(PURE) HODGE STRUCTURES

- **Def** R noeth subring of $\mathbb{R}$

  A pHS of weight k is
  - A V_R fin. gen. R-mod
  - A decreasing filtration F (Hodge filt.)

  (*) s.t. $F^p \cap \overline{F}^q = 0$ and $F^p \oplus \overline{F}^q = V_\mathbb{C}$ $\forall p+q=k+1$

  (this implies $V_\mathbb{C} = \bigoplus_{p+q=k} (F^p \cap \overline{F}^q)$ $=: V^{p,q}$)

  $\overline{V^{p,q}} = V^{q,p}$

- Abstracting, we can define a $\mathbb{C}$-HS

  V $\mathbb{C}$-vector space with two decreasing filtration $F, \overline{F}$ st. * still holds

  (F and $\overline{F}$ are no more conjugate !!)

- **Def** A HS is
  - V_R a fin. gen R-mod
  - an increasing filtration W ("weight filtration") of $V_R \otimes (R \otimes \mathbb{Q})$
  - A decreasing filtration F of $V_\mathbb{C}$ st.

    F induces on each $Gr^W_\ell (V_R \otimes (R \otimes \mathbb{Q}))$ a pHS of weight ℓ.

We have $Gr^{p,q} := [Gr^W_{p+q}(V_\mathbb{C})]^{p,q}$

We want to find a bigrading of $V_\mathbb{C} = \bigoplus S^{p,q}$

- s.t. $\begin{cases} W_k^\mathbb{C} = \bigoplus_{p+q \le k} S^{p,q} \\ F^p = \bigoplus_{r \ge p} S^{r,s} \end{cases}$

i.e. such that $S^{p,q}$ maps isomorphically to $Gr^{p,q}$

If we ask $S^{p,q} = \overline{S^{q,p}}$ then $V_k = \bigoplus_{p+q=k} S^{p,q}$ would be pure of weight k and V is semisimple. This is too much. We just need

$$S^{p,q} = \overline{S^{q,p}} \pmod{W_{p+q-1}}$$

Deligne Weak Splitting

$$S^{p,q} = (F^p \cap W_{p+q}) \cap (\bar{F}^q \cap W_{p+q} + \sum_{j\geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j})$$

Consequences 1.) Any morphism of MHS is strict

i.e. $f: V \to V'$ $\quad f(V) \cap F^p(V') = F^p(V)$

$f(V) \cap W_p(V') = W_p(V)$

2.) MHS form an abelian category

3) The functors Gr^F_k, Gr^K_W and $Gr^F_k Gr^k_W$ are exact.

ANOTHER DESCRIPTION OF $\mathbb{C}$-MHS.

Let $(V; W, F, \bar{F})$ a $\mathbb{C}$-MHS

$V = \bigoplus S^{p,q} = \bigoplus \overline{S^{p,q}}$ $\quad (S^{p,q} \equiv \overline{S^{q,p}} \pmod{W_{p+q-1}})$

$S^{p,q} \xrightarrow{\sim} Gr^{p,q} \xleftarrow{\sim} \overline{S^{q,p}}$

$Gr^W V = \bigoplus Gr^W_k V$ is a direct sum of pHS.

~~eg~~ We can glue all these isomorphisms to obtain $a_F : V \xrightarrow{\sim} Gr$. This preserves W and F, but not $\bar{F}$

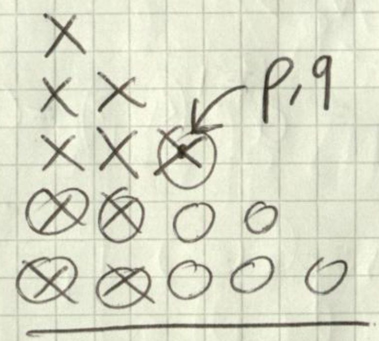
Analogously we can define $a_{\bar F} : V \xrightarrow{\sim} Gr\,V$

$d = a_{\bar F}\, a_F^{-1}$, $d : Gr \to Gr$.

<u>Prop</u> $(d-1)\, Gr^{p,q} \subseteq \bigoplus_{\substack{r<p \\ s<q}} Gr^{r,s}$ ⊛

<u>Sketch</u> $I(p,q) = \left\{ (r,s) \;\middle|\; \begin{matrix} r+s \le p+q \\ (r,s) = (p,q) \vee r<p \end{matrix} \right\} = X$

$K(p,q) = \{ (r,s) \mid$ " " $\vee\, s<q \} = O$



One can check that $S^{p,q} \subseteq \bigoplus_{(r,s)\in K(p,q)} \overline{S^{r,s}} =: \overline{S^K}$

$d\,Gr^{p,q} = a_{\bar F} S^{p,q} \subseteq Gr^K$

$\Rightarrow d\,Gr^K = Gr^K$. Analogously $d\,Gr^I = Gr^I$

$\Rightarrow d\,Gr^{p,q} \subseteq Gr^{I\cap K}$.

