, hence we get a priviliged trivialization of $\mathcal{V} \to \Delta^*$ and a priviliged extensions $\mathcal{V}^c \to \Delta$ still endowed with a trivialization $\mathcal{V}^c \cong V \otimes_{\mathbb{C}} \mathcal{O}_\Delta$

Prop $\nabla^c N = 0$, $\nabla^c S = 0$

In other words N and S are constant in the trivialization associated to ∇^c

Thm (Schmid)

1) $\{\mathcal{F}^{\cdot}\}$ extends to $\mathcal{V}^c \to \Delta$ as a filtration by subbundle
2) Let $\{\mathcal{F}^{\cdot}_{nil}\}$ be the unique filtration of $\mathcal{V}^c \to \Delta$ satisfying: $\begin{cases} \mathcal{F}_{nil}(0) = \overset{\mathcal{F}_{\lim}}{\overset{\parallel}{\mathcal{F}(0)}} \\ \nabla^c \mathcal{F}_{nil} \subset \mathcal{F}_{nil} \end{cases}$

The data $\mathbb{V}_{nil} = (V_{\mathbb{R}}, \mathcal{V}, \nabla, \mathcal{F}_{nil}, S)$ defines a $\mathbb{R}$-PVHS of weight k with unipotent monodromy with the same Hodge numbers as $\mathbb{V}$ on Δ_ε^* for some $\varepsilon > 0$

Rmk 1) V_{nil} extends to $\mathbb{C}^*$

2) $\mathbb{V}_{nil}$ is completely determined by the following data $((V_{\mathbb{R}})_1, V, N, F_{lim}, S)$

$\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$, $\nabla = d + \frac{1}{2i\pi} N \frac{dz}{z}$

$F^{\cdot}_{nil} = F_{lim} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$, $S = S \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$ $\quad (V_{\mathbb{R}})_z = \exp\left(\frac{1}{2\pi i} \log(z) N\right) V_{\mathbb{R}}$

QUESTION When does such data $(V_{\mathbb{R}}, V, N, F^{\cdot}, S)$ define a $\mathbb{R}$-PVHS of weight k with nilpotent monodromy in a nbhd of 0 in $\mathbb{C}^*$?

When it is the case, the corresponding nilpotent orbit

Need 1) N is nilpotent
2) $N(F^{\cdot}) \subset F^{\cdot -1}$
3) $S(N\cdot,\cdot) + S(\cdot, N\cdot) = 0$
4) N and S are real
5) $S(F^p, F^{k-p+1}) = 0 \ \forall p$

1-5) are necessary but not sufficient.

Lemma (Deligne) V f.d. k-vector space (k field) $N \in \mathrm{End}_k V$ nilpotent

$\exists!$ unique increasing exhaustive filtration $W_{\cdot}$ $(= W_{\cdot}(N))$ on V s.t.

1) $N(W_\ell) = W_{\ell-2}$ $\qquad$ 2) $N^\ell : Gr^W_\ell V \xrightarrow{\sim} Gr^W_{-\ell} V$

Moreover if $P_\ell(N) = \ker(N^{\ell+1} : Gr^W_\ell V \to Gr^W_{-\ell-2} V)$ then

$$Gr^W_\ell V = \bigoplus_{j \geq 0} N^j(P_{\ell+2j}(N))$$

Thm (Schmid) If $(\underset{\substack{\uparrow \\ \text{weight}}}{k}, V_{\mathbb{R}}, V, N, F; S)$ define a nilpotent orbit, then

1) $N^{k+1} = 0$

2) $\forall z \in \mathbb{C}^*$, $(\exp(\frac{1}{2i\pi} N \log(z)) V_{\mathbb{R}}, V, F, W[-k])$ is a $\mathbb{R}$-MHS graded-polarized by (N,S) i.e.:
- N is real
- N is type $(-1,-1)$ $(N(F^\cdot) \subset F^{\cdot -1}, N(W_\cdot) \subseteq W_{\cdot -1}$
- the pure HS of weight $k+\ell$ induced by F on $P_\ell(N)$ is polarized by $S_\ell(\cdot,\cdot) = S(\cdot, N^\ell \cdot))$

Thm (Cattani-Kaplan-Schmid)

Let $(k, V_{\mathbb{R}}, V, N, F; S)$ as before. Assume that this data satisfy 2), ..., 5)

$+ N^{k+1} = 0$ + $(W[-k], F^\cdot)$ is a $\mathbb{R}$-MHS on $V_{\mathbb{R}}$ graded polarized by (N,S)

Then it defines a nilpotent orbit.

