Holonomic D-modules

- Last time: filtration on D_X (locally)

  $F_m\mathcal{D}_X = \{P \in D_X \mid P \text{ has order } \alpha\}$ ⤳ can be globalized.

- $F_m(\mathcal{D}_X) = 0$ for $m<0$
- $F_m(\mathcal{D}_X) = \{P \in \mathcal{D}_X \mid [P, \mathcal{O}_X] \subset F_{m-1}(\mathcal{D}_X)\}$
- $F_0(\mathcal{D}_X) = \mathcal{O}_X$ & $F_1(\mathcal{D}_X) = T_X$. (tgt sheaf)

- $F_{m_1}(\mathcal{D}_X) \cdot F_{m_2}(\mathcal{D}_X) \subset F_{m_1+m_2}(\mathcal{D}_X)$
- $[F_{m_1}(\mathcal{D}_X), F_{m_2}(\mathcal{D}_X)] \subset F_{m_1+m_2-1}(\mathcal{D}_X)$

Form $Gr^F(\mathcal{D}_X)$; commutative ring.

$Gr_0^F(\mathcal{D}_X) = \mathcal{O}_X$ & $Gr_1^F(\mathcal{D}_X) = T_X$, so $T_X \longrightarrow Gr^F(\mathcal{D}_X)$: $\mathcal{O}_X$-linear.

So, $S^{\cdot}_{\mathcal{O}_X}(T_X) \longrightarrow Gr^F(\mathcal{D}_X)$

Locally, is $\mathcal{O}_X \otimes \mathbb{C}[\xi_1, \dots, \xi_n]$; $\xi_i \longmapsto \partial_i$

And $Gr^F(\mathcal{D}_X) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geq 0}} \mathcal{O}_X \partial_X^\alpha$

Principal symbol map: $\sigma_m : F_m(\mathcal{D}_X) \longrightarrow Gr^F_m(\mathcal{D}_X) \subseteq Gr^F(\mathcal{D}_X) \cong S_{\mathcal{O}_X} T_X$

$$\sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \longmapsto \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

$\pi : T^*X \longrightarrow X$; $S^{\cdot}_{\mathcal{O}_X} T_X = \pi_* \mathcal{O}_{T^*X}$; sections of this $\Leftrightarrow$ regular functions on cotangent bundle.

& $\{-,-\}$ Poisson bracket on T^*X from canonical symplectic structure on T^*X

Locally: $P \in F_m$, $Q \in F_n$; $\sigma([P,Q]) = \{\sigma(P), \sigma(Q)\}$

Start with non-comm $\mathcal{D}_X \to$ commutative, but $\{-,-\}$ "remembers" some of the non-commutativity.

Characteristic variety:

$\mathcal{M}$ coherent $\mathcal{D}$-module, M is locally gen. by $u_1, \dots, u_N$

$F_m(\mathcal{M}) = \sum_{\nu=1}^{N} F_m(\mathcal{D}_X).u_\nu$ filtration

Defn: $\mathcal{M} \in \mathcal{D}_X$-mod is called filtered if $\exists$ subsheaves $\{F_m(\mathcal{M})\}$ s.t.:

- $F_i \subseteq F_{i+1}$, - $F_i = 0$ for $i \ll 0$, $\mathcal{M} = \bigcup F_i$, $(F_i\mathcal{D}_X)(F_j\mathcal{M}) \subseteq F_{i+j}\mathcal{M}$.

This is "good" if $F_i\mathcal{M}$ coherent over $\mathcal{O}_X$, and $(F_j\mathcal{D}_X)(F_i\mathcal{M}) = F_{i+j}\mathcal{M}$ for $i \gg 0$.

[Fact: M coherent over $\mathcal{D}_X \Leftrightarrow \exists$ good filtration]

Let F be a coherent $\mathcal{D}_X$-module $\mathcal{M}$. Choose a good filtration Support of $\mathrm{Gr}^F\mathcal{M}$ regarded as coherent $\mathrm{Gr}\,\mathcal{D}_X$-module is closed subvariety of T^*X, called char. variety $\mathrm{Ch}(M)$

- $\mathrm{Ch}(M)$ indep. of choice of good filtration (?) - though the sheaf depends on it.