<u>Thm</u> The category of $\mathbb{C}$-MHS is equivalent to the category of $(B = \bigoplus B^{p,q}, d)$ where $d : B \to B$ has ⊛

<u>Proof Sketch</u> We define an inverse

$(B,d) \rightsquigarrow (B, W, F, \bar F)$

where $W_i = \bigoplus_{p+q\le i} B^{p,q}$ $F^i = \bigoplus_{p\ge i} B^{p,q}$ $\tilde F^i = \bigoplus_{q\ge i} B^{p,q}$

$\overline{F} = d^{-1}(\tilde{F})$

Infact a_F preserves W and F and sends $a_F(\overline{F}) = d^{-1} a_{\overline{F}}(F) = d^{-1}(\tilde{F})$

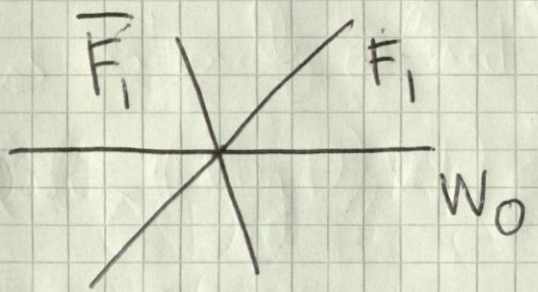
<u>EXAMPLE</u> Want to classify $\mathbb{C}$-MHS with

$V^{0,0} = V^{1,1} = \mathbb{C} \iff V^{1,1} \xrightarrow{d-1} V^{0,0}$ $\quad d = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

If $d - 1 = 0$ then $V = \mathbb{C}^{1,1} \oplus \mathbb{C}^{0,0}$

If $d - 1 = 1$ then $V = \langle v_1, v_0 \rangle$

$W_0 = \langle v_0 \rangle \quad F_1 = \langle v_1 \rangle \quad \overline{F}_1 = d^{-1}(\langle v_1 \rangle) = \langle v_1 - v_0 \rangle$



V semisimple $\iff F_1 = \overline{F}_1$

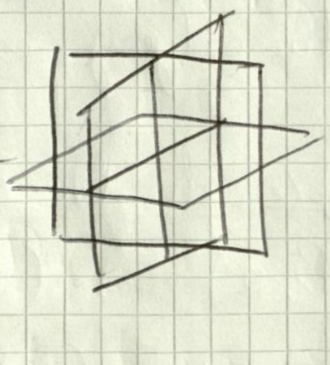
MHS ON THE COHOMOLOGY OF A SMOOTH VARIETY

Let U a smooth~~variety~~ alg. variety

<u>Thm</u> (Nagata / Hironaka)

U is Zariski open in X smooth complete alg. variety and $D = X \setminus U$ is a normal crossing divisor with smooth components

D is locally $\{z_1 \cdots z_h = 0\}$



LOGARITHMIC DE RHAM COMPLEX

3

$\Omega_X(\log D)$ = diff. form ω such that both ω and $d\omega$ have poles at most of order 1 along D.

Locally $\Omega^\bullet_X(\log D)_p = \Omega^\bullet_X\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k} \rangle$.

<u>Prop</u> $\Omega^\bullet_X(\log D) \xrightarrow{\sim} j_*\Omega^\bullet_U \xrightarrow{\sim} Rj_*\mathbb{C}_U$

So we can use $\nwarrow$ to compute the cohomology of U.

We define
$$W_m\Omega^p_X(\log D) = \begin{cases} 0 & m<0 \\ \Omega^{m-p}_X \wedge \Omega^p_X(\log D) & 0 \le m \le p \\ \Omega^p_X(\log D) & m \ge p \end{cases}$$

$$F^m\Omega^\bullet_X(\log D) = \left(\Omega^\bullet_X(\log D)\right)^{\ge m}$$

<u>Claim</u> These induce a MHS on the cohomology.

$$D = \bigcup_{i=1}^k D_i \qquad D_I = \bigcap_{i\in I} D_i \qquad \widetilde{D}_k = \coprod_{|I|=k} D_I$$

$$a_I : D_I \hookrightarrow X \qquad a_k : \widetilde{D}_k \longrightarrow X$$

$$\mathrm{res}_I : \Omega^\bullet_X(\log D) \to \Omega_{D_I}(\log D_I')[-m] \quad |I| = m$$

$$\omega = \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_m}}{z_{i_m}} \wedge \eta + \eta' \longmapsto \eta$$

$\rightsquigarrow \mathrm{res}_I : Gr^W_m(\Omega_X(\log D)) \to \Omega^\bullet_{D_I}[-m]$

