Lecture 0.5 Categories of braid groups.

\[ x \mapsto \pi_1(x) \leq _2 \text{ fundamental groupoid } \Pi(x) \leq_0 \]

objects: points of \( X \)
morphisms: paths \( x \to y / \text{homology} \).
\[ \text{End}(x) = \pi_1(x, x) \]
\[ \pi(X) \leq_2 \text{ fundamental 2-groupoid } \]

\[ \pi(X) \text{ fundamental co-groupoid} \]

Goal: diagrammatic description of \( \pi(X) \leq_2 \) "2-version of group presentation".

Assume \( X \) is a 2-complex:

objects: 0-cells: \( X_0 \).
1-cells: arrows \( x \xrightarrow{f} x' \) in \( X_1 \). "colours". (without loops),
(mean we don't need joins).
2-cells: loops \( x_0 \xrightarrow{c} x_1 \xleftarrow{x_2} \ldots x_m = x_0 \). \( r \in X_2 \).

To describe \( c \) "we dualize":

\[ y \xrightarrow{t} x \]
identity on \( y \).

Example: \[ x \xrightarrow{1} y \]

\[ \text{morphisms are composed out of relators.} \]

\[ \text{endomorphism of identity} \]
modulo relations: \[
\begin{align*}
\gamma & = x \\
0 & =
\end{align*}
\]

"cancelling spiders":

\[
\begin{align*}
\text{Thin (?) : } & \quad \mathcal{G}_X \cong \pi_2(\mathcal{G}) \\
& \quad \text{(equivalence of 2-cats).} \quad \text{See Penn}
\end{align*}
\]

"Techniques of geometric topology".
Beautiful book!!

Exercise: Explore the functor

\[
\begin{align*}
\text{Now assume that } W \text{ is finite rank 3 crossed group.}
\end{align*}
\]

Consider the 2-complex \( X \)

- objects \( w \in W, \ X_0 \)
- \( \x 
\rightarrow \x \) if \( x < x \).
- \( \x \):

\[
\begin{align*}
\rightarrow \end{align*}
\]

\( \pi_2(X) \cong \mathcal{G} \) is equivalent to the category with objects \( w \in W \),
morphisms \( \x \rightarrow \x \).

2-morphisms: planar diagrams \( \epsilon \) with generators:

\[
\begin{align*}
\text{Also: } & \quad \text{End}(\x) = \pi_2(\mathcal{G}|(W, S), \x) = \\
& \quad \begin{cases}
\mathbb{Z} & \text{if } W \text{ is finite} \\
\mathbb{Z} \times \mathbb{Z} & \text{if } W \text{ is infinite}
\end{cases}
\end{align*}
\]

This explains where EK hom comes from.

Any closed diagram in \( W \) is an element of \( \mathcal{G} \) dual crossed graph.
Braid groups:

Now consider the variant where we forget the same category where we forget the labels in regions.

- one object
- generators: \( s \)
- morphisms: \( s \)-valent vertices.
- relations: \( s^3 = 1 \)\( \bigotimes \bigotimes \) \( v = 0 \) \( \bigcirc = 0 \)

Then \( B \) is a 2-groupoid of \( \text{Bw}. \)

Note: Gives a simple criterion for a braid group to act on a category.

Proof uses the following result ofDlgue, which we look to recover!