E.g. $X = \mathbb{C} = \mathbb{A}^1$; $\lambda \in \mathbb{C}$; consider $z \cdot \frac{df}{dz} = \lambda f$

$\mathcal{M}_\lambda = \mathcal{D}/\mathcal{D}(z\partial - \lambda)$; $Eu = z\partial$; $\mathcal{D} = \langle z^p E^q, \partial^p E^q \mid p, q \in \mathbb{Z} \rangle_{\mathbb{C}}$

$\mathcal{D}(z\partial - \lambda) = \langle z^p \partial^q (E-\lambda), \partial^p E^q (E-\lambda) \rangle_{\mathbb{C}}$

$\mathcal{M}_\lambda = \langle z^p, \partial^p \rangle$ & $[E, z] = z\partial z - z^2\partial = z$

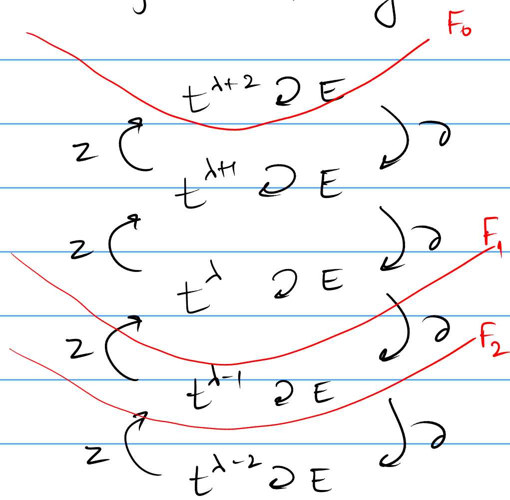
& $[E,\partial] = z\partial^2 - \partial z \partial = -\partial$.

So: $Ez = z(E+1)$ & $E\partial = \partial(E-1)$

& z^n is an eigenvector of E with eigenvalue $\lambda+n$.

∂^n " " " " " " " $\lambda - n$.

Let $t^\nu :=$ ev of E w/ eigenvalue ν



If $\lambda \notin \mathbb{Z}$, E is an iso.
$\Rightarrow M_\lambda$ is irreducible.

} beyond a certain point we can only "go down by 1"

Or: $F_m = 0$ for $m < 0$

$F_0 = \langle t^{\lambda+p} \mid p \geq 0 \rangle$

$F_k = \langle t^{\lambda+p} \mid p \geq -k \rangle$

$Gr_0 M_\lambda = \langle t^{\lambda+p} \mid p \geq 0 \rangle$

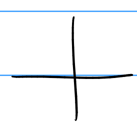
$Gr_p M_\lambda = \langle z^{\lambda-p} \rangle \bmod \langle z^{p+q} \mid q > -p \rangle \quad p > 0$

$\mathrm{Ann}(Gr\, M_\lambda) \subset Gr\, \mathcal{D}_X$?

- z annihilates all $Gr_p M_\lambda$ for $p \neq 0$

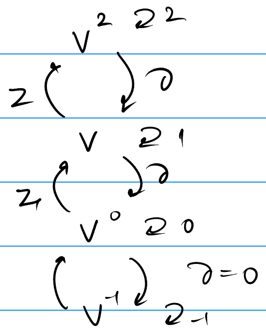
∂ annihilates almost all of $Gr_0 M_\lambda$

$z \cdot \xi$ annihilates everything :

$Ch(M_\lambda) = \{ z\xi = 0 \} =$  $\subseteq \mathbb{C}^2$

Assume $\lambda = -n \in \mathbb{Z}_+$; choose a basis of eigenvectors v^m

$z v^m = v^{m+1}$



$\partial v^m = \partial z v^{m-1} = [\partial, z]\, v^{m-1} + E v^{m-1}$

$= m v^{m-1}$

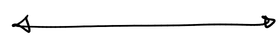
$$0 \longrightarrow \mathbb{C}[x] \longrightarrow \mathbb{C}[x, x^{-1}] \longrightarrow \mathbb{C}[x,x^{-1}]/\mathbb{C}[x] \longrightarrow 0$$

$$\parallel$$

$$\mathcal{D}.\delta$$

$Ch(\mathbb{C}[x])$ good filtration $F_0 = \mathbb{C}[x]$ & $F_k = \mathbb{C}[x]\ \forall\ k \geq 0$

$Gr(\mathbb{C}[x]) =$ concentrated in degree zero & has annihilator $\langle \xi \rangle \Rightarrow Ch =$

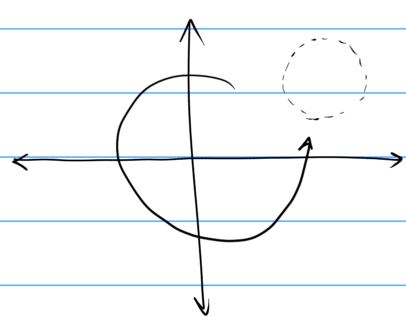


$M_\mu \cong M_\lambda$ iff $\lambda - \mu \in \mathbb{Z}$

$Ch(\mathcal{D}.\delta) =$

Rmk : Char. variety of M, a coherent $\mathcal{D}_X$-mod = zero section of T^*X iff M already coherent. (= vector bundle + flat conn.)