$\rightsquigarrow \mathrm{res}_m = \bigoplus_{|I|=m} Gr\,\mathrm{res}_I : Gr^W_m(\Omega_X(\log D)) \to \Omega^\bullet_{\widetilde{D}_m}[-m]$

Prop Res_k is a quasi isomorphism

$\left(\beta \mapsto \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_k}}{z_{i_k}} \wedge \beta\right.$ is an inverse$\left.\right)$

~~We~~ If we keep track of the Hodge filtration

~~Res~~: $F^p(Gr^W_m(\Omega(\log D)) \leftrightarrow a_{m*} F^{p-m}(\Omega^{\sim}_{D_m}[-m])$

So $\mathrm{res}_m : (Gr^W_m(\Omega^{\cdot}(\log D)), F) \to (a_{m*}\Omega^{\sim}_{D_m}[-m], F(-m))$

is a filtered q.i. and its q cohomology ~~is a~~ has a pHS of weight $q-2m$

W is rationally defined

$(\Omega_X(\log D), W) \xleftarrow{\alpha} (\Omega_X(\log D), \tau) \xrightarrow{\beta} (Rj_*\underline{\mathbb{C}}, \tau)$

are filtered q.i.

We already know for β.

Need to show

$\mathcal{H}^m(Rj_*\mathbb{C}_U)_p = H^m(V \setminus V \cap D) \xrightarrow{\mathrm{res}} Gr^W(\Omega_X(\log D))_p$

V small nbhd of p $\quad \searrow^{\mathrm{res}} \quad Gr^W(\Omega_X(\log D))_p \cong H^0(\widetilde{D}_m)$ is an iso

For example for $m=1$

$$\bigoplus_{i=1}^{k} \mathbb{C}\,\frac{dz_i}{z_i} \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathbb{C}_{D_i}$$

On $Rj_*\underline{\mathbb{C}}$ τ is rationally defined, so is W.

Thm The ~~MHS~~ cohomology of a smooth variety has a MHS 4

Proof (Sketch)

$E_r^{p,q}$ spectral sequence of the filtered complex $(R\Gamma(\Omega_X(\log D)), W)$.

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr^W_{-p}\Omega^\bullet_X(\log D))$$
$$= \mathbb{H}^{p+q}(X, a_{p*}\Omega_{\widetilde{D}_p}[-p])$$
$$= H^{2p+q}(\widetilde{D}_p, \mathbb{C}) \Rightarrow E_\infty^{p,q} = Gr^W_q H^{p+q}(U, \mathbb{C})$$

but with a pHS of weight $2p+q-2p = q$

$d_1 : H^{2p+q}(\widetilde{D}_p, \mathbb{C}) \to H^{2p+2+q}(\widetilde{D}_{p+1}, \mathbb{C})$

is the Gysin map, one can check is a morphism of HS.

So $E_2^{p,q}$ is pHS of weight q.

HARD STEP ("Lemma des deux filtrations")

d_2 is compatible with the Hodge filtrations.
But then d_2 is a morphism between pHS of different weights, so it is 0. The same holds for d_i, $i \geq 3$.

So $E_\infty^{p,q} = E_2^{p,q}$ has a pHS of weight q.

We also obtain

$$W_m H^k(U,\mathbb{C}) = \mathrm{Im}\left(\mathbb{H}^k(X, W_{m-k}\Omega^\bullet_X(\log D)) \to H^k(U;\mathbb{C})\right)$$

$$F^p H^k(U,\mathbb{C}) = \mathrm{Im}\left(\mathbb{H}^k(X, F^p\Omega^\bullet_X(\log D)) \to H^k(U;\mathbb{C})\right)$$

<u>Cor</u> U smooth, then $W_m H^k(U) = 0 \ \forall m < k$

$$W_k H^k(U) = \mathrm{Im}\left(H^k(X) \to H^k(U)\right)$$

<u>AN EXAMPLE</u> Assume $D = X \setminus U$ is a smooth hypersurface

Gysin sequence (comes from $i_! i^! \to \mathrm{Id} \to j_* j^* \xrightarrow{+1}$)

$$\to H^{m-2}(D) \xrightarrow{G} H^m(X) \xrightarrow{i^*} H^m(U) \xrightarrow{res} H^{m-1}(D) \xrightarrow{G} H^{m+1}(X)$$

$\uparrow$ pHS of $w = m$ $\qquad\qquad\qquad$ $\uparrow$ pHS of weight $m-1$

i^* is a MHS-morphism

We know that res is a morphism of MHS if we consider $H^{m-1}(D)$ as pure of weight $m+1$.

