Riemann-Hilbert correspondence - Williamson

22 Apr 2015

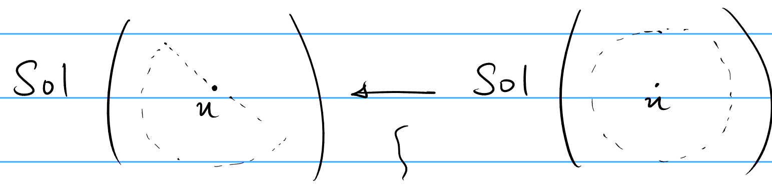
$P = a_m \partial^m + a_{m-1}\partial^{m-1} + \cdots + a_0$ diff op

"Solution complex" Sol : $\mathcal{O}_{an} \xrightarrow{P\cdot} \mathcal{O}_{an}$

<u>Cauchy</u>: If $a_m(u) \neq 0$, $\exists$ nbhd of u st Sol is concentrated in degree 0 & is a local system. [existence & uniqueness of soln]

But Sol will "jump" at zeroes of a_m [a_m not identically 0]

$a_m(u) \neq 0$



[in this case, SS is $a_m \xi^m$]

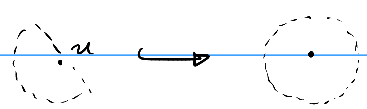
iso iff

$a_m(u) \neq 0 \quad \Longleftrightarrow \quad T^*_u \not\subset SS(P)$

<u>Rmk</u>: This holds for holonomic modules in any dimension.

X arbitrary; $\xi \in T^*_u X$ and $f : X \to \mathbb{C}$ with $df = \xi$, $f(u) = 0$

$f^{-1}(B(0,\varepsilon) \cap (\mathrm{Re} < 0)) \subset f^{-1}(B(0,\varepsilon))$



$D = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ & $\mathcal{O}$ = germs of hol. functions at 0

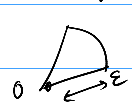
$\dot{D} = D \setminus \{0\}$ local coordinate z,

$K = \mathcal{O}[\frac{1}{z}]$ = germs of meromorphic fns at 0

$\tilde{\mathcal{O}}$ = "multi-valued functions on $\dot{D}$"

= regular functions on $\tilde{\dot{D}}$ = universal cover.

$f \in \tilde{\mathcal{O}}$ has <u>moderate growth</u> if: for any branch on any segment

$z \in$  $|f(z)| < C \cdot \frac{1}{|z^n|}$ for some z, n.

Fact: If f holomorphic on $\dot{D}$, moderate growth $\Leftrightarrow$ f meromorphic at 0.

E.g. $(z^\alpha,\ \alpha \in \mathbb{C})$, $(\log z)^m$, $m \geq 0$

But $\exp\left(\frac{1}{z^m}\right)$ does not.

Thm of Fuchs: If $P = a_m \partial^m + \cdots + a_0$; $a_i \in K$ (meromorphic)

All solutions of $P\tilde{u} = 0$ for $\tilde{u} \in \tilde{\mathcal{O}}$ have moderate growth iff $\frac{a_i}{a_m}$ has poles of order $\leq (m-i)$ $\forall i$

E.g. $\left(\partial - \frac{\lambda}{z}\right)$ has solns z^λ ; but $\left(\partial - \frac{1}{z^m}\right)$ for $m \geq 2$ has solns

$-(m-1)^{-1} \exp\left(\frac{1}{z^{m-1}}\right)$; not moderate growth.

* Multiply by $\frac{z^m}{a_m}$: $P' = z^m \partial^m + a'_{m-1}\partial^{m-1} + \cdots + a'_0$

Condition becomes : a'_i has zero of at least order i

ie : $P'' = \sum g_i E^i$, where E is the Euler operator $z\partial$

$\underset{\sim}{g_i}$ ↳ holomorphic.

Standard trick : Rewrite eqn as : $\frac{d}{dz} f = Af$, where $f = (f_1, \ldots, f_n)$

$A \in M_n(K)$

Regular $\Leftrightarrow$ equivalent to a system $\frac{d}{dz} h = \frac{B}{z} h$, where B hol.