Dependence on the coordinate

X complex manifold of dim 1, $p \in X$

Let $\mathbb{V} \in \mathbb{R}$-PVHS on $X^* = X \setminus \{p\}$ with unipotent local monodromy around p

Let $z : X \to \mathbb{C}$ hol. function satisfying $z(p) = 0$ $(dz)_p : TX_p \cong \mathbb{C}$

$(V_{\mathbb{R}}, \mathcal{V}, \nabla, F; S)$

In general the construction depends on the choice of the coordinate (just on the first derivative) but it is canonical on $(dz)_p(TX_p) \setminus \{0\}$)

SOME EXAMPLES

WEEK 13

E elliptic curve. Fix a basis δ, γ for $H_1(E,\mathbb{Z})$ s.t. $\delta \circ \gamma = 1$

δ^*, γ^* dual basis in $H^1(E,\mathbb{Z})$

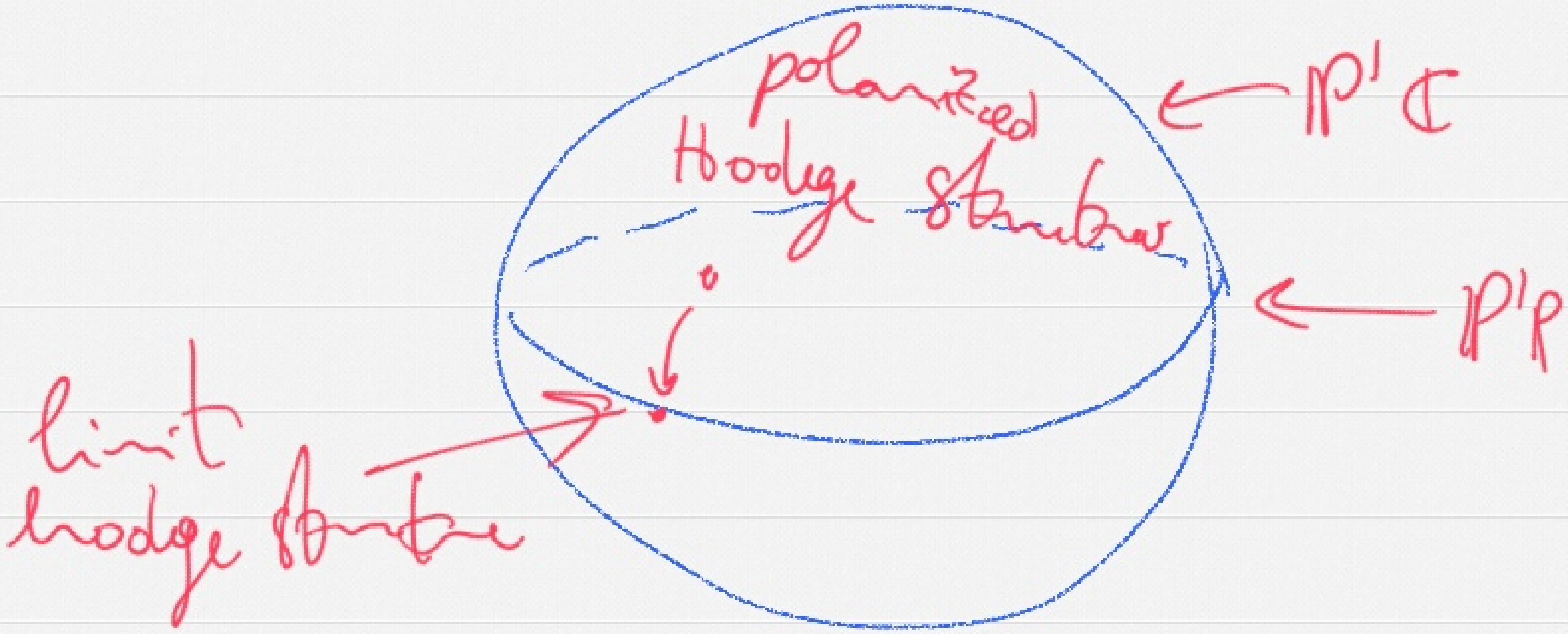
Given $j \in H^1_{DR}(E)$ $\quad j = \left(\int_\delta j\right)\delta^* + \left(\int_\gamma j\right)\gamma^*$

If ω is a hol. 1-form $\quad F^1$ = line in $H_\mathbb{C}$ generated by $\left(\int_\delta \omega\right)\delta^* + \left(\int_\gamma \omega\right)\gamma^*$

Hodge structures on H is determined by $F^1 \subset H_\mathbb{C}$: Hodge structures on $H \hookrightarrow \mathbb{P}^1\mathbb{C}$