Also G is a morphism of ~~..~~ MHS

$$\Rightarrow \ W_{m-1} H^m(U) = 0$$

we already know it

$$W_m H^m(U) = \mathrm{Im}\left(H^m(X) \to H^m(U)\right)$$

$$W_{m+1} H^m(U) = H^m(U)$$

In particular if U is a smooth curve

$$D = \{p_1, \dots, p_m\}$$

$$0 \to H^1(X) \to H^1(U) \to H^0(D) \xrightarrow{G} H^2(X) \to 0$$

$$W_1 H^1(U) = H^1(X)$$

$Gr^W_1 H^1(U) = W_2 H^1(U) / W_1 H^1(U)$ is gen. by the difference of $\dfrac{dz}{z-p_1}, \dfrac{dz}{z-p_2}, \dots, \dfrac{dz}{z-p_m}$

~~FIXED PART THEOREM~~ (or INVARIANT CYCLE THM) 5

- <u>Recall</u> $f : X \to Y$ smooth ~~projective~~ proper morphism between smooth ~~projective~~ varieties

$R^k f_* \underline{\mathbb{C}}_X$ is the loc. sys associated to the local system
$(U \to H^k(f^{-1}(U)))$

$$H^0(R^k f_* \underline{\mathbb{C}}_X) \cong [H^k(f^{-1}(p))]^{\pi_1(Y,p)}$$

$H^k(X) \twoheadrightarrow H^0(Y, R^k f_* \mathbb{C}_X)$ is surjective
(this comes from the degeneracy of the <u>Leray SS at page 2</u>)

MORE GENERAL SITUATION

$f : X \to Y$, X smooth projective.

$\exists U$ open Zariski open $\subseteq X$ s.t. $f : U \to Y$ is smooth (and projective...)

$$H^k(\cancel{X}) \to H^k(U) \twoheadrightarrow H^0(Y, R^k f_* \underline{\mathbb{C}}_U) \hookrightarrow H^k(f^{-1}(p))$$

To show $H^k(X) \to H^0(Y, R^k f_* \underline{\mathbb{C}}_U)$ is surjective
we need to show that $H^k(X)$ and $H^k(U)$ have the same image in $H^k(f^{-1}(p))$

But this is clear since only $W_k H^k(U)$ is relevant and $H^k(X) \twoheadrightarrow W_k H^k(\cancel{U})$..

We want to find a non semisimple MHS with $\mathbb{C}^{0,0}$, $\mathbb{C}^{1,1}$ and nothing else arising from a geometrical setting.

Let $\lambda \neq 1$. $H^*(\mathbb{C}^*, \{1,\lambda\})$ Relative cohomology

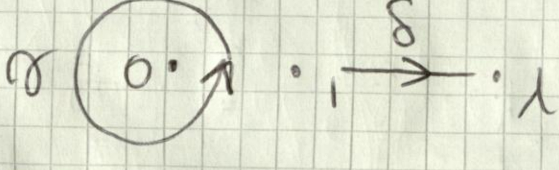
$j : \mathbb{C}^* \setminus \{1,\lambda\} \hookrightarrow \mathbb{C}^*$ $\quad = H^*(\mathbb{C}^*, j_! j^* \mathbb{Q}_{\mathbb{C}^*})$ has a MHS

which fits in a long exact sequence

$$0 \to H^0(\mathbb{C}^*) \to H^0(\{1,\lambda\}) \to \underbrace{H^1(\mathbb{C}^*, \{1,\lambda\})}_{=H(\lambda)} \to H^1(\mathbb{C}^*) \to 0$$

$$0 \to \underbrace{\tilde{H}^0(\{1,\lambda\})}_{=\mathbb{C}^{0,0}} \to H^1(\mathbb{C}^*, \{1,\lambda\}) \to \underbrace{H^1(\mathbb{C}^*)}_{=\mathbb{C}^{1,1}} \to 0$$

We want to show this doesn't split (or equivalently that $F^1 \neq \overline{F^1}$)



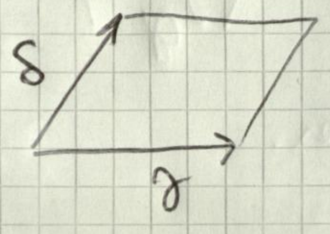
$H^1(\mathbb{C}^*, \{1,\lambda\})$ is dual of $H_1(\mathbb{C}^*, \{1,\lambda\}) = \mathbb{C}\gamma \oplus \mathbb{C}\delta$ and a $\mathbb{Z}$-basis is given by $\{\gamma^*, \delta^*\}$

$$F^1 H(\lambda) \xrightarrow{\sim} F^1 H^1(\mathbb{C}^*) = \mathbb{C}\frac{1}{2\pi i}\frac{dz}{z}$$

$$\frac{1}{2\pi i}\frac{dz}{z} = \frac{1}{2\pi i}\left(\int_\gamma \frac{dz}{z}\right)\gamma^* + \frac{1}{2\pi i}\left(\int_\delta \frac{dz}{z}\right)\delta^* \text{ so}$$

it $\gamma^* + \frac{\log\lambda}{2\pi i}\delta^* \Rightarrow \boxed{F^1 \neq \overline{F^1}}$

