Variations of Hodge structures on the punctured disc - Yohan   June 10

I Reminder on PVHS

Hodge structures: Let $R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$

Defn: A R-Hodge structure of wt k consists of:

① V a finitely generated k-module

② A bigrading $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$

②' A finite decreasing filtration $F^{\cdot}$ of $V_{\mathbb{C}}$ satisfying
$V_{\mathbb{C}} = F^p \oplus \overline{F}^{k-p+1}$ $\forall p$.

A polarization of the R-Hodge structure $(V_R, F^{\cdot})$ of wt k is a $(-1)^k$-symmetric bilinear form $V_R \otimes_R V_R \xrightarrow{S} R$ satisfying

① $S_{\mathbb{C}}(V^{p,q}, V^{r,s}) = 0$ unless $p = s$ & $q = r$.

② $\forall\, 0 \neq v \in V^{p,q}$, $i^{p-q} S_{\mathbb{C}}(v, v) > 0$.

E.g.: $X^n/\mathbb{C}$ a smooth projective variety; $L \to X$ a (very) ample line bundle

$H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z}) \to \mathbb{Z}$ [cup product & $H^{2n} \simeq \mathbb{Z}$]
$(v, v) \mapsto v \wedge v$

Must restrict to $PH^n(X; \mathbb{Z}) := \ker\left(- \wedge c_1(L)\right) : H^n(X, \mathbb{Z}) \to H^{n+2}(X, \mathbb{Z})$.
(positivity property fails otherwise)

Defn: Let (V, F, S) be a R-P-Hodge structure.

1) Weil operator: linear operator on $V_{\mathbb{C}}$ such that $C|_{V^{p,q}} =$ multiplication by i^{p-q}. [it is a real operator]

2) $(X, Y) \mapsto S(CX, \overline{Y})$ defines a hermitian symmetric positive def form on $V_{\mathbb{C}}$ ["the Hodge norm"]

PVHS ($R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$)

Defn: A R- VHS of wt k over a complex manifold X is:

(i) $\mathcal{L}$ an R- local system

(ii) $\mathcal{F}^{\cdot}$: a finite decreasing filtration of $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_X$ by holomorphic subbundles such that:

(a) $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$

(b) $\forall\, x \in X$, $(\mathcal{L}_x, \mathcal{F}^{\cdot})$ is a R-HS of wt k.

A polarization is a $(-1)^k$- symmetric map that induces a polarization of each fibre.

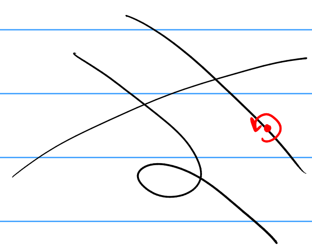
From the polarization, one gets a canonical hermitian $\mathcal{C}^\infty$ metric ("the Hodge metric"); $\mathcal{V}^\infty := C^\infty \otimes_{\mathcal{O}_X} \mathcal{V}$

II Degenerations of PVHS

X a complex manifold $D \subset X$ a normal crossing divisor.

$\mathbb{V} = R$-PVHS on $X - D$

Thm (Landman, Borel): The eigenvalue of the local monodromies around D have modulus 1



$$\mathbb{V} = (\mathcal{L}_R, \mathcal{V}, \nabla, \mathcal{F}^{\cdot}, s)$$

In particular, if $\mathcal{L}_{\mathbb{R}}$ has an integral structure [ie $\mathcal{L}_{\mathbb{R}} = \mathcal{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$] then the local monodromies are quasi-unipotent.

Nilpotent orbit I

(X, D) ; $\mathbb{V} = R$-PVHS on $X - D$ with quasi-unipotent local monodromies around D

$\mathbb{V} = (\mathcal{L}, \mathcal{V}, \nabla, \mathcal{F}, s)$

Let $\mathcal{V}^{Del}$ = Deligne canonical extension of $(\mathcal{V}, \nabla)$ to X.
(i.e. $\nabla : \mathcal{V}^{Del} \to \mathcal{V}^{Del} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$ + eigenvalues of the residues belong to
$[0,1)$. [quasi-unipotent $\Rightarrow$ real, rational residues].

Thm (Schmid): The Hodge filtration extends to X as a filtration of $\mathcal{V}^{Del}$ by sub-$\mathcal{O}_X$-modules. When local monodromies are unipotent, $\tilde{\mathcal{F}}^{\cdot}$ is a filtration of $\mathcal{V}^{Del}$ by sub-bundles (i.e. locally direct factors).

$$\tilde{\mathcal{F}}^p = (j_* \mathcal{F}^p) \cap \mathcal{V}^{Del}$$

$X = \Delta$ & $D = \{0\}$

$\mathbb{V}$ on Δ^*

$\mathcal{V}^{Del}$ on Δ

$\parallel$

$V \otimes_{\mathbb{C}} \mathcal{O}_\Delta$

<u>Nilpotent orbit II</u>

$X = \Delta$, $D = \{0\}$, z_1 = coordinate.

① Let $(\mathcal{V}, \nabla)$ on Δ^* with unipotent monodromy.
Let $T \in \mathrm{End}(\mathcal{V}, \nabla)$ be the monodromy.

$N := \log T$

<u>Claim</u>: $\mathcal{V} \to \Delta^*$ has a privileged trivialization.

Trivialization on $\mathcal{V}$ $\longleftrightarrow$ flat connection on $\mathcal{V}$ with trivial monodromy.

$\mathcal{V} = V \otimes \mathcal{O}_X$

<u>Define</u> $\nabla^c := \nabla - \frac{1}{2\pi i} N(z) \frac{dz}{z_1}$. It is a connection on $\mathcal{V} \to \Delta^*$.

Prop: ∇^c has trivial monodromy.

$$\{\text{flat multisection of } \nabla\} \longleftrightarrow \{\text{flat [multi]section of } \nabla^c\}$$

$$h(z) \longmapsto \exp\left(-\frac{1}{2\pi i}\log(z)\cdot N\right) h(z)$$

Cor: ∇^c determines a trivialization of $\mathcal{V}$: $\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^*}$, hence an extension $\mathcal{V}'$ on Δ.

Claim: $\nabla^c N = 0$ (clearly $\nabla N = 0$).

Prop: $\mathcal{V}'$ is the Deligne canonical extension of $(\mathcal{V}, \nabla) \to \Delta^*$.

Rmk: Let $\mathbb{V} = (\mathcal{L}_{\mathbb{R}}, \mathcal{V}, \nabla, \mathcal{F}^\cdot, S)$ be a $\mathbb{R}$-PVHS with integral structure on Δ^* with unipotent monodromy.

By Schmid, $\mathcal{F}^\cdot$ extends to Δ as a filtration by sub-bundles of $\mathcal{V}^{Del}$. Let $\mathcal{F}^\cdot_{nil}$ be the filtration of $\mathcal{V}^{Del}$ which satisfies

1) $\mathcal{F}^\cdot_{nil}(-) = \mathcal{F}^\cdot(0)$
2) $\nabla^c \mathcal{F}^p_{nil} = 0$, $\forall p$

Thm (Schmid): The data $(\mathcal{L}_{\mathbb{R}}, \mathcal{V}, \nabla, \mathcal{F}^\cdot_{nil}, S)$ define a $\mathbb{R}$-PVHS with integral structure in a nbhd of 0 in Δ^*.