Parity sheaves and the decomposition theorem

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Tuesday Problem Sheet

1. Let $\mathcal{A}$ be a Krull-Remak-Schmidt category. Show that the Krull-Remak-Schmidt theorem holds in $\mathcal{A}$.

2. Let $\mathcal{A}$ be an abelian category, and let $D(\mathcal{A})$ denote its derived category. Given a complex

$$F = \dots \to F^{i-1} \stackrel{d_{i-1}}{\to} F^i \stackrel{d_i}{\to} F^{i+1} \to \dots$$

consider the complexes:

$$\tau_{\le i} F = \dots \to F^{i-1} \stackrel{d_{i-1}}{\to} \ker d_i \to 0 \to \dots$$
$$\tau_{>i} F = \dots \to 0 \to F^i / \ker d_i \to F^{i+1} \to \dots$$

Describe the cohomology of $\tau_{\le i} F$ and $\tau_{>i} F$ in terms of F. Show that $\tau_{\le i}$ and $\tau_{>i}$ define endofunctors of $D(\mathcal{A})$ and that one has a functorial distinguished triangle

$$\tau_{\le i} \to \mathrm{id} \to \tau_{>i} \stackrel{[1]}{\to} .$$

The functors $\tau_{\le i}$ and $\tau_{>i}$ are called *truncation functors.*

In the lectures we saw that one can calculate the stalks of intersection cohomology complexes (with $\mathbb{Q}$-coefficients) using resolutions. In the following exercise we will explore the Deligne construction, which gives an explicit construction of the intersection cohomology complex.

3. Fix X a stratified variety as in lectures, and let $X_{\ge i}$ denote the union of all strata of dimension $\ge i$. Consider the chain of inclusions

$$X_{\ge d_X} \stackrel{j_{d_X}}{\hookrightarrow} X_{\ge d_X - 1} \stackrel{j_{d_X - 1}}{\hookrightarrow} X_{\ge d_X - 2} \hookrightarrow \dots \stackrel{j_1}{\hookrightarrow} X_{\ge 0} = X.$$

Given any local system $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$, where $X_\lambda \subset X$ is a stratum of dimension d_X, show that the complex

$$(\tau_{\le -1} \circ j_{1*}) \circ (\tau_{\le -2} \circ j_{2*}) \circ \dots (\tau_{\le -d_X + 1} \circ j_{(d_X - 1)*}) \circ (\tau_{\le -d_X} \circ j_{d_X *}) \mathcal{L}$$

satisfies the conditions characterising $\mathrm{IC}(\overline{X_\lambda}, \mathcal{L})$. (*Hint:* Check the conditions by induction, remembering that $i^! j_* = 0$!)

In practice it is very difficult to calculate examples using the Deligne construction. Here we see two examples where calculations are feasible.

4. Let V be a k-vector space, and $\rho : \pi_1(\mathbb{C}^\times, 1) = \mathbb{Z} \to GL(V)$ be a representation. Consider the corresponding local system $\mathcal{L}_\rho$ on $\mathbb{C}^\times$ with stalk V at 1 and monodromy given by ρ. Show that

$$H^i(\mathbb{C}^\times, \mathcal{L}_\rho) = \begin{cases} 0 & \text{if } i \neq 0, 1, \\ V^\rho & \text{if } i = 0, \\ V_\rho & \text{if } i = 1. \end{cases}$$

Here V^ρ and V_ρ denote the invariants and covariants of ρ. They are defined by the long exact sequence:

$$0 \to V^\rho \to V \stackrel{\rho(1) - \mathrm{id}}{\longrightarrow} V \to V_\rho \to 0.$$

Use the Deligne construction to conclude that $\mathrm{IC}(\mathbb{C}, \mathcal{L}_\rho)_0 = V^\rho[1]$. Hence give a proof of the decomposition theorem for the map $\mathbb{C} \to \mathbb{C} : z \mapsto z^m$.

5. Suppose that $X \subset \mathbb{P}(V)$ is a smooth projective variety and let $Y \subset V$ denote by cone over X. Describe the stalk of $\mathrm{IC}(Y)$ at 0 in terms of the cohomology of X.

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