SMOOTH AND ~~PRO~~ PROJECTIVE CASE



$$H^1(E, \mathbb{C}) = H^{0,1} \oplus H^{1,0}$$

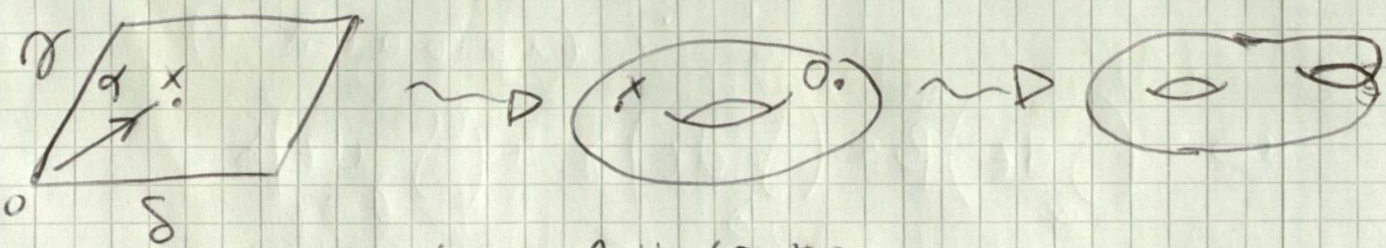
We have a basis of the $H^1(E,\mathbb{Z}) \subseteq H^1(E,\mathbb{C})$ given by γ^*, δ^*

$H^{1,0} = \mathbb{C}\,dz$. Let $\omega \in H^{1,0}$ $\quad dz = \underbrace{\left(\int_\gamma dz\right)}_{= \gamma} \gamma^* + \underbrace{\left(\int_\delta dz\right)}_{= \delta} \delta^*$

Writing dz, up to a scalar, in the $\mathbb{Z}$-basis determines up to a scalar ~~basis of $H_1(E,\mathbb{Z}) \cong \Gamma$~~ so, determines E.

SINGULAR CASE.

Let $0 \neq x \in E$. We glue together x and 0



γ, δ, α is a basis of $H_1(E, \mathbb{Z})$

$\gamma^*, \delta^*, \alpha^*$ $\mathbb{Z}$-basis of $H^1(E,\mathbb{Z})$. $\quad 0 \to \mathbb{Z}_X \to \pi_* \mathbb{Z}_E \to \mathbb{Z}_p \to 0$

$$0 \to H^0(E) \to H^0(p) \xrightarrow{d} H^1(X) \to H^1(E) \to 0$$

is an exact sequence of MHS.

$W_0 H^1(X) = \operatorname{Im} d$

$W_1 H^1(X) = H^1(X)$ $\quad F^1$ is unidimensional and maps to

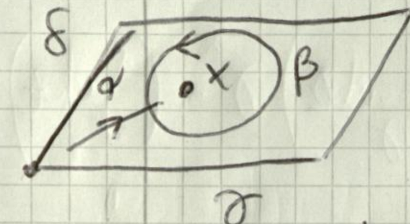
$$F^1 H^1(X) \xrightarrow{\sim} F^1 H^1(E) = H^{1,0}(E) = \mathbb{C}\, dz$$

Let $\omega \in F^1 H^1(X)$

~~Let~~ Writing $\omega_{\cancel{dz}} = \left(\int_\gamma dz\right)\gamma^* + \left(\int_\delta dz\right)\delta^* + \left(\int_\alpha dz\right)\alpha^*$

determines Γ and x up to a scalar.

NON-PROJECTIVE CASE



Let $U = E \setminus \{0, x\}$

$$H^2(E) \cong \mathbb{C}^{1,1}$$

$$H_1(U) \cong H^1_c(U) \cong \mathrm{Hom}(H^1(U), H^2_c(U))$$

A $\mathbb{Z}$-basis of $H^1_c(U)$ is given by β~~α~~, γ, δ

We have

$$0 \to H^0(E) \to H^0(\{0,x\}) \to H^1_c(U) \to H^1(E) \to 0$$
$$dz \in H^1(E)$$

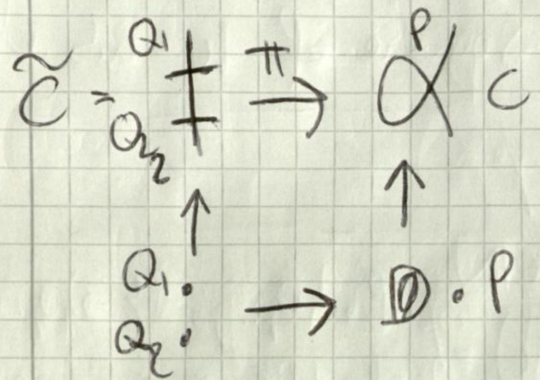
Let ~~case~~ $\mathbb{C}\omega = F^1$. So $i_!\omega = dz$ (up to a constant).

$$\omega = \left(\int_\alpha dz\right)\beta + \left(\int_\delta dz\right)\gamma + \left(\int_\gamma dz\right)\delta$$

and determines Γ and x up to a constant.

