Lecture 6.1

Hecke Algebras w/ unequal parameters and folding.

There's a generalization of $H$, by first generalizing the length function $L^2$. Instead of requiring $L(s) = 1 \forall s \in S$, instead, $L(s) \in \mathbb{Z}$. However, we want $L(t)$ well-defined, so $L$ must satisfy the basic relation, i.e. $L(st_m) = L(ts_m) \Rightarrow L(s) = L(t)$ when $m$ is odd.

Now define $H_L$ w/ generators $H_s$ as before, but replace the old quadratic relation $H_s - t)(H_s + t^r) = 0$ with a new one $(H - t)(H + t^r) = 0$.

The basic theory of the Hecke algebra extends to $H_L$; basic, given by $W$.

- bar maintain $H_s = H_s^2$, $v = v^2$

- $x$, $y$ in $W$ with $H_x = H_x^2$ and $H_y = H_y$ mod $v$. $\mathbb{Z}v[x, y] < H_L$

The $KL$ polyhedron is even degenerate (after napping $y^2 = H_L$)

(Though some facts get weird when $L(s) < 0$, $L(t) > 0$.)

However, behavior of $KL$ basis is quite different!

Ex: $S = \{s, t\}$, $m_{st} = 4$. $L(s) = 1$, $L(t) = 0$. $H_s = 1 + V$, $H_t = H_t + V^2$.

However, much of this
I wish to do
will be decided...
Where do weigted Cox sets come from? One nice set of examples: folding

\[ \text{Ex 1: } s + v \mapsto w \]

Then \( W^* = \langle s, u, v, w \rangle \) is type \( B_{k+1} \).

\[ (W, l) = \]

with natural length \( l \) (cf. \( L(s) = 2, L(u) = 2, L(v) = 1 \)).

However, you don't get \( H^*(w) \) or \( H^*(w, s) \) either, but fixed

\[ (w, v^0_s) \]}

Clearly, \( \sigma^* C \) \( H_{(w, s), l} \) too, but fixed

\[ \sigma^* C \]}

\[ H_{x, y}^* \]}

while \( H_{x, y}^* \) should have size \( H_{x, y}^* \).

Ex 1: \( \sigma^* C \) \( \text{Ob} = (w, S_{\text{Bin}}) \)

On color level, wrong to just look at "fixed points" \( M \) or \( M = \sigma(M) \).

Should look at \( \text{equivalent objects} \)

\[ \sigma_{\text{Bin}} = \text{Ob} = (M, \quad M \rightarrow \sigma M) \]

\[ \text{(so clearly } \sigma_{\text{Bin}}) \]

\[ M_{x, y}^* (M, \phi) (N, \psi) = \{ M \rightarrow \sigma M \} \]

\[ \phi \leftarrow \lambda \psi \]

\[ M \rightarrow \sigma M \]

\[ N \rightarrow \sigma N \]

Example objects:

\[ (B_1, 1) \quad (B_2, B_1, 1) \quad (B_2, B_1, -1) \quad (B_2 \otimes B_1, (s, 0) \times (a, b) = 1) \]

All isomorphic? 

Not isomorphic. 

\[ B_2 \not\cong B_1 \]

\[ a \not\equiv b \]

of order scalars.
There's an action of $\mathbb{Z}/2\mathbb{Z}$ on objects: $(M, \nu) \leftrightarrow (M, -\nu)$

$$(b_\nu, 1) \leftrightarrow (b_\nu, -1)$$

The Grothendieck group will be too big - 2 copies of $W$, one of the rest. But you can "weight" or "twist" the Grothendieck group by a character of $\mathbb{Z}/2\mathbb{Z}$!

Def: \[
\mathbb{S}_{\text{Bino}}^{\text{inv}} = \mathbb{S}_{\text{Bino}}^{\text{inv}} / [\mathbb{M}, \mathbb{E}] = [\mathbb{M}, -\mathbb{E}] \quad \text{basis same as } (H^{w, S_{\text{Bino}}})
\]

\[
\mathbb{S}_{\text{Bino}}^{\text{inv}} = \mathbb{S}_{\text{Bino}}^{\text{inv}} / [\mathbb{M}, \mathbb{E}] = - [\mathbb{M}, -\mathbb{E}] \quad \text{basis just } \nu.
\]

Why bases? Space $\text{End}(M) = k$ for an adec. $\text{End}(M)$.

Then $(M \pm 1)$ and $(M \otimes M, \mathbb{E})$ are adec. in $\mathbb{S}_{\text{Bino}}^{\text{inv}}$. \(\otimes_{\text{adeq}}\)

Thm: Assume the Soergel conjecture holds. Then $\mathbb{S}_{\text{Bino}}^{\text{inv}}$ categories

\((H^{w, S_{\text{Bino}}})^2\) and $\mathbb{S}_{\text{Bino}}^{\text{inv}}$ categories

Ex: (hypothetical) $$(H^{w, S_{\text{Bino}}})^2 = (v^2 + q^2 + v) H^{w, S_{\text{Bino}}}

\mathbb{S}_{\text{Bino}} \cong \mathbb{S}_{\text{Bino}}(2) \oplus \mathbb{S}_{\text{Bino}}(3) \oplus \mathbb{S}_{\text{Bino}}(2) \oplus \mathbb{S}_{\text{Bino}}(3)

\text{but } (\mathbb{S}_{\text{Bino}}(2), \mathbb{S}_{\text{Bino}}(2)) \cong (\mathbb{S}_{\text{Bino}}(2)^2) \oplus (\mathbb{S}_{\text{Bino}}(2)^2) \oplus (\mathbb{S}_{\text{Bino}}(3))

\text{they cancel in Groth } \mathbb{P}_{\text{no}}

To get KL poly, can count light leaves, like in usual way, but weigh by the trace of $\sigma$. for [Jaya]
\[ \text{Ex}_{Buv(Buv Buv Buv, Buv)} \]

we know \[ B_{Buv}(uv^2) \subseteq B_{Buv}A_{Buv} \]

\[ +1 \]

projection

\[ \frac{1}{2} + \frac{1}{2} \]

inclusion

\[ \text{remapping maps on} \]

\[ \frac{1}{2} + \frac{1}{2} \]

\[ \text{traw} - 1 \]

\[ \text{traw} + 1 \]

so get KL poly \[ V^3 - V. \]

Any ideas on how to prove in this talk?