4.3 Lightning introduction the category $\mathcal{O}$ and the Kazhdan-Lusztig conjecture

$\mathfrak{g}$ complex semi-simple Lie algebra, finite dimensional.

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra, $\underset{}{\overset{\text{roots}}{R}} \subset \mathfrak{h}^*$ $\quad \underset{\text{simple}}{\Delta} \subset \underset{\text{positive}}{R^+} \subset R \qquad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

W Weyl group, $S \subset W$ simple reflections.

$\mathcal{O}$ = full subcategory of $\mathfrak{g}$-mod which is

- $\mathfrak{h}$-semi-simple ("weight")
- finitely generated as $U(\mathfrak{g})$-mod
- $\mathfrak{n}_+$ locally finite.

= full subcat of weight modules generated by Verma modules.
$\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$.

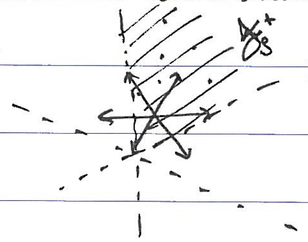
Harish-Chandra isomorphism: $Z \xrightarrow{\sim} S(\mathfrak{h})^{(W\cdot)} \subset$ Poly. functions on $\mathfrak{h}^*$. $\quad S(\mathfrak{h})^{(W\cdot)}$ = W-invariant pol. func. on $\mathfrak{h}^*$.

$z \longmapsto p_z \qquad$ ($p_z(\lambda)$ = scalar with which z acts on $\Delta(\lambda)$)

Hence $\operatorname{Spec} Z = \mathfrak{h}/(\cdot W)$.

Verbal: because every indecomposable $V \in \mathcal{O}$ admits a central character we have a "block decomposition"

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}/(\cdot W)} \mathcal{O}_\lambda . = \bigoplus_{\lambda \in X^+_\rho} \mathcal{O}_\lambda$$



From now on we assume λ is ~~dominant~~ in $X^+_{\rho,\mathbb{Z}}$

$\mathfrak{h}^{*+}_\rho := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda + \rho, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\}$

$X^+_{\rho,\mathbb{Z}} = \mathfrak{h}^{*+}_\rho \cap X$.

Jantzen's translation functors:

Let V be a finite dimensional $\mathfrak{g}$-module.

Then $(- \otimes_{\mathbb{C}} V, - \otimes_{\mathbb{C}} V^*)$ exact biadjoint functors on $\mathfrak{g}$-mod.