Juteau

~~K~~ KEY EXAMPLE



The sheaf $\mathbb{Z}_C$ is quasi-isomorphic to the complex

$$\left[\pi_*\mathbb{Z}_{\tilde{C}} \oplus \mathbb{Z}_P \to \mathbb{Z}_{Q_1} \oplus \mathbb{Z}_{Q_2}\right]$$

Here we can define a Hodge structure.
To deal with the general case we need to introduce some language.

CUBICAL ~~VARIETY~~ CATEGORY

Def $\Box$ Obj $\Box$ = $I \subseteq \mathbb{N}$

Morph $\mathrm{Hom}(I, S) = \begin{cases} \text{has a unique element if } I \subseteq S \\ \emptyset \text{ o/w} \end{cases}$

~~Ex~~ $\Box_k \subseteq \Box$ subcategory with obj $I \subseteq \{0, \dots, k-1\}$

Def Given $\mathcal{C}$ cat. A cubical object is a contravariant functor $\Box \to \mathcal{C}$. A k-cubical object $\Box_k \to \mathcal{C}$

Ex. 3-cubical object

$$\begin{tikzcd}
& X_{\{0,1,2\}} \arrow[rr] \arrow[dl] \arrow[dd] & & X_{\{1,2\}} \arrow[dl] \arrow[dd] \\
X_{\{0,2\}} \arrow[rr, crossing over] \arrow[dd] & & X_{\{2\}} & \\
& X_{\{0,1\}} \arrow[rr] \arrow[dl] & & X_{\{1\}} \arrow[dl] \\
X_{\{0\}} \arrow[rr] & & X_{\emptyset} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

~~The example is a 1-cubical variety.~~

Rmk $(k+1)$-cubical objects = morphism between k-objects

$(k+2)$-cub. obj = comm. square of k-cub. obj.

SEMISIMPLICIAL CATEGORY

Def Δ Obj $= k \in \mathbb{N}$

Morp $= \mathrm{Hom}(j,k) = \left\{ \begin{array}{l} f : \{0,\dots,j-1\} \to \{0,\dots,k-1\} \\ \text{strictly increasing} \end{array} \right\}$

$\Delta_k \subseteq \Delta$ full subcat. with obj $j \leq k$.

$$0 \rightrightarrows 1 \Rrightarrow 2 \;\cdots$$

Def A k-semisimplicial object is a ~~funct~~ contravariant functor $\Delta_k \to \mathcal{C}$.

$(k+1)$ CUBICAL OBJECT DEFINES A k-SEMISIMP OBJ.

$$X_k = \coprod_{|I|=k+1} X_I$$

$\beta : \{0,\dots,j-1\} \to \{0,\dots,k-1\}$ strictly increasing then

$$X(\beta)|_{X_I} = d_{\beta(s)I} \quad X_I \to X_{\{\beta(0),\dots,\beta(j-1)\}}$$

Obs We have not used $X_\emptyset$!

Def "augmentation" $X_\cdot \to Y$

i.e. A morphism of k-semisimp. object.

(where $X_\emptyset$ is seen as the constant semisimplicial object)

So $\{X_I\}$ cubical $\rightsquigarrow$ defines augmented semisimplicial

$$X_\cdot \to X_\emptyset$$

EXAMPLE X top. space. $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ covering. 2

$U_\emptyset = X$ ~~$\{U_I\}$~~ $U_I = \bigcap_{i \in I} U_i$ defines a cubical space.

The ass. ~~augmented~~ semisimp. space is called the nerve

$$N(\mathcal{U})_k = \coprod_{|I| = k+1} U_I. \quad \text{We have } N(\mathcal{U})_\cdot \xrightarrow[\text{augmentation}]{} X$$

SHEAVES ON SEMISIMPLICIAL SPACES

Def It is an semisimp. obj in the category

Obj $(X, \mathcal{F})$ X top. space, $\mathcal{F}$ sheaf on X

Map $f : (X, \mathcal{F}) \to (Y, \mathcal{G}) = \begin{cases} f : X \to Y \\ f^\# : \mathcal{G} \to f_* \mathcal{F}. \end{cases}$

~~$f : X \to Y$~~

We have a sheaf $\mathcal{F}_n$ on each X_n.