(↳ meromorphic gauge transf)

(ie logarithmic poles)

$\Leftrightarrow$ all solutions have moderate growth.

Defn: A meromorphic connection is a fin. diml K-vector space M, together with a $\mathbb{C}$-linear map $\nabla : M \to M \cdot dz$

s.t. $\nabla(hm) = dh \cdot m + h \cdot \nabla(m)$ $\forall h \in \mathcal{O}, m \in M$.

Ex: Meromorphic connection $\Leftrightarrow$ $\mathcal{D}$-module on D s.t. mult. by z_1 is an isomorphism.

A meromorphic connection has regular singularities if $\exists$ $(E = z_1 d)$-stable $\mathcal{O}$-lattice $L \subset M$. If L is such a lattice, can choose a basis of L; the E-stable condition says: matrix of ∇ in this basis has poles of order ≤ 1.

Structure thm for R.S. meromorphic conns

Let ∇ be a mer. connection. Choose an E-stable lattice $L \subset M$ s.t. ∇ has connection matrix $\frac{A(z)}{z_1} dz_1$; $A(z_1)$ hol.

Then ∇ is equivalent to a mer. connection with connection matrix

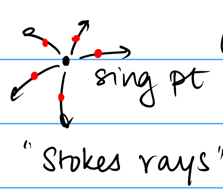
(*) $\frac{A(0)}{z_1} dz_1$; $A(0)$ = "residue" of r.s. meromorphic connection.

A basis of solns of * is given by $\exp(A(0) \log z_1)$
& monodromy of * is given by $\exp(2\pi i A(0))$

[Γ, Γ' constant mat.]

Two regular singular connections of the form $\frac{\Gamma}{z_1} dz$, $\frac{\Gamma'}{z} dz$ are equivalent iff $\exp(2\pi i \Gamma)$, $\exp(2\pi i \Gamma')$ are conjugate

Moral: Structure of rs meromorphic connections is "linear alg".
Outside of RS, things are complicated
Boalch "wild character varieties"



(monodromy around each pt)

Riemann-Hilbert for local systems.

- X analytic variety

RH: $\{\text{integrable}_{\wedge}^{\text{hol.}}\text{ conn. on } X\} \xleftrightarrow{\sim}$ loc sys. of $\mathbb{C}$-vector spaces

$$(\mathcal{F}, \nabla) \longmapsto \mathcal{F}^{\nabla}$$

By GAGA, also holds for X projective in alg. world.

Fails for X non-projective

E.g. on $\mathbb{C}$, consider $\mathcal{D}/(\partial-1)$; solns $\mathbb{C}\exp(z)$

$\mathcal{D}/\partial$; solns $\mathbb{C}$

They both give rise to the trivial local system; and are analytically isomorphic but not algebraically isom.

So in general: Many alg conn. give rise to the same analytic connection.

* In above example: At ∞, the first mod looks like $\partial \pm \frac{1}{z^2}$; not RS., or $\exp(z)$ essential singularity at ∞.

But $\mathcal{D}/\partial$ does have RS. at ∞.

For algebraic connections on X, $\exists$ natural meromorphic extension to any normal crossings compactification $\overline{X}$ of X, and the notion of RS is well-defined

Thm (Deligne): Any (holomorphic, integrable) conn. on X has a unique ext. to a meromorphic conn. with rs on any n.c compactification $\overline{X}$.

Thm: $\exists$ equivalence $\left\{\begin{array}{c}\text{algebraic connections} \\ \text{with RS}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{local systems} \\ \text{on } X\end{array}\right\}$

Goal: Explain why RH holds for curves.
X smooth curve ; $\mathcal{D}$-modules with holomorphic RS

$$\mathcal{D}\text{-mod}^{rs}_{hol} \xrightarrow[\sim]{DR} \mathrm{Perv}(X) \subset D^b_c(X, \mathbb{C})$$

Warning: Discussion is not along the lines of the general pf.

