V-filtration

06 May 2015

X complex mfld; $f : X \to \Delta$; smooth above $\Delta^* = \Delta \setminus \{0\}$

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{k} & X & \xleftarrow{i} & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H} & \to & \Delta & \xleftarrow{i} & \{0\} \end{array}$$

$K \in Perv(X, \mathbb{Q})$

$\psi_f(K) := i^{-1} R k_* k^{-1} K$

$P_{\psi_f}(K) = \psi_f(K)[-1] \in Perv(X_0, \mathbb{Q})$

$$\begin{array}{ccccc} i^{-1}K & \to & \psi_f K & \xrightarrow{can} & \phi_f K \\ \downarrow & & \downarrow{\scriptstyle T-1} & & \downarrow{\scriptstyle var} \\ 0 & \to & \psi_f K & \xrightarrow{id} & \psi_f K \end{array}$$

$\phi_f K = \bigoplus_{\lambda \in \mathbb{C}} \phi_{f,\lambda} K$; $\psi_f K = \bigoplus_{\lambda \in \mathbb{C}} \psi_{f,\lambda} K$ generalized eigenspaces

Var (as opposed to var): $\phi_{f,1} K \to \psi_{f,1} K(-1)$ [Tate twist]

such that $\mathrm{Var} \circ can = \frac{1}{2\pi i} \log T$ $\to$ makes sense for unipotent T;

$\log T = (T-1) - \frac{1}{2}(T-1)^2 \cdots$ [ok b/c $(T-1)^N = 0$ for $N \gg 0$]

$\phi_{f,\lambda} K \cong \psi_{f,\lambda} K$ for all $\lambda \neq 1$

I The KM filtration

$t : X \to \mathbb{C}$ smooth & holomorphic; ∂_t a vf with $[\partial_t, t] = 1$

Defn: The KM filtration on $\mathcal{M} \in \text{mod-}\mathcal{D}_X$ wrt $X_0 = t^{-1}(0)$ is:
an increasing exhaustive filtration

$V_\bullet M$; $\bullet \in \mathbb{Z}$, such that:

(1) $V_\bullet M$ coherent over $V_0 \mathcal{D}_X = \{P \in \mathcal{D}_X \mid P I_{X_0} \subset I_{X_0}\}$

$= \mathcal{O}_X \langle \mathcal{D}_{X_0}, Eu = t\partial_t \rangle$ [$\to$ locally]

(2) $V_k M \cdot t \subseteq V_{k-1} M \cdot t$

$V_k M \cdot \partial_t \subseteq V_{k+1} M$

(3) $\mathcal{V}_k M \cdot t = \mathcal{V}_{k-1} M$ if $k \ll 0$

(4) Each eigenvalue of $E = t\partial_t$ on $gr_k^{\mathcal{V}}(M)$ has $Re(\lambda) \in (k-1, k]$

(5) $gr_k^{\mathcal{V}} M$ decomposes into a sum of generalized eigenspaces wrt E.

<u>Thm</u>: If $\mathcal{M}$ is (regular) holonomic, then such a filtration exists & is unique; moreover $gr_k^{\mathcal{V}} \mathcal{M}$, $k \leq 0$ are (regular) holonomic on X_0

<u>Exercise 1</u>:

(a) $\cdot t : \mathcal{V}_k \mathcal{M} \longrightarrow \mathcal{V}_{k-1} \mathcal{M}$ is an iso for $k<0$. (surj. for $\overset{k\leq 0?}{\text{all (?)}}$ k)

(b) $\cdot \partial_t : gr_k^{\mathcal{V}} \mathcal{M} \longrightarrow gr_{k+1}^{\mathcal{V}} \mathcal{M}$ is an iso for all $k \neq -1$

(c) M is generated as a mod-$\mathcal{D}_X$ by $\mathcal{V}_0 M$.

