Parity sheaves and the decomposition theorem

Ruhr-Universität Bochum

Thursday Problem Sheet

1. Let X_λ be smooth and $\mathcal{L}, \mathcal{L}'$ be local systems on X_λ.

a) Show that one has an isomorphism

$$\mathcal{H}om(\mathcal{L}, \mathcal{L}') \cong \mathcal{L}^\vee \otimes \mathcal{L}'.$$

b) Take global sections to deduce that one has

$$\mathrm{Hom}^i(\mathcal{L}, \mathcal{L}') = H^i(X_\lambda, \mathcal{L}^\vee \otimes \mathcal{L}').$$

c) Deduce that condition (P) in lectures is equivalent to

$$H^{\mathrm{odd}}(X_\lambda, \mathcal{L}) = 0 \qquad \text{for all } \mathcal{L} \in \mathrm{Loc}(X_\lambda) \qquad (P')$$

d) Assume that $\pi_1(X_\lambda)$ is finite. Show that (P') is equivalent to (P''): $\mathrm{Loc}(X_\lambda)$ is semi-simple and $H^{\mathrm{odd}}(U) = 0$ where U denotes the universal cover of X_λ.

2. Assume that X_λ satisfies condition (P) from lectures. Assume that $\mathcal{F} \in D^b_c(X_\lambda)$ is such that $\mathcal{H}^i(\mathcal{F}) \in \mathrm{Loc}(X_\lambda)$ for all $i \in \mathbb{Z}$ and $\mathcal{H}^{\mathrm{odd}}(\mathcal{F}) = 0$ for odd i. Show that one has an isomorphism

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(\mathcal{F})[-i].$$

Discuss the canonicity of this decomposition. For example, can you give a condition on $H^*(X_\lambda, \mathcal{L})$ for this decomposition to be canonical?

3. Suppose that the parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$ exists. Show that $\mathbb{D}\mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}(\lambda, \mathcal{L}^\vee)$.

4. Let $G = GL_3$, $\mathfrak{g} = \mathfrak{gl}_3(\mathbb{C})$ and $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Calculate the stalks of the parity sheaves on $\mathcal{N}$ with coefficients in a field of characteristic 2 and 3. (*Hint:* Consider the resolution $\widetilde{\mathcal{O}_{\min}} \to \overline{\mathcal{O}_{\min}}$ given in lectures. Identify $\widetilde{\mathcal{O}_{\min}}$ with $T^*\mathbb{P}^2$, and $\widetilde{\mathcal{O}_{\min}} \to \overline{\mathcal{O}_{\min}}$ with the contraction of the zero section. Hence calculate the stalks of $\mathcal{E}(\overline{\mathcal{O}_{\min}})$. Now use the decomposition of $\pi_* \underline{k}_{\widetilde{\mathcal{N}}}$ given in today's lecture.)

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