DIRECT IMAGE THM FOR CURVES

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$\bar{S}$ compact RS. $S \overset{j}{\hookrightarrow} \bar{S}$, $\# \bar{S} \setminus S < \infty$

$H_{\mathbb{Z}}$: $\mathbb{Z}$-local system on S + PVHS$(\mathcal{V}, \nabla, \dots)$ of wt m on $H = H_{\mathbb{Z}} \otimes \mathbb{C}$

<u>Thm</u> $H^1(\bar{S}, j_* H)$ has a hodge structure pure of wt $m+1$

<u>Local theory</u> $z \in \bar{S} \setminus S$: $0 \in \mathbb{D} = \{ t \in \mathbb{C} \mid |t| < 1 \}$ $(\mathcal{V}, \nabla, \dots)$ PVHS on $\dot{\mathbb{D}}$ of wt m.

Meromorphic Deligne ext: $\tilde{\mathcal{V}} \subseteq j_* \mathcal{V}$

finite free $\mathcal{O}_{\mathbb{D}}[t^{-1}]$-module, reg. hol. $\mathcal{D}_{\mathbb{D}}$-mod with $DR(\tilde{\mathcal{V}}) \cong Rj_* H$

<u>Note</u>: $j_* H \subseteq Rj_* H$ is the smallest perverse subsheaf which is $\cong$ over $\mathbb{D}^*$

$\rightsquigarrow j_* H = DR(\mathcal{D}_{\mathbb{D}} \tilde{\mathcal{V}}^{>-1}) =: DR(\tilde{\mathcal{V}}_{min})$

<u>Rmk</u> Borel $\Rightarrow$ Monodromy on H is quasi-unipotent $\rightsquigarrow \exists$ refined KM-filt. $\tilde{\mathcal{V}}^{\cdot}$ with $\cdot \in \mathbb{Q}$

$$\begin{array}{c} \psi_t = \tilde{\mathcal{V}}^0 / t \tilde{\mathcal{V}}^0 \\ \partial_t \downarrow \ \uparrow t \\ \phi_t = \tilde{\mathcal{V}}^1 / t \tilde{\mathcal{V}}^1 \end{array} \quad \rightsquigarrow \quad j_* H = DR\left(\left[\begin{array}{c} \psi_t \\ \downarrow \uparrow \\ \operatorname{im} \partial_t \end{array} \right] \right)$$

M hol. $\mathcal{D}_{\mathbb{D}}$-module $\quad V^\lambda DR(M) := [V^\lambda M \to \Omega^1 \otimes_{\mathcal{D}} V^{\lambda-1} M] \rightsquigarrow$

$$\rightsquigarrow V^0 DR(M) \xhookrightarrow{\text{q. iso.}} DR(M) \quad (\text{since } \partial_t : gr_V^k M \xrightarrow{\sim} gr_V^{k-1} M \text{ for } <0)$$

$$M = \tilde V_{min} \rightsquigarrow \tilde V^1_{min} = \partial_t \tilde V^0 + \tilde V^{>-1}$$

$$V^0 DR \tilde V_{min} = [\tilde V^0 \xrightarrow{\partial_t} \Omega^1 \otimes (\partial_t \tilde V^0 + \tilde V^{>-1})]$$

$$\cong [\tilde V^0 \xrightarrow{t\partial_t} \Omega^1 \otimes (t\partial_t \tilde V^0 \oplus \tilde V^{>0})]$$

$$\cup| \text{ q. iso}$$

$$[M_0 \tilde V^0 \xrightarrow{t\partial_t} \Omega^1 \otimes M_{-2} \tilde V^0]$$

Here $M_\cdot \tilde V^b$ pullback of weight filt. for $(t\partial_t)_{nil} = N$ on $\tilde V^{\geq b} / \tilde V^{>b}$

$$gr^m V \xrightarrow[\sim]{N^m} gr^{-m} V \qquad \psi_t^0 \tilde V / M_0 \psi_t^0 \tilde V \xrightarrow{\sim} N \psi_t^0 \tilde V / M_{-2} \psi_t^0 \tilde V$$

$$V_i \xrightarrow{N} V_{i-2}$$

$$\rightsquigarrow DR(\tilde V_{min}) \xrightarrow{\text{q. iso.}} [M_0 \tilde V_{min} \xrightarrow{t\partial_t} \Omega^1 \otimes M_{-2} \tilde V^0_{min}]$$