<u>Prop</u>: $\mathcal{V}_k \mathcal{M} = \mathcal{V}_0 \mathcal{M} \cdot t^{-k}$ $\forall$ k

Here if $k > 0$, then $\mathcal{V}_0 \mathcal{M} \cdot t^{-k} = \{ m \in \mathcal{M} \mid m t^k \in \mathcal{V}_0 \mathcal{M} \}$

<u>Pf</u>: If $k < 0$ $\rightsquigarrow$ Exercise (a) + (proof thereof) induction

If $k = 0$ $\rightsquigarrow$ ok

if $k > 0$ $\rightsquigarrow$ Induction

$$\mathcal{V}_k M = \mathcal{V}_{k-1} M \partial_t + \mathcal{V}_{k-1} M \quad \text{by Ex. (b)}$$

$$= (\mathcal{V}_0 M) \underbrace{t^{-k} \partial_t}_{} + (\mathcal{V}_0 M) t^{-k+1} \subset \mathcal{V}_0 \mathcal{M} t^{-k}$$

(use that $\partial t^k = k t^{k-1} + t^k \partial$)

<u>Conversely</u>: Let $m \in \mathcal{V}_0 \mathcal{M} t^{-k}$; then $m t^k \in \mathcal{V}_0 M$.

$\Rightarrow m t^k \partial_t^k \in \mathcal{V}_k \mathcal{M}$

Use $t^k \partial_t^k = (E+k) \cdots (E+1)$

So, get $(E+k)(E+k-1) \cdots (E+1) \in \mathcal{V}_k \mathcal{M}$

$\exists\ k_0 \in \mathbb{Z}$ st $m \in \mathcal{V}_{k_0}\mathcal{M}$; choose it minimal.

Suppose $k_0 > k$; then $\bar{m} \neq 0$ in $gr^{\mathcal{V}}_{k_0}$

But $\bar{m}(E+k)\cdots(E+1) = 0 \longrightarrow$ contradiction!

$\uparrow$ in $gr^{\mathcal{V}}_{k_0}\mathcal{M}$

Example 1 : $\mathcal{M} = \omega_X \in$ mod-$\mathcal{D}_X$

Then $\mathcal{V}_0\omega_X = \omega_X$; $\mathcal{V}_{-k}\omega_X = \omega_X t^k \ \forall\ k \geq 0$.

$$\rightsquigarrow gr^{\mathcal{V}}_{-1}\omega_X = \omega_X t \big/ \omega_X t^2 \simeq \omega_X \otimes \underbrace{\mathfrak{m}/\mathfrak{m}^2}_{\mathcal{O}_{X_0}|_{X_0}} = \omega_{X_0}.$$

Example 2 : $\operatorname{supp}\mathcal{M} \subset X_0$; $\mathcal{V}_{-1}\mathcal{M} = 0$

$$\mathcal{V}_k\mathcal{M} = \bigoplus_{r=0}^{k}\mathcal{M}_r \ ; \ \mathcal{M}_r = \{m \in \mathcal{M} \mid mE = rm\}$$

Prop : The KM filtration is unique if it exists

Pf : Suppose $\mathcal{V}_\bullet$ & $\mathcal{W}_\bullet$ satisfy (1) – (5)

$\mathcal{V}_\bullet \cap \mathcal{W}_\bullet$ satisfies (1) to (5) $\Rightarrow$ WLOG $\mathcal{V}_\bullet \subseteq \mathcal{W}_\bullet$

(1) $\Rightarrow$ $\mathcal{W}_0\mathcal{M} \subseteq \mathcal{V}_N\mathcal{M}$ $\quad N \geq 0$.

Prop $\Rightarrow$ $\mathcal{W}_k\mathcal{M} \subset \mathcal{V}_{k+N}\mathcal{M} \ \forall\ k$.

Choose N minimal $\Rightarrow \exists\ m \in \mathcal{W}_k$ with $m \notin \mathcal{V}_{k+N-1}\mathcal{M}$

Suppose $N \geq 1$

(5) $\Rightarrow$ WLOG $mE = \lambda m + m'$; $m' \in \mathcal{W}_{k-1}\mathcal{M} \subset \mathcal{V}_{k-1+N}$; $\operatorname{Re}\lambda \in (k-1, k]$

Contradiction if we go to $gr^{\mathcal{V}}_{k+N}\mathcal{M}$: $\underset{\neq 0}{\bar{m}}E = \lambda\bar{m} \Rightarrow \operatorname{Re}\lambda \in (k+N-1, k+N]$

General case :

$f : X \to \mathbb{C}$ non-constant holomorphic. Consider graph embedding $(id, f) : X \hookrightarrow X \times \mathbb{C}$; $\mathcal{M}_f := (id \times f)_*\mathcal{M} = \mathcal{M}[\partial_t]$ w/ different action of ∂_t

Let $t =$ coord on $\mathbb{C}$ above ; $\partial_t = \frac{d}{dt} \Rightarrow [\partial_t, t] = 1$.