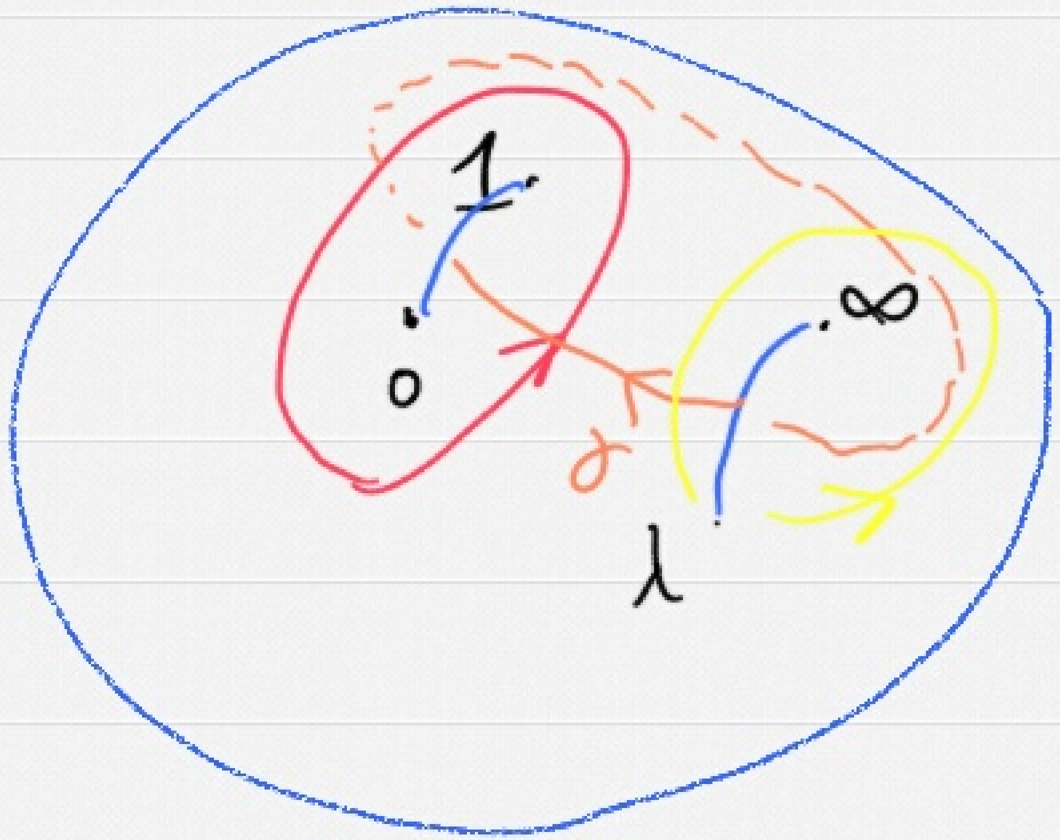
Hodge structure $= \{F_1 \mid \overline{F_1} \neq F_1\} = \mathbb{P}^1\mathbb{C} \setminus \mathbb{P}^1\mathbb{R}$

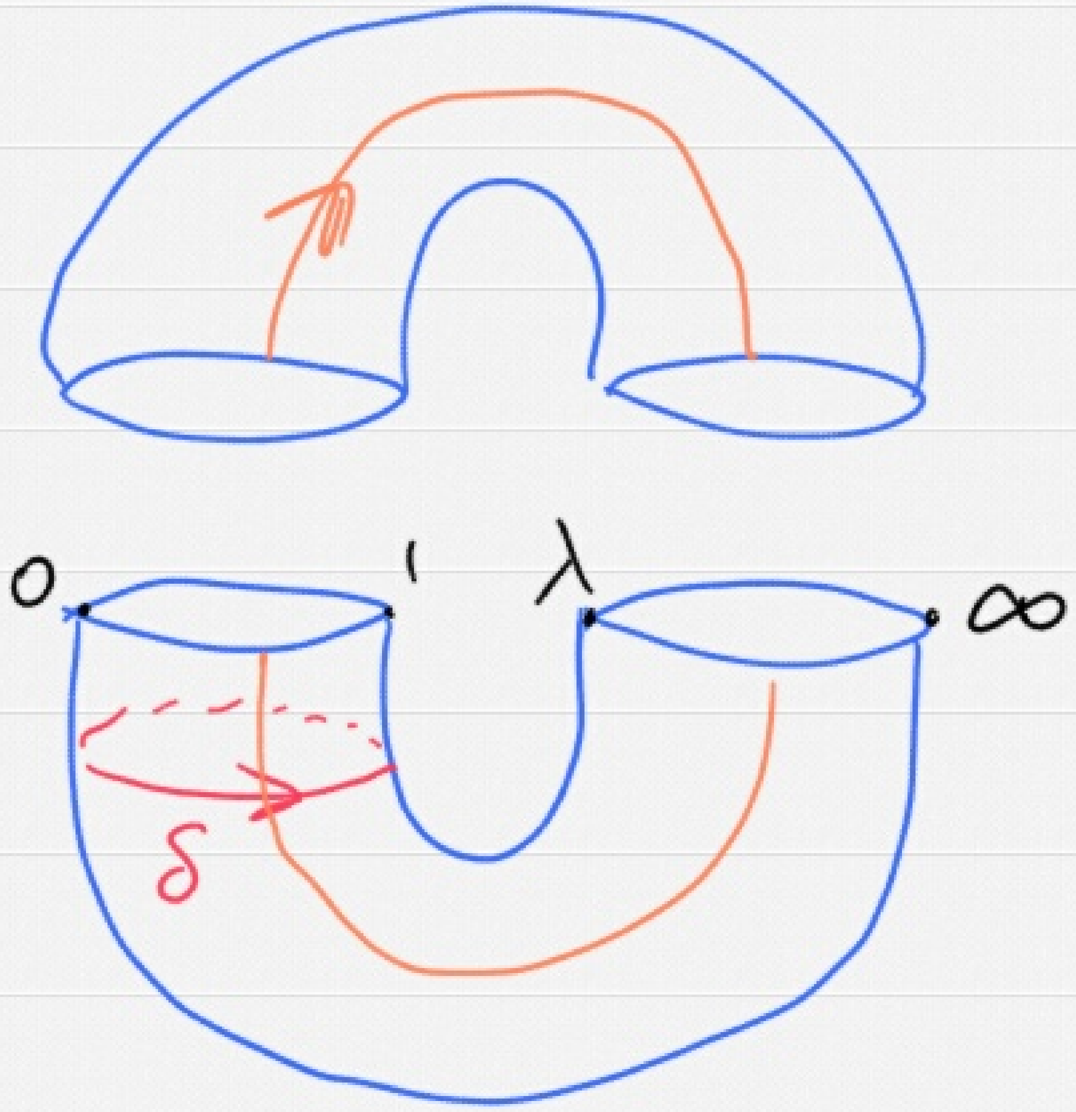
Expand $\quad i\int \omega \wedge \overline{\omega} = 2\int |f(z)|^2 dx \wedge dy > 0 \Rightarrow \omega = A\delta^* + B\gamma^* \Rightarrow \mathrm{Im}\left(\frac{B}{A}\right) > 0$