Defn: X is a curve ; $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{hol}$ has regular sing if:
at each singular pt z_0 of M:
1) Either: the corresp. meromorphic conn $K \otimes_{\mathcal{O}} \mathcal{M}$ has RS, or
2) Equivalently: in a nbhd of 0 we can choose an $E = z_0\partial$-stable good filtration in $\mathcal{M}$.

(Claim (1) $\Leftrightarrow$ (2).)

Because both LHS & RHS are stacks of abelian categories, enough to check locally.

Let $\mathcal{C} = \mathcal{D}_D\text{-mod}$ & $\mathcal{C}_0 =$ those $\mathcal{D}$-mods with only singularities at 0.
ie, $T^*_{z_1} \not\subset SS(\mathcal{M})$ if $z_1 \neq 0$

$\mathcal{C}_0^{rs} = \mathcal{M}$ in $\mathcal{C}_0$ with regular singularities

Let $\mathcal{O}_{alg} = \mathbb{C}[z_1] \subset \mathcal{O}$
Lemma: If $\mathcal{M} \in \mathcal{C}_0^{rs}$, we can find $\mathcal{M}_{alg} \subset \mathcal{M}$ such that
$\mathcal{O} \otimes_{\mathcal{O}_{alg}} \mathcal{M}_{alg} \cong \mathcal{M}$.

Idea: $\exists$ d.Δ: $i_* i^! \mathcal{M} \to \mathcal{M} \to j_* \mathcal{M}|_{\dot{D}} \xrightarrow{+1}$; $j_* \mathcal{M}|_{\dot{D}} = K \otimes_{\mathcal{O}} \mathcal{M}$

corresp. to (under DR)
$i_* i^! \to id \to j_* j^* \xrightarrow{+1}$
$i: \{0\} \hookrightarrow D \hookleftarrow \dot{D} : j$

Get : $0 \to H^0(i_+ i^! \mathcal{M}) \to \mathcal{M} \to K \underset{\mathcal{O}}{\otimes} \mathcal{M} \to H^1(i_+ i^! \mathcal{M}) \to 0$

$$\begin{array}{ccc} \| & \| & \| \\ \oplus\,\delta & \text{mer conn} & \oplus\,\delta \\ \text{(alg)} & \text{w/ rs (alg)} & \text{(alg)} \end{array}$$

get $\mathcal{M}_{alg}$ (inverse image of alg lattice)

<u>Lemma</u> : $\mathcal{M}_{alg} = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{M}_\lambda$; $\mathcal{M}_\lambda$ = gen. eigenspace for $E = z\partial$ Euler operator.

$[E, z] = z$,
$[E, \partial] = -\partial \Rightarrow z : M_\lambda \longrightarrow M_{\lambda+1}$
$\partial : M_\lambda \longrightarrow M_{\lambda-1}$

$z : M_\lambda \longrightarrow M_{\lambda+1}$ is an iso if $\lambda \neq -1$
$\partial : M_\lambda \longrightarrow M_{\lambda-1}$ is an iso if $\lambda \neq 0$

$$\begin{array}{c} \vdots \\ \mathcal{M}_{i+1} \\ \downarrow\uparrow \\ \mathcal{M}_i \\ \downarrow\uparrow \\ \vdots \end{array} \qquad \begin{array}{c} \vdots \\ \mathcal{M}_1 \\ \downarrow\uparrow \\ \mathcal{M}_0 \\ \downarrow\uparrow \\ \mathcal{M}_{-1} \\ \downarrow\uparrow \\ \vdots \end{array}$$

$\Psi = \bigoplus_{Re\,\lambda \in [0,1)} M_\lambda$ & $\phi = \bigoplus_{Re\,\lambda \in [-1,0)} M_\lambda$

<u>Claim</u> : $\mathcal{M} \longmapsto \left\{ \begin{array}{c} \Psi(\mathcal{M}) \\ \partial \downarrow \quad \uparrow z \\ \phi(\mathcal{M}) \end{array} \right\}$ is an equiv. of categories

where $A = \left\{ \begin{array}{c} V \\ \downarrow\uparrow \\ W \end{array} \right\}$

& Ψ, ϕ correspond to nearby & vanishing cycles resp. under DR.