So KM filtration for $\mathcal{M}_f$ exists if $\mathcal{M}$ is regular holonomic.

Also, $gr_k^{\nu} \mathcal{M}$ lives on $X \times \{0\} \cap \operatorname{supp} \mathcal{M}_f = f^{-1}(0)$ on $X \simeq X \times \{0\}$.

* Assume $\mathcal{M}$ reg. holonomic
For $\alpha \in \mathbb{C}$, set $\mathcal{M}_{f\alpha} = \ker(E - \alpha \cdot \mathrm{Id})^m \subset gr_k^{\nu} \mathcal{M}$; $k = \lceil \alpha \rceil$; $m \gg 0$.

<u>Facts</u> : Put $\lambda = e^{2\pi i \alpha}$, then

$DR(\mathcal{M}_{f\alpha}) \simeq {}^p\psi_{f,\lambda}(DR(\mathcal{M}))$ for $-1 \leq Re(\alpha) < 0$.

$DR(\mathcal{M}_{f,\alpha}) \simeq {}^p\phi_{f,\lambda}(DR(\mathcal{M}))$ for $-1 < Re(\alpha) \leq 0$.

+ action of T.

T corresponds to $e^{2\pi i E}$

can : $\psi_{f,1} K \to \phi_{f,1} K$ & Var : $\phi_{f,1} K \to \psi_{f,1} K(-1)$.

Correspond to : $\partial_t : \mathcal{M}_{f,-1} \to \mathcal{M}_{f,0}$ & $t : \mathcal{M}_{f,0} \to \mathcal{M}_{f,-1}$.
Var $\circ$ can $\simeq t\partial_t = E = \frac{1}{2\pi i} \log T$.

<u>The V-filtration</u>
Motivation : $M = (\underbrace{\mathcal{M}, F.\mathcal{M}}, K)$ where $K \in Perv(X, \mathbb{Q})$
filtered regular holonomic right $\mathcal{D}$-module
[corresponds to Hodge filtration].

[Have $DR(M) \simeq K \otimes_{\mathbb{Q}} \mathbb{C}$]

(1) The monodromy T of VHS on punctured disc is always quasi-unipotent ; ie $\lambda = \sqrt{1}$, ie $\alpha \in \mathbb{Q}$

(2) The Hodge filtration on the nearby cycles should be the

limit mixed Hodge structure; it should be compatible with the generalized eigenspace decomp of ψ_f.

<u>Defn</u>: (1) The rational V-filtration on $\mathcal{M}_f$ is an increasing filtration $V_\bullet \mathcal{M}_f$ indexed by $\mathbb{Q}$, s.t: for $\alpha \in \mathbb{Q}$; $\lceil \alpha \rceil = k$, then

- $V_\alpha \mathcal{M} \subseteq V_k \mathcal{M}_f$ [$\leftarrow$ KM filtration] is the preimage of $\bigoplus_{k-1<\beta\leq\alpha} \mathcal{M}_{f,\beta} \subseteq gr_k^V$

- If $\mathcal{M}$ is filtered $F_\bullet \mathcal{M}$, then we define

$$F_p\, gr_\alpha^V \mathcal{M}_f := (F_p \mathcal{M}_f \cap V_\alpha \mathcal{M}_f) \big/ (F_p \mathcal{M}_f \cap V_{<\alpha} \mathcal{M}_f).$$

where $F_p \mathcal{M}_f = \bigoplus_{i=0}^{\infty} F_{p-1} \mathcal{M} \partial_t^i$ ($\mathcal{M}_f = \mathcal{M}[\partial_t]$ as above)