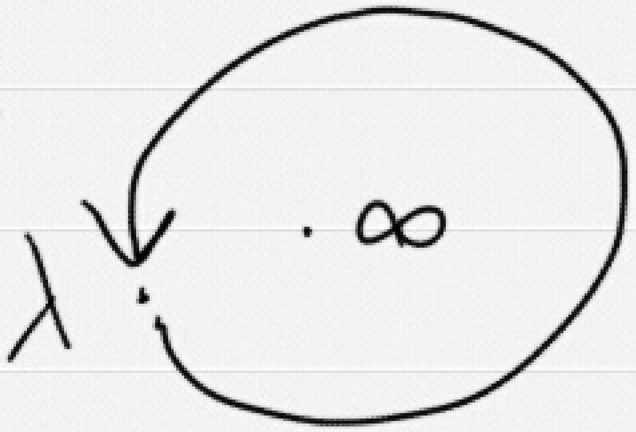
We calculate in an example. $\{y^2 = x(x-1)(x-\lambda)\}$ family of ell. curves.

2:1 ramified cover of $\mathbb{P}^1\mathbb{C}$



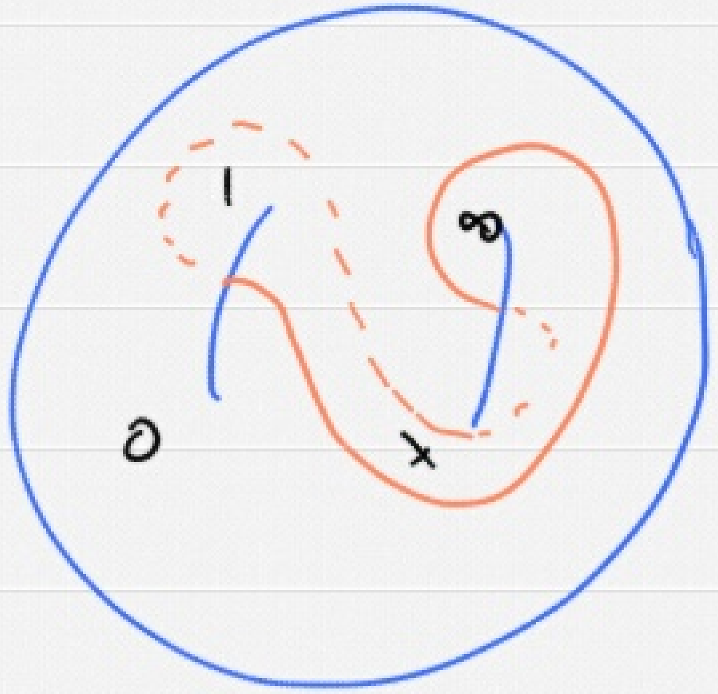


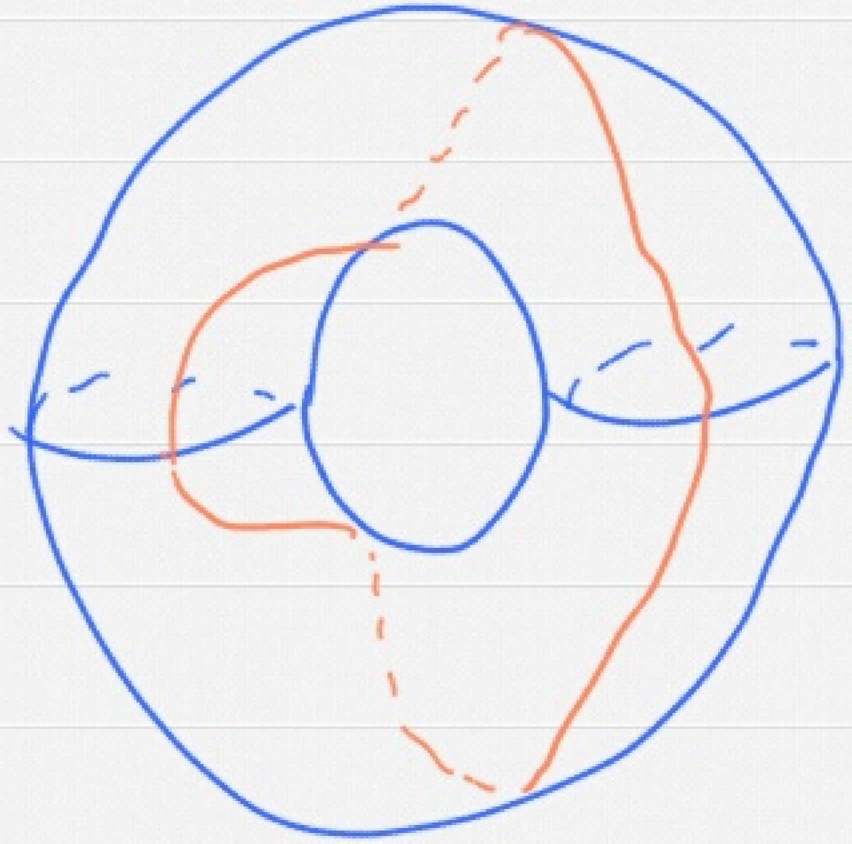
Purely topological fact



induces $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ on $H_1(E,\mathbb{Z})$

EXPLANATION





DOUBLE
DEHN TWIST

Nilpotent orbit thm Asymptotic behaviour of period map is determined by local (topological) monodromy

As. of period map $\omega = \frac{dx}{y}$ spans $F^1 \subset H^1_{\mathbb{C}}$

we want to understand the behaviour of the line F^1 as $\lambda \to \infty$

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \underset{\sim}{\text{for } \lambda \text{ large}} \int \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi i}{i\sqrt{\lambda}} = \frac{2\pi}{\sqrt{\lambda}}$$

Trick $\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \frac{2i}{\sqrt{\lambda}} \log \lambda +$ terms hol. in λ^{-1}

Hence $\underset{\underset{\mathbb{P}^1\mathbb{C}}{\cap}}{\tau}(\lambda) = \dfrac{\frac{2i}{\lambda}\log(\lambda)}{\frac{2\pi}{\sqrt{\lambda}}} + \ldots = \underbrace{\frac{i}{\pi}\log\lambda}_{\text{"nilpotent orbit"}} + \left(\begin{matrix}\text{terms hol.}\\ \text{at } \infty\end{matrix}\right)$

$\cong$ space of Hodge structures

Rewrite with $\mu^{-1} = \lambda$ $\quad \tau(\mu) = \frac{1}{\pi i}\log\mu + \begin{matrix}\text{holomorphic}\\ \text{in } \mu\end{matrix}$

