BERNSTEIN-SATO POLYNOMIAL & V-FILTRATION

27/05
AZILATA

HISTORY Gelfand's question from '54

$f(x)$ polynomial

$$f_+(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & \text{o/w} \end{cases}$$

If $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$s \longmapsto \int_{\mathbb{R}^n} f_+^s(x)\varphi(x)\,dx$$

is analytic in s when $\mathrm{Re}(s) > 0$

Q: Does it extend to other values of s?

BS-POLYNOMIAL

For any polynomial f: Let s be a formal parameter.

For some P, diff. operator depending polynomially in s and some polynomial $b(s)$ in s

$$P \cdot f^{s+1} = b(s) f^s$$

Use this equation to continue

$$s \longmapsto \int f_+^s(x)\varphi(x)\,dx = \int \frac{P f_+^{s+1}}{b(s)}\varphi(x)\,dx$$

By integrating by parts: extends meromorph. (in s) to $Re(s) > -1$. By induction extends anywh.

Motivation #2 The existence of $b(s)$ gives us the existence of V-filtrations

Proof of existence of P, b

1) $X = \mathbb{A}^n$ ✓

2) More gen. case

Recall that D_X has an order filtration, but $D_X = D_{\mathbb{A}^n}$ has another one $\to$ Bernstein filt.

$$D_{\mathbb{A}^n} = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / \sim \qquad |x_i| = |\partial_i| = 1$$ Graded pieces are finite dim.

FACT If $X = \mathbb{A}^n$, then $M \in D_X$-mod is holonomic iff it is holonomic wrt. Bernstein filtration

$f \in \mathbb{C}[x]$. Consider $D_X(s) \cong D_X \otimes_{\mathbb{C}} \mathbb{C}(s)$

$M \subseteq \mathcal{O}_X(s)[f^{-1}] \cdot f^s$ generated by the symbol f^s.

If $g \in \mathcal{O}_X$ $g f^s = g f^s$. If $\xi \in T_X$, $\xi \cdot f^s = s \cdot f^{-1}(\xi f) f^s$. M is a $D_X(s)$-module.

Claim M is holonomic as a $D_X(s)$-module; D_X has a filtration.

Consequence of claim Holonomic module $\Longrightarrow$ finite length

So M has fin. length

$$Mf \supseteq Mf^2 \supseteq \dots \supseteq Mf^n \supseteq \dots$$

this stabilizes $\Rightarrow$ $f^{s+k} \in Mf^{k+1} = D_X(s)$-submod. gen. by f^{s+k+1}

Then there is some $L \in D_X(s)$ s.t. $L \cdot s^{k+s+1} = f^{s+k}$ $\Rightarrow$ clear denominators in s

$$P f^{s+k+1} = b'(s) f^{s+k} \xrightarrow{\text{shifting } s+k \to s}$$

$$P f^{s+1} = b(s) f^s$$

Claim M is holonomic, in fact $N := \mathcal{O}_X(s)[f^{-1}] f^s$ is also hol. as $D_X(s)$-modules

Consider a filtration on N: $F_i N = \{ P \cdot f^{s-i} \mid p \in \mathbb{C}(s)[X], \deg p \leq i(1+\deg f) \}$

Compare with Bernstein filtration $x_j \cdot p f^{s-i} = x_j p \cdot f \cdot f^{s-i-1}$

Count degrees: $\deg(x_j p f) \leq (i+1)(1+\deg f)$

Rmk Graded piece of $F_i N$ are fin. dim. In fact $Gr_i N$ has dim $=$ dim of polynomials of degree $i(1+\deg f)$

LEMMA If $\exists$ a filtration on N such that $\dim Gr_i N \leq \frac{i^{m-1} e}{(m-1)!} + O(i^{m-2})$, then N is holonomic

In our case we have $\binom{i(1+\deg f)+m-1}{m-1} = \frac{i^{m-1}(1+\deg f)^{m-1}}{(m-1)!} + O(i^{m-2})$

$\Rightarrow N$ is holonomic $\Rightarrow$ So is M since $M \subseteq N$

V-FILTRATION

t-coord.

$i : X \hookrightarrow X \times \mathbb{C}$

$x \mapsto (x, f(x))$

$\mathcal{O}_X$

$\downarrow$

$i_+ \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathbb{C}[\partial_t]$ as $\mathcal{O}_X$-mod.

As a $\mathcal{D}$-mod, action depends on the embedding

$i_+ \mathcal{O}_X = \mathcal{O}_{X \times \mathbb{C}}$-module generated by "$\delta_{t-f}$" i.e.

$(t-f)\delta_{t-f} = 0$; freely gen. as an $\mathcal{O}_X$-mod by $\delta_{t-f}, \partial_t \delta_{t-f}, \partial^2 \delta_{t-f}, \dots$

$N = i_+ \mathcal{O}_X$

$\cup|$

$M = \mathcal{D}_X[\partial t]\,\delta_{t-f}$ a $\mathcal{D}_X$-submod. δ_{t-f} $t \cdot \delta_{t-f} = f \cdot \delta_{t-f}$ $-\partial_t t \cdot \delta_{t-f}$

$\wr$ $\updownarrow$ $\updownarrow$ $\updownarrow$

$\mathcal{D}_X(s)$-mod. gen. by the symbol f^s f^s $f \cdot f^s$ $s \cdot f^s$

$b(s)$ can be thought as the minimal polynomial of s acting on the quotient module $\mathcal{D}_X[s] \cdot f^s \big/ \mathcal{D}_X[s] \cdot f^{s+1}$

$\forall m \in A$

$P(m f^{s+1}) = b(s)(m f^s)$

On the other side $b(s)$ is the min. pol. of $-\partial_t t$ on M/tM

Let A any hol. $\mathcal{D}$-mod. on sing X. We can construct $\mathcal{M} = A \otimes_{\mathcal{D}_X} \mathcal{D}_X(s) f$

It is enough to show that $\mathcal{M}$ is hol.

<u>Idea</u>: Find some $M' \subseteq M$ holonomic s.t. $M'|_U = M|_U$ where $U = X \setminus \{f=0\}$