Let $\mathbb{V} = (V_{\mathbb{R}}, \mathcal{V}, \nabla, F^{\cdot}, S)$ be a $\mathbb{R}$-PVHS of weight k with unipotent monodromy on $\Delta_t^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let $T \in \mathrm{End}(\mathcal{V}, \bar{\mathcal{V}})$ be the monodromy corresponding to the counterclockwise generator loop of Δ^* and set $N = \log(T) \in \mathrm{End}(\mathcal{V}, \bar{\mathcal{V}})$. **17/06**

Consider the connection $\nabla^c = \nabla - \frac{1}{2i\pi} N(z) \frac{dz}{z}$ on $\mathcal{V} \to \Delta^*$.

Its monodromy is trivial, hence we get a priviliged trivialization of $\mathcal{V} \to \Delta^*$ and a priviliged extensions $\mathcal{V}^c \to \Delta$ still endowed with a trivialization $\mathcal{V}^c \cong V \otimes_{\mathbb{C}} \mathcal{O}_{\Delta}$

<u>Prop</u> $\nabla^c N = 0$, $\nabla^c S = 0$

In other words N and S are constant in the trivialization associated to ∇^c

<u>Thm</u> (Schmid)

1) $\{\mathcal{F}^{\cdot}\}$ extends to $\mathcal{V}^c \to \Delta$ as a filtration by subbundle
2) Let $\{\mathcal{F}_{nil}\}$ be the unique filtration of $\mathcal{V}^c \to \Delta$ satisfying: $\begin{cases} \mathcal{F}_{nil}(0) = \mathcal{F}(0) \overset{\text{"}}{} \mathcal{F}_{lim} \\ \nabla^c \mathcal{F}_{nil} \subset \mathcal{F}_{nil} \end{cases}$

The data $\mathbb{V}_{nil} = (V_{\mathbb{R}}, \mathcal{V}, \nabla, \mathcal{F}_{nil}, S)$ defines a $\mathbb{R}$-PVHS of weight k with unipotent monodromy with the same Hodge numbers as $\mathbb{V}$ on Δ_{ε}^* for some $\varepsilon > 0$

Rmk 1) V_{nil} extends to $\mathbb{C}^*$

2) $\mathbb{V}_{nil}$ is completely determined by the following data $((V_{\mathbb{R}})_1, V, N, F_{lim}, S)$

$\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$, $\nabla = d + \frac{1}{2i\pi} N \frac{dz}{z}$

$F^{\cdot}_{nil} = F_{lim} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$, $S = S \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}$ $\qquad (V_{\mathbb{R}})_z = \exp\left(\frac{1}{2\pi i} \log(z) N\right) V_{\mathbb{R}}$

QUESTION When does such data $(V_{\mathbb{R}}, V, N, F^{\cdot}, S)$ define a $\mathbb{R}$-PVHS of weight k with nilpotent monodromy in a nbhd of 0 in $\mathbb{C}^*$?

When it is the case, the corresponding nilpotent orbit

Need
1) N is nilpotent
2) $N(F^{\cdot}) \subset F^{\cdot -1}$
3) $S(N\cdot, \cdot) + S(\cdot, N\cdot) = 0$
4) N and S are real
5) $S(F^p, F^{k-p+1}) = 0 \ \forall p$

1-5) are necessary but not sufficient.

Lemma (Deligne) V f.d. k-Vector space (k field) $N \in \mathrm{End}_k V$ nilpotent

$\exists!$ unique increasing exhaustive filtration $W_{\cdot}$ $(= W_{\cdot}(N))$ on V s.t.

1) $N(W_\ell) = W_{\ell-2}$ $\qquad$ 2) $N^\ell : Gr^W_\ell V \xrightarrow{\sim} Gr^W_{-\ell} V$

Moreover if $P_\ell(N) = \ker(N^{\ell+1} : Gr_\ell^W V \to Gr_{-\ell-2}^W V)$ then

$$Gr_\ell^W V = \bigoplus_{j\geq 0} N^j(P_{\ell+2j}(N))$$

Thm (Schmid) If $(k, V_{\mathbb{R}}, V, N, F; S)$ define a nilpotent orbit, then
weight

1) $N^{k+1} = 0$

2) $\forall z \in \mathbb{C}^*$, $(\exp(\frac{1}{2i\pi} N \log(z)) V_{\mathbb{R}}, V, F, W[-k])$ is a $\mathbb{R}$-MHS graded-polarized by (N, S) i.e.:
- N is real
- N is type $(-1,-1)$ $(N(F^\cdot) \subset F^{\cdot -1}, N(W_\cdot) \subseteq W_{\cdot -1})$
- the pure HS of weight $k+\ell$ induced by F on $P_\ell(N)$ is polarized by $S_\ell(\cdot,\cdot) = S(\cdot, N^\ell \cdot)$

Thm (Cattani-Kaplan-Schmid)

Let $(k, V_{\mathbb{R}}, V, N, F; S)$ as before. Assume that this data satisfy 2), ..., 5)

$+ N^{k+1} = 0$ $+$ $(W[-k], F^\cdot)$ is a $\mathbb{R}$-MHS on $V_{\mathbb{R}}$ graded polarized by (N, S)

Then it defines a nilpotent orbit.

Dependence on the coordinate X complex manifold of dim 1, $p \in X$

Let $\mathbb{V} \in \mathbb{R}$-PVHS on $X^* = X \setminus \{p\}$ with unipotent local monodromy around p

Let $z : X \to \mathbb{C}$ hol. function satisfying $z(p) = 0$ $(dz)_p : TX_p \cong \mathbb{C}$

$(V_{\mathbb{R}}, \mathcal{V}, \nabla, F; S)$

In general the construction depends on the choice of the coordinate (just on the first derivative)

but it is canonical on $(dz)_p(TX_p) \setminus \{0\}$)