Endow $\mathbb{D}$ with Poincaré metric: $\quad re^{i\theta} \quad \mathbb{D} \xleftarrow[z \mapsto z+2]{} \mathbb{H} \ni z = x + iy$

$$e^{i\pi\tau} \longleftarrow\!\shortmid \tau$$

$$\frac{dr^2 + r^2 d\theta^2}{r^2 (\log r)^2} \longleftarrow\!\shortmid \frac{dx^2 + dy^2}{y^2}$$

dvol = assoc. volume element

Prop i) $f \in L^2(d\,vol) \iff (\log r)^{-1} f \in L^2(d\theta \frac{dr}{r})$

ii) $\omega = f\frac{dr}{r} + g\, d\theta \in L^2(d\,vol) \iff f \,\&\, g \in L^2(d\theta \frac{dr}{r})$

$(\log r)^{\ell/2} r^b = [\tilde{\mathcal{V}}_{(2)} \xrightarrow{\nabla} (\Omega^1 \otimes_{\mathcal{O}} \tilde{\mathcal{V}})_{(2)}] \qquad \ell \in \mathbb{Z}$

Def $(DR(\tilde{\mathcal{V}}))_{(2)} = [\tilde{\mathcal{V}}_{(2)} \xrightarrow{\nabla} (\Omega^1 \otimes_{\mathcal{O}} \tilde{\mathcal{V}})_{(2)}]$

$\omega \in \Omega^{\cdot} \otimes_{\mathcal{O}} \tilde{\mathcal{V}}$ local section in $L^2 \iff \omega \,\&\, \nabla\omega$ are loc L^2-integr

(L^2: only condition at 0!)

holo L^2-de Rham complex

Thm (Schmid) $\quad s$: local section of $\tilde{\mathcal{V}}$; then

$s \in M_\ell \tilde{\mathcal{V}}^b \setminus M_{\ell-1}\tilde{\mathcal{V}}^b + \tilde{\mathcal{V}}^{>b} \iff \|s\|$ has order of smooth $|t|^b \log|t|^{\ell/2}$

$\rightsquigarrow (\Omega^0 \otimes \tilde{\mathcal{V}})_{(2)} = M_0 \tilde{\mathcal{V}}^0$

$(\Omega^1 \otimes \tilde{\mathcal{V}})_{(2)} = M_{-2} \tilde{\mathcal{V}}^{-1} \qquad$ **Remark** $t : M_{-2}\tilde{\mathcal{V}}^{-1} \xrightarrow{\sim} M_{-2}\tilde{\mathcal{V}}^0$

$\underline{\text{Thm}}$ $j_+ \mathcal{H} \xrightarrow[\sim]{q.iso.} DR(\tilde{V}_{min}) \xrightarrow{\sim} DR(\tilde{V})_{(2)}$

HODGE FILTRATION

$\underline{\text{Thm}}$: $j_+ F^p \mathcal{V} \cap \tilde{\mathcal{V}}^0$ are loc. free $\mathcal{O}_{\mathbb{D}}$-mod

$\underline{\text{rel}}$ $F^p \tilde{\mathcal{V}}^{>-1} : j_* F^p \mathcal{V} \cap \tilde{\mathcal{V}}^{>-1}$ are loc. free $\mathcal{O}_{\mathbb{D}}$-mod

$F^p \tilde{\mathcal{V}}_{min} := \sum_{j \geq 0} \nabla^j F^{p+j} \tilde{\mathcal{V}}^{>-1}$ fullfills Griffith theorem

$F^p \tilde{\mathcal{V}}_{(2)} = j_* F^\ell \mathcal{V} \cap \tilde{\mathcal{V}}_{(2)} = F^p \tilde{\mathcal{V}}_{min} \cap \tilde{\mathcal{V}}_{(2)}$

$F^i DR(\tilde{\mathcal{V}}_{min}) := [\Omega^0 \otimes F^i \tilde{\mathcal{V}}_{min} \longrightarrow \Omega^1 \otimes F^{i-1} \tilde{\mathcal{V}}_{min} \to \dots]$

$F^i DR(\tilde{V})_{(2)} \quad \dots \quad F^i \tilde{V}_{(2)} \dots \quad F^{i-1} \tilde{V}_{(2)}$

$\underline{\text{Thm}}$ $(DR(\tilde{V})_{(2)}, F^\cdot) \xrightarrow[\sim]{q.iso.} (DR(\tilde{V}_{min}), F^\cdot)$ filtered q.iso.