$x \cdot \frac{\partial f}{\partial x} = \lambda \cdot f(x) \rightsquigarrow f(x) = e^{\lambda \cdot (\log x)}$



get $e^{\lambda \cdot (\log x + 2\pi i)}$

& $e^{\lambda \cdot 2\pi i} = 1$ iff $\lambda \in \mathbb{Z}$

simple $\mathcal{D}$-module $\rightsquigarrow$ non-trivial monodromy.

Thm (Bernstein inequality) : Let $F \neq 0$ be a coherent $\mathcal{D}_X$-module
Then $\dim ch(F) \geq \dim X$.
$Ch(\mathcal{D}_X) = T^*X$.

Define : A coherent $\mathcal{D}_X$-module $M \neq 0$ is called holonomic if
the dimension of $ch(M)$ = dimension of X.

Rmk: If $Ch(M)$ "small", $\rightsquigarrow$ we have many principal symbols
$\rightsquigarrow$ many partial diff. operators $\rightsquigarrow$ less solutions of $Pu = 0$.

Regular holonomic $\mathcal{D}$-modules :
Say $X \subseteq \mathbb{C}$ open; $P(x, \partial)$ ordinary diff op of order m.
$P = \sum_{k=0}^{m} a_k(x) \partial_x^k$ | $\mathcal{M} = \mathcal{D}/\mathcal{D}P$

If $a_m(x_0) \neq 0$, then locally $\mathcal{M} = \mathcal{O}_X^m$
ie m linearly indep. solutions of $Pu = 0$.
But if $a_m(x_0) = 0$, then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ loc. const sheaf on $U \setminus \{x_0\}$ (ie locally around x_0)
but monodromy may cause problems.

Thm : The following are equivalent : [say $a_m(x_0) = 0$]
1) Let $r = \mathrm{ord}_{x=x_0}(a_m(x))$; then $\mathrm{ord}_{x=x_0}(a_j(x)) \geq r - (m-j)$

2) $Pu = 0$ has m linearly indep solutions of the form
$(x-x_0)^\lambda \sum_{j=0}^{s} u_j(x) (\log(x-x_0))^j$; $s \in \mathbb{Z}_{\geq 0}$; $\lambda \in \mathbb{C}$ & $u_j(x)$ holomorphic

Defn : If x_0 fulfils one of the statements, then x_0 is called
a regular singularity

Say $x_0 = 0$, regular singularity of P.
$\to$ replace P by $x^e P$ or $\partial^e P$ $\rightsquigarrow$ to assume $r = m$.

$$P = \sum_{j=0}^{m} c_j(x)\, x^j \partial^j \ ; \quad c_m(0) \neq 0 \quad ; \text{may assume } \ c_m(0) = 1.$$

$$P = (x\partial)^m + \sum_{j=0}^{m-1} c_j(x)(x\partial)^j \ ; \ \text{but } x^j \partial^j = (x\partial)(x\partial - 1) \cdots (x\partial - j + 1)$$

<u>Define</u> F.M. $\quad \mathcal{M} = \mathcal{D}/_{\mathcal{D}\cdot P} = \mathcal{D}u \ ; \ u = 1 \ (\mathrm{mod}\ \mathcal{D}P)$

$$F_k \mathcal{M} = \sum_{j=0}^{m-1} F_k(\mathcal{D})\,(x\partial)^j u$$

$(x\xi) \cdot F_k \mathcal{M} \subseteq F_k \mathcal{M}$ ie $(x\xi)\cdot(\mathrm{Gr}\,\mathcal{M}) = 0$

In 1d situation: we always have $(x\xi)^N \cdot \mathrm{Gr}\, M = 0 \quad N \gg 0$
But condition $\Leftrightarrow N = 1$

<u>Defn</u>: $\mathcal{M}$ (holonomic module) is called regular holonomic if it admits a good filtration s.t. $\mathcal{I} \cdot \mathrm{Gr}\, M = 0$ where $\mathcal{I}$ = (radical) ideal of char. variety.

E.g. $\mathrm{Sol}(z^2\partial - 1)$: $e^{1/z}$ not a regular singularity

$$D^b_{rh}(\mathcal{D}_X) \xrightarrow{\ \sim\ } D^b_c(\mathbb{C}_{X^{an}})$$

$$\mathcal{M} \longmapsto R\mathcal{H}om_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \mathcal{M}^{an})$$

(can also have $D^b_h(\mathcal{D}_X)$)