Ex ~~ϵ~~ $X_\cdot \to Y$ aug. $\mathcal{F}$ sheaf on Y

Then $\epsilon_\cdot^* \mathcal{F}$ defines ~~a~~ a sheaf on $X_\cdot$

COHOMOLOGY AND DIRECT IMAGE $\epsilon : X_\cdot \to Y$

We need to use injective resolution, like $\mathcal{C}^p_{\mathrm{God}}(\cdot)$ ← GODEMENT

This defines a double complex

$$\begin{array}{ccccc}
\uparrow & & \uparrow & & \\
\epsilon_* \mathcal{C}^1(\mathcal{F}^0) & \to & \epsilon_* \mathcal{C}^1(\mathcal{F}^1) & \to & \cdots \\
\uparrow & & & & \\
\epsilon_* \mathcal{C}^0(\mathcal{F}^0) & \to & \epsilon_* \mathcal{C}^0(\mathcal{F}^1) & \to & \epsilon_* \cdots
\end{array}$$

Def $R\epsilon_* \mathcal{F}_\cdot = \mathrm{Tot}(\epsilon_* \mathcal{C}^\cdot(\mathcal{F}^\cdot))$

If $Y = \{pt\}$ this then $H^k(X_\cdot, \mathcal{F}_\cdot) = \mathbb{H}^k(\mathrm{Tot}(\epsilon_* \mathcal{C}^\cdot(\mathcal{F}^\cdot)))$

EXAMPLE $N(\mathcal{U})_\cdot \to X$, $\mathcal{F}$ sheaf on X

Then $H^k(N(\mathcal{U})_\cdot, \mathcal{F}) \cong H^k(X, \mathcal{F})$

(Mayer-Vietoris double complex)

Def $\epsilon : X_\cdot \to Y$ is of cohomological descent if the can. morphism $\mathbb{Z}_Y \to R\epsilon_* \mathbb{Z}_{X_\cdot}$ is a quasi-isomorphism.

This allows to compute $H^k(Y) = H^k(X_\cdot, \mathbb{Z}_{X_\cdot})$.

Rmk ~~lon~~ Our key example is was of coh. des.

Let X compact variety, $\tilde{X} \to X$ res. of singularity, D discriminant, i.e. minimal closed variety s.t. $\tilde{X} - f^{-1}(D) \to X - D$ is an isomorphism.

Prop
$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] & X \\ f^{-1}(D) \arrow[u, "i"] \arrow[r] \arrow[ur, dashed, "h"] & D \arrow[u, "i"] \end{tikzcd}$$
is of coh. descent (or, more precisely, its associated augmented semisimplicial ~~m~~ object)

Proof
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & i_* i^* \mathcal{F} \arrow[d] \\ Rf_* f^* \mathcal{F} \arrow[r] & Rh_* h^* \mathcal{F} \end{tikzcd}$$

If $p \notin D$ vertical lines are isomorphisms 3
If $p \in D$ horizontal lines are isomorphisms.

$$\boxed{\begin{array}{c}\left(\mathrm{Cone}\,(\mathrm{Cone}\,(1), \mathrm{Cone}\,(2)) \text{ is acyclic}\right) \\ \iff \mathrm{Cone}\,(1) \simeq \mathrm{Cone}\,(2).\end{array}}$$

If D is smooth we have already concluded.
Then X comp. variety of dim n. $\exists$ cubical hyperresolution $X_\bullet \to X$?
~~How to deal with the general case? (X compact)~~

Step 1 X cpt. $\exists f: \tilde{X} \to X$ resolution of sing. of X st. $\dim f^{-1}(D) < \dim X$

Step 2 Let $X_\bullet$ a n-cubical variety. $\exists \tilde{X}_\bullet \to X_\bullet$ resolution of sing. (i.e. proper morphism st.
- $\tilde{X}_I \to X_I$ is a r.s. for every I
- $\dim f_I^{-1}(D_I) < \dim X_I \quad \forall I$

where $D_\bullet \subseteq X_\bullet$ is the minimal closed cubical subvar. st. $X_I - f^{-1}(D_I) \to X_I - D_I$ is an iso.

No?

Sketch $\Sigma_\emptyset = \{\text{irr. comp. of } X_\emptyset\} \cup \{\text{closure } f(X_{I,\alpha})\}$ where $f: X_{I,\alpha} \to X_\emptyset$

$X_{I,\alpha}$ irred comp. of X_I $\quad X_{I,\alpha}$ irr. com

$$\tilde{X}_\emptyset = \coprod_{I,\alpha} \left(d_{I,\emptyset}(X_{I,\alpha})\right) \qquad d_{I,\alpha}: X \to X_\emptyset$$

"Resolution of irr. comp. of $X_\emptyset$ and of the closure of the image of all irr. comp. of X_I".