On H^1 $\quad T = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ $\quad N = \begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}$

$$\begin{pmatrix}1 & 0\\ \frac{1}{\pi i}\log z & 1\end{pmatrix}\begin{pmatrix}1\\ \alpha\end{pmatrix} = \begin{pmatrix}1\\ \frac{1}{\pi i}\log z + \alpha\end{pmatrix}$$

<u>Schmid's N.O.T.</u> $\tau(\mu)$ is asymptotic to $z \longmapsto \exp\left(\frac{1}{2\pi i}(\log z)N\right)\cdot\begin{pmatrix}1\\ \alpha\end{pmatrix} \in \mathbb{P}^1\mathbb{R}$

(or motivated hol. fcts) $\qquad$ (or multivalued hol. fct)

What is the limit Hodge structure?

The weight filt. is determined by N

$$0 = W_{-2} \subseteq \underset{\underset{\displaystyle W_0}{\shortparallel}}{W_{-1}} = \ker N = \mathbb{Q}\gamma^* \qquad \text{Here } \underset{\underset{\displaystyle \delta^* + \alpha\gamma^*}{\shortparallel}}{\overline{F_1}} = F_1$$

$$\overset{\cap}{W_1} = H^1 = \mathbb{Q}\delta^* \oplus \mathbb{Q}\gamma^*$$

DIRECT IMAGE THM FOR CURVES

KAISER 24/06

$\overline{S}$ compact RS. $S \overset{j}{\hookrightarrow} \overline{S}$, $\forall\, \overline{S}\setminus S < \infty$

$\mathcal{H}_{\mathbb{Z}}$: $\mathbb{Z}$-local system on S + PVHS $(\mathcal{V}, \nabla, \dots)$ of wt m on $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes \mathbb{C}$

<u>Thm</u> $H^1(\overline{S}, j_*\mathcal{H})$ has a hodge structure pure of wt $m+1$

<u>Local theory</u> $z \in \overline{S}\setminus S$: $0 \in \mathbb{D} = \{t \in \mathbb{C} \mid |t| < 1\}$ $(\mathcal{V}, \nabla, \dots)$ PVHS on $\dot{\mathbb{D}}$ of wt m.