HODGE THEORY / S : $F^p := F\tilde{\mathcal{V}}$; $\mathcal{J}^p = F^p/_{F^{p+1}}$

$$\nabla F^p \subseteq \Omega^1 \otimes F^{p-1}$$

$\mathcal{E}(\mathcal{V}) = \mathcal{C}^\infty(S) \otimes_{\mathcal{O}} \mathcal{V} \cong \bigoplus_k K^{k,m-k}$ orthog. dec. of $\mathcal{C}^\infty$ bundles

$\cup$

$\mathcal{E}(F^p)$ — — — $\otimes F^p$ — — $= \bigoplus_{k \geq p} K^{k,m-k}$

$$\mathcal{E}^r(\mathcal{V}) = \Omega^r_{\mathcal{C}^\infty}(S) \otimes_{\mathcal{O}_S} \mathcal{V} = \bigoplus_{p+q=r} \underbrace{\mathcal{E}^{p,q}(\mathcal{V})}_{\Omega^{p,q}_{\mathcal{C}^\infty} \otimes_{\mathcal{O}_S} \mathcal{V}}$$

$D \downarrow = d \otimes 1 + 1 \otimes \nabla$

$\mathcal{E}^{r+1}(\mathcal{V})$

By $(*)$ & $\overline{(*)}$: $D\mathcal{E}^{r,s}(\mathcal{K}^{p,q}) \subseteq \mathcal{E}^{r+1}(\underset{\partial'}{K^{p,q}} \oplus \underset{\nabla'}{K^{p-1,q-1}} \oplus \underset{\bar{\partial}'}{\mathcal{E}^{r,s+1}}(K^{p,q} \oplus \underset{\bar{\nabla}'}{K^{p+1,q+1}})$

$$D = \partial' + \bar{\partial}' + \nabla' + \bar{\nabla}'$$

<u>Remark</u> ∇' & $\bar{\nabla}'$ are linear map

$Gr^p_F \mathcal{E}^\cdot(\mathcal{V})$ is double complex of sheaves

$K^{r,s} = \mathcal{E}^{r,s}(K^{p-r,m-p+r})$ with diff. $D'' = \underset{\text{s-dir}}{\bar{\partial}'} + \underset{\text{r-dir}}{\nabla'}$

fix n $\longrightarrow$ Doulbeault complex for $\mathcal{G}^{p,\cdot} = gr^p_{F^\cdot}\,\Omega^\cdot(0)$

$\rightsquigarrow Gr^p_{F^\cdot}\,\Omega(\mathcal{V}) \xrightarrow[\sim]{q.\,iso.} Gr^q_{F^\cdot}\,\mathcal{E}(\mathcal{V})$

$\rightsquigarrow \Omega^\cdot(\mathcal{V}) \xrightarrow[\sim]{q.\,iso.} \mathcal{E}^\cdot(\mathcal{V})$ fine sheaves

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We want to pass to $\overline{S}$. We need to substitute with

$\mathcal{L}^i(\mathcal{V})_{(2)}$ = sheaf on $\overline{S}$ of loc. L^2-integrable $\mathcal{V}$-valued i forms ϕ, $D\phi \in L^2$

ZUCKER: Is a fine sheaf.

Thm: $\Omega^\cdot(\mathcal{V}) \xrightarrow[q.\,iso.]{\sim} \mathcal{L}(\mathcal{V})_{(2)}$

- $H^\cdot(\overline{S}, Gr^p_{F^\cdot}\,\Omega(\mathcal{V})_{(2)}) \cong H^1(\Gamma(\overline{S}, gr^p_{F^\cdot}\,\mathcal{L}(\mathcal{V})_{(2)})$

Have $Gr^p_F\,\mathcal{L}(\mathcal{V}) = \bigoplus_q \left[\mathcal{L}^{0,q}(\mathcal{G}^p)_{(2)} \oplus \mathcal{L}^{1,q}(\mathcal{G}^{p+1})_{(2)}\right]$

Cor $H^1(\overline{S}, j_*\mathcal{H}) = H^1(\overline{S}, \Omega(\mathcal{V})_{(2)}) = H^1(\Gamma(\overline{S}, \mathcal{L}(\mathcal{V})_{(2)})$

Now: Harmonic Hodge theory $\longrightarrow H^1_{(2)}(S, \mathcal{V})$

Thm M complete Kähler mfld, $\mathcal{H}$: local system underlying a PVHS $(\mathcal{V},\nabla,\dots)$ of wt m. **Assume**: $\dim H^{i}_{(2)}(M,\mathcal{H})<\infty$

Then $H^{i}_{(2)}(M,\mathcal{H})=\bigoplus_{p+q=i+m} h^{p,q}_{(2)}(M)$ pure Hodge structure of wt $m+i$

$h^{p,q}_{(2)}(M):=\{\Delta_{D''}\text{-harmonic } \mathcal{V}\text{-valued } L^2\text{-form of type } (p,q)\}$

Sketch of proof Consider ext. of D: $A^{p}_{c}(S,\mathcal{V})\to A^{p+1}_{c}(S,\mathcal{V})$ to $L^{p}_{2}(S,\mathcal{V}):=$ H-space of measurable forms with finite.

and the same for D': $A^{p,q}_{c}\longrightarrow A^{p+1,q}_{c}$

D'': . . .

Lemma $\exists!$ ext. to a closed operator

δ formal adj. of D, $\delta=\delta'+\delta''$ have unique closed

Laplacian: $\Delta_D=D\delta+\delta D$, $\Delta_{D'}$, $\Delta_{D''}$ and $\Delta_D=\Delta_{D'}+\Delta_{D''}$

Kähler id. $\Delta_{D'}=\Delta_{D''}\rightsquigarrow\Delta_D=2\Delta_{D''}$

$\text{domain}(\Delta^p_D) = \{ \varphi \in L^p_2 \mid \varphi \in D(\delta) \cap D(D),\ \delta\phi(D),\ D\phi \in D(\delta) \}$

<u>Lem</u> $(\Delta^p_D, \text{domain})$ is a self-adjoint extension.

<u>Lem</u> M has a complete metric

$\rightsquigarrow \Delta^p_{D,D',D''}$ are ess. self-adjoint, i.e. closure are unique self-adjoint extensions.

$\rightsquigarrow$ domain(Δ^p_-) has $\pm D, D', D''$ coincide

Also domain for $\Delta' = \bigoplus_{p,q=v} \Delta^{p,q}$ open

$$\partial(\Delta^*) = \bigoplus \mathcal{D}(\Delta^{p,q}) \rightsquigarrow$$

$$\rightsquigarrow \text{for } h^p = \{ \phi \in \mathcal{D}(\Delta^*) \mid \Delta^p_D \phi = 0 \}$$
$$\overset{\shortparallel}{\bigoplus h^{p,q}}$$

Δ^p_D self adj. $\rightsquigarrow L^p_2 = h^p \oplus \overline{(\text{image } \Delta^p_D)}$

<u>Lemma</u> Δ^p_D has closed image $\Leftrightarrow$ D^p & D^{p+1} have closed images

$D^p : \mathcal{D}(D^p) \to \ker D^{p+1} \subset L^{p+1}_2$ (the first one is a Hilbert with the graph norm so it is bounded) coker = $H^p_{(2)}(M,\mathcal{H})$ by def! Banach's thm $\Rightarrow$ image of D^p is closed.

$\rightsquigarrow \phi \in L^p_2 \rightsquigarrow \phi = h + \Delta^p_D \eta = \underset{\substack{\uparrow \\ \text{harmonic}}}{h} + D\delta\eta + \delta D\eta$

If $D\phi = 0 = D\varphi D\eta$

$0 = \langle D\delta D\eta, D\eta \rangle = \langle \delta D\eta, \delta D\eta \rangle = \|\ \ \| \rightsquigarrow \delta D\eta = 0$

$\rightsquigarrow h^p_{(2)} = H^p_{(2)}(M,\mathcal{H})$. Regularity thm $\rightarrow h^p \subseteq A^p_{(2)}$

Can do also Hodge decomposition for $\Gamma(\bar{S}, Gr^p_F \mathcal{L}(V)_{(2)}) = \bigoplus_{P+Q=p} h^{P,Q}$

$\rightsquigarrow H^1(\bar{S}, F^{\cdot}\Omega(V)_{(2)}) \hookrightarrow H^1(\bar{S}, \Omega^{\cdot}(V)_{(2)})$

Hodge-de Rham SS. deg at E_1