Suppose we have defined $\widetilde{X}_S$ for $S \subsetneq I$.

$$\begin{tikzcd}
\widetilde{X}^S_{I,\alpha} \arrow[r] \arrow[d] & X_{I,\alpha} \arrow[d] \\
\overline{\widetilde{d_{IS}(X_{I,\alpha})}} \arrow[r] & \overline{d_{IS}(X_{I,\alpha})}
\end{tikzcd}$$

$\rightsquigarrow$ $\widetilde{X}_{I,\alpha}$ is a resolution of $X_{I,\alpha}$ which dominates all the $\widetilde{X}^S_{I,\alpha}$.

<u>Step 3</u> I will illustrate it for X surface.

We start ~~X~~ with
$$\begin{tikzcd}
\widetilde{X} \arrow[r, "f_0"] & X \\
f_0^{-1}(D_0) \arrow[u] \arrow[r] & D_0 \arrow[u]
\end{tikzcd}$$
discriminant square

$\dim D \leq 1$

$\dim f^{-1}(D) \leq 1$

$$\begin{tikzcd}
\widetilde{f^{-1}(D_0)} \arrow[r, "f_1"] \arrow[d, "\pi_1"'] & f_0^{-1}(D_0) \arrow[r] \arrow[d, "\pi_1"'] & \widetilde{X} \arrow[d, "f"] \\
\widetilde{D_0} \arrow[r] \arrow[u] & D_0 \arrow[r] \arrow[u] & X \\
f_1^{-1}(D_1') \arrow[r] \arrow[u] & D_1' \arrow[u] \arrow[ru] & \\
f_1^{-1}(D_1) \arrow[r] \arrow[u] & D_1' \arrow[ruu, "\pi_1"'] &
\end{tikzcd}$$

discriminant square of the 1-cubical of $f_0^{-1}(D_0) \to D_0$

Downstairs everything is of dim 0

Being of cd. depends only on the cones $C(f_i)$

$$C(\pi_i) = \mathrm{Cone}(\mathbb{Z}_Y \to R\pi_* \pi^* \mathbb{Z})$$

But $C(\pi_1) \cong C(f_0)$ because 1 is a dis. square. □

RELATIVE CASE

4

Thm X compact variety, $T \subseteq X$ closed subvar.
Then $\exists f: \tilde{X} \to X$ resolution of sing. st.
$f^{-1}(D)$ is a NCD on $\tilde{X}$

We can redo everything in the relative case:

Thm X variety of dim n, $T \subseteq X$ subvariety.
$\exists f: \tilde{X}_\cdot \to X$ cubical hyperresolution
s.t. $f_I^{-1}(T)$ is a NCD on each irr. comp. of $\tilde{X}_I$.

CONCLUSION V alg. variety. $\subseteq Y$ compact variety
We find a hypercubical resolution
$\epsilon: X \to Y$ of $(V, Y \setminus V)$, $D = f^{-1}(T)$
Let $X_\cdot$ the associated semisimplicial variety.
We have a sheaf $\Omega_{X_\cdot}(\log D_\cdot^\cdot)$
We define filtrations on ~~$R\epsilon_* \Omega_{X_\cdot}(\log D)$~~

$$R\epsilon_* \Omega_{X_\cdot}(\log D_\cdot^\cdot) \cong R\epsilon_* Rj_*^\cdot \underline{\mathbb{C}}_U = R\epsilon_*^\cdot \underline{\mathbb{C}}_U = \underline{\mathbb{C}}_V$$

by $R\epsilon_* \Omega_{X_\cdot}(\log D_\cdot) = \bigoplus_{p,q \geq 0} (R\epsilon_q)_* \Omega^p_{X_\cdot}(\log D_\cdot)$

The Hodge filt. $F^p(R\epsilon_*)$ is just $R\epsilon_*(F^p)$.

$$W_m(R\epsilon_*) = \bigoplus_{q \geq 0} (R\epsilon_q)_* W_{m+q}(\Omega^\cdot_{X_q}(\log D_q)).$$

One has $Gr^W_m(R\epsilon_*) = \bigoplus_{q \geq 0} (R\epsilon_q)_* Gr^W_{m+q}(\Omega_{X_q}(\log D_q))$

Recall from last ~~lesson~~ lecture: their cohomology is a a ~~pHS~~ pHS of weight $m+q$, but here we ~~have~~ the Hodge filtration is shifted, so they are a pHS of weight m.

~~Indeed~~ As in the last lecture this implies that

$$Gr^W_m H^m(R\epsilon_*) = Gr^W_m H^k(U) \text{ is pHS of weight } m$$