Meromorphic Deligne ext: $\tilde{\mathcal{V}} \subseteq j_*\mathcal{V}$

finite free $\mathcal{O}_{\mathbb{D}}[t^{-1}]$-module, reg. hol. $\mathcal{D}_{\mathbb{D}}$-mod with $DR(\tilde{\mathcal{V}}) \cong Rj_*\mathcal{H}$

<u>Note</u>: $j_*\mathcal{H} \subseteq Rj_*\mathcal{H}$ is the smallest perverse subsheaf wich is $\approx$ over $\mathbb{D}^*$

$\rightsquigarrow j_*\mathcal{H} = DR(\mathcal{D}_{\mathbb{D}}\tilde{\mathcal{V}}^{>-1}) =: DR(\tilde{\mathcal{V}}_{min})$

<u>Rmk</u> Borel $\rightsquigarrow$ Monodromy on $\mathcal{H}$ is quasi-unipotent $\rightsquigarrow \exists$ refined KM-filt. $\tilde{\mathcal{V}}^{\cdot}$ with $\cdot \in \mathbb{Q}$

$$\begin{tikzcd} \psi_t = \tilde{\mathcal{V}}^0 / t\tilde{\mathcal{V}}^0 \arrow[d, shift right, "\partial_t"'] \\ \phi_t = \tilde{\mathcal{V}}^1 / t\tilde{\mathcal{V}}^1 \arrow[u, shift right, "t"'] \end{tikzcd} \rightsquigarrow j_*\mathcal{H} = DR\left(\left[\begin{tikzcd} \psi_t \arrow[d, shift right] \\ \mathrm{im}\,\partial_t \arrow[u, shift right] \end{tikzcd}\right]\right)$$

M hol. $\mathcal{D}_{\mathbb{D}}$-module $\quad V^\lambda DR(M) := [V^\lambda M \longrightarrow \Omega^1 \otimes_{\mathcal{D}} V^{\lambda-1} M] \rightsquigarrow$

$\rightsquigarrow V^0 DR(M) \xhookrightarrow{q.iso.} DR(M)$ (since $\partial_t : gr^k_V M \xrightarrow{\sim} gr^{k-1}_V M$ for <0)

$$M = \tilde{V}_{min} \rightsquigarrow \tilde{V}^1_{min} = \partial_t \tilde{V}^0 + \tilde{V}^{>-1}$$

$$V^0 DR \tilde{V}_{min} = [\tilde{V}^0 \xrightarrow{\partial_t} \Omega^1 \otimes (\partial_t \tilde{V}^0 + \tilde{V}^{>-1})]$$

$$\cong [\tilde{V}^0 \xrightarrow{t\partial_t} \Omega^1 \otimes (t\partial_t \tilde{V}^0 \oplus \tilde{V}^{>0})]$$

$\cup\!\mid$ q.iso

$$[M_0 \tilde{V}^0 \xrightarrow{t\partial_t} \Omega^1 \otimes M_{-2} \tilde{V}^0]$$

Here $M_\cdot \tilde{V}^b$ pullback of weight filt. for $(t\partial_t)_{nil} = N$ on $\tilde{V}^{>b}/\tilde{V}^{>b}$

$$gr^m V \xrightarrow[\sim]{N^m} gr^{-m} V \qquad \psi_t^0 \tilde{V} / M_0 \psi_t^0 \tilde{V} \xrightarrow{\sim} N \psi_t^0 \tilde{V} / M_{-2} \psi_t^0 \tilde{V}$$

$$V_i \xrightarrow{N} V_{i-2}$$

$$\rightsquigarrow DR(\tilde{V}_{min}) \xrightarrow{q.iso.} [M_0 \tilde{V}_{min} \xrightarrow{t\partial_t} \Omega^1 \otimes M_{-2} \tilde{V}^0_{min}]$$

Endow $\mathbb{D}$ with Poincaré metric: $re^{i\theta} \quad \mathbb{D} \xleftarrow{} {}_{z \mapsto z+2}\backslash \mathbb{H} \ni z = x + iy$

$$e^{i\pi\tau} \longleftarrow\!\shortmid \tau$$

$$\frac{dr^2 + r^2 d\theta^2}{r^2 (\log r)^2} \longleftarrow\!\shortmid \frac{dx^2 + dy^2}{y^2}$$

dvol = assoc. volume element

Prop i) $f \in L^2(d\mathrm{vol}) \iff (\log r)^{-1} f \in L^2(d\theta \frac{dr}{r})$

ii) $\omega = f\frac{dr}{r} + g\, d\theta \in L^2(d\mathrm{vol}) \iff f \,\&\, g \in L^2(d\theta \frac{dr}{r})$

$(\log r)^{\ell/2} r^b = [\tilde{V}_{(2)} \xrightarrow{\nabla} (\Omega^1 \otimes_b \tilde{V})_{(2)}]$ $\ell \in \mathbb{Z}$

Def $(DR(\tilde{V}))_{(2)} = [\tilde{V}_{(2)} \xrightarrow{\nabla} (\Omega^1 \otimes_b \tilde{V})_{(2)}]$

$\omega \in \Omega^i \otimes_b \tilde{V}$ local section in $L^2 \iff \omega$ & $\nabla\omega$ are loc L^2-integer

(L^2: only condition at 0!)

holo L^2-de Rham complex

Thm (Schmid) s: local section of $\tilde{V}$; then

$s \in M_\ell \tilde{V}^b \setminus M_{\ell-1}\tilde{V}^b + \tilde{V}^{>b} \iff \|s\|$ has order of smooth $|t|^b \log|t|^{\ell/2}$

$\rightsquigarrow (\Omega^0 \otimes \tilde{V})_{(2)} = M_0 \tilde{V}^0$

$(\Omega^1 \otimes \tilde{V})_{(2)} = M_{-2} \tilde{V}^{-1}$ Remark $t : M_{-2}\tilde{V}^{-1} \xrightarrow{\sim} M_{-2}\tilde{V}^0$

Thm $j_+ H \overset{q.iso.}{\xrightarrow{\sim}} DR(\tilde{V}_{min}) \xrightarrow{\sim} DR(\tilde{V})_{(2)}$

HODGE FILTRATION

Thm: $j_+ F^p V \cap \tilde{V}^0$ are loc. free $\mathcal{O}_D$-mod

Def $F^p \tilde{V}^{>-1} := j_* F^p V \cap \tilde{V}^{>-1}$ are loc. free $\mathcal{O}_D$-mod

$F^p \tilde{V}_{min} := \sum_{j \geq 0} \nabla^j F^{p+j} \tilde{V}^{>-1}$ full fills Griffith theorem

$F^p \tilde{V}_{(2)} = j_* F^\ell V \cap \tilde{V}_{(2)} = F^p \tilde{V}_{min} \cap \tilde{V}_{(2)}$

$F^i DR(\tilde{V}_{min}) := [\Omega^0 \otimes F^i \tilde{V}_{min} \longrightarrow \Omega^1 \otimes F^{i-1} \tilde{V}_{min} \to \dots]$

$F^i DR(\tilde{V})_{(2)} \quad \dots \quad F^i \tilde{V}_{(2)} \dots \quad F^{i-1} \tilde{V}_{(2)}$

Thm $(DR(\tilde{V})_{(2)}, F^\cdot) \overset{q.iso.}{\xrightarrow{\sim}} (DR(\tilde{V}_{min}), F^\cdot)$ filtered q.iso.

HODGE THEORY/S : $F^p := F\tilde{\mathcal{V}}$; $\mathcal{G}^p = F^p/F^{p+1}$

$$\nabla F^p \subseteq \Omega^1 \otimes F^{p-1}$$

$\mathcal{E}(\mathcal{V}) = \mathcal{C}^\infty(S) \otimes_{\mathcal{O}} \mathcal{V} \cong \bigoplus_k K^{k,m-k}$ orthog. dec. of $\mathcal{C}^\infty$ bundles

$\cup$

$\mathcal{E}(F^p)$ - - - $\otimes F^p$ - - $= \bigoplus_{k \geq p} K^{k,m-k}$

$$\mathcal{E}^r(\mathcal{V}) = \Omega^r_{\mathcal{C}^\infty}(S) \otimes_{\mathcal{O}_S} \mathcal{V} = \bigoplus_{p+q=r} \underbrace{\mathcal{E}^{p,q}(\mathcal{V})}_{\Omega^{p,q}_{\mathcal{C}^\infty} \otimes_{\mathcal{O}_S} \mathcal{V}}$$

$D \downarrow = d \otimes 1 + 1 \otimes \nabla$

$\mathcal{E}^{r+1}(\mathcal{V})$

By $(*)$ & $\overline{(*)}$: $D\mathcal{E}^{r,s}(\mathcal{K}^{p,q}) \subseteq \mathcal{E}^{r+1}(\underset{\partial'}{K^{p,q}} \oplus \underset{\nabla'}{K^{p-1,q-1}} \oplus \underset{\bar{\partial}'}{\mathcal{E}^{r,s+1}}(K^{p,q} \oplus \underset{\bar{\nabla}'}{K^{p+1,q+1}})$

$$D = \partial' + \bar{\partial}' + \nabla' + \bar{\nabla}'$$

Remark ∇' & $\bar{\nabla}'$ are linear map

$Gr^p_F \cdot \mathcal{E}^\cdot(\mathcal{V})$ in double complex of sheaves

$K^{r,s} = \mathcal{E}^{r,s}(K^{p-r,m-p+r})$ with diff. $D'' = \overset{s\text{-dir}}{\overset{\uparrow}{\bar{\partial}'}} + \overset{r\text{-dir}}{\overset{\uparrow}{\nabla'}}$

fix n $\longrightarrow$ Doulbeault complex for $\mathcal{G}^{n} = gr_F^{n}\,\Omega^\cdot(\mathcal{O})$

$\rightsquigarrow Gr_F^p\,\Omega(\mathcal{V}) \xrightarrow[\sim]{q.iso.} Gr_F^q\,\mathcal{E}(\mathcal{V})$

$\rightsquigarrow \Omega^\cdot(\mathcal{V}) \xrightarrow[\sim]{q.iso.} \mathcal{E}^\cdot(\mathcal{V})$ fine sheaves

---

We want to pass to $\overline{S}$. We need to substitute with

$\mathcal{L}^i(\mathcal{V})_{(2)}$ = sheaf on $\overline{S}$ of loc. L^2-integrable $\mathcal{V}$-valued i forms ϕ, $D\phi \in L^2$

ZUCKER: Is a fine sheaf.

Thm: $\Omega^\cdot(\mathcal{V}) \xrightarrow[q.iso.]{\sim} \mathcal{L}(\mathcal{V})_{(2)}$

- $H^\cdot(\overline{S}, Gr_F^p\,\Omega(\mathcal{V})_{(2)}) \cong H^1(\Gamma(\overline{S}, gr_F^p\,\mathcal{L}(\mathcal{V})_{(2)}))$

Have $Gr_F^p\,\mathcal{L}(\mathcal{V}) = \bigoplus_q \left[\mathcal{L}^{0,q}(\mathcal{G}^p)_{(2)} \oplus \mathcal{L}^{1,q}(\mathcal{G}^{p+1})_{(2)}\right]$

Cor $H^1(\overline{S}, j_*\mathcal{H}) = H^1(\overline{S}, \Omega(\mathcal{V})_{(2)}) = H^1(\Gamma(\overline{S}, \mathcal{L}(\mathcal{V})_{(2)}))$

Now: Harmonic Hodge theory $\longrightarrow H^1_{(2)}(S, \mathcal{V})$

Thm M complet Kähler mfld, $\mathcal{H}$: local system underlying a PVHS $(\mathcal{V}, \nabla, \ldots)$
of wt m. **Assume**: dim $H_{(2)}(M, \mathcal{H}) < \infty$

Then $H^{i}_{(2)}(M, \mathcal{H}) = \bigoplus_{p+q=i+m} h^{p,q}_{(2)}(M)$ pure Hodge structure of wt m+i)

$h^{p,q}_{(2)}(M) := \{ \Delta_{D''}$ - harmonic $\mathcal{V}$-valued L^2-form of type $(p,q) \}$

Sketch of proof Consider ext. of D: $A^{p}_{c}(S, \mathcal{V}) \to A^{p+1}_{c}(S, \mathcal{V})$
to $L^{p}_{2}(S, \mathcal{V})$:= H-space of measurable forms with finite.
and the same for D': $A^{p,q}_{c} \longrightarrow A^{p+1,q}_{c}$
D'': . .

Lemma ∃! ext. to a closed operator

δ formal adj. of D, $\delta = \delta' + \delta''$ have unique closed
Laplacian: $\Delta_D = D\delta + \delta D$, $\Delta_{D'}$, $\Delta_{D''}$ and $\Delta_D = \Delta_{D'} + \Delta_{D''}$
Kähler id. $\Delta_{D'} = \Delta_{D''} \rightsquigarrow \Delta_D = 2\Delta_{D''}$

domain$(\Delta^p_D) = \{ \varphi \in L^p_2 \mid \varphi \in D(\delta) \cap \mathcal{D}(D),\ \delta\phi(D),\ D\phi \in \mathcal{D}(\delta) \}$

Lemma $(\Delta^p_D$, domain$)$ is a self-adjoint extension.

Lemma M has a complete metric

$\rightsquigarrow \Delta^p_{D,D',D''}$ are ess. self-adjoint, i.e. closure are unique self-adjoint extensions.

$\rightsquigarrow$ domain(Δ^p_-) has $\cdot \pm D, D', D''$ coincide

Also domain for $\Delta'_\cdot = \bigoplus_{P,Q=v} \Delta^{P,Q}$ open

$$\partial(\Delta^*) = \bigoplus \mathcal{D}\left(\Delta^{P,Q}\right) \rightsquigarrow$$

$$\rightsquigarrow \text{for } h^P = \{ \phi \in \mathcal{D}(\Delta^*) \mid \Delta^p_D \phi = 0 \}$$

$$\overset{\shortparallel}{\bigoplus h^{p,q}}$$

Δ^p_D self adj. $\rightsquigarrow L^p_2 = h^p \oplus \overline{(\text{image } \Delta^p_D)}$

Lemma Δ^p_D has closed image $\Leftrightarrow$ D^p & D^{p+1} have closed images

$D^p : \mathcal{D}(D^p) \rightarrow$ has $D^{p+1} \subset L^{p+1}_2$ (the first one is a Hilbert with the graph norm so it is bounded) coker = $H^p_{(2)}(M,\mathcal{H})$ by def! Banach's thm $\Rightarrow$ image of D^p is closed.

$\rightsquigarrow \phi \in L^p_2 \rightsquigarrow \phi = h + \Delta^p_D \eta = \underset{\substack{\uparrow \\ \text{harmonic}}}{h} + D\delta\eta + \delta D\eta$

If $D\phi = 0 = D\varphi D\eta$

$0 = \langle D\delta D\eta, D\eta\rangle = \langle \delta D\eta, \delta D\eta \rangle = \| \quad \| \rightsquigarrow \delta D\eta = 0$

$\rightsquigarrow h^p_{(2)} = H^p_{(2)}(M,\mathcal{H})$. Regularity thm $\rightsquigarrow h^p \subseteq A^p_{(2)}$

Can do also Hodge decomposition for $\Gamma(\bar{S}, Gr^p_F \mathcal{L}(V)_{(2)})$

$$Gr^p_F \mathcal{L}(V)_{(2)} = \bigoplus_{P+Q=p} h^{P,Q}$$

$\rightsquigarrow H^1(\bar{S}, F^\cdot \Omega(V)_{(2)}) \hookrightarrow H^1(\bar{S}, \Omega^\cdot(V)_{(2)})$

Hodge-de Rham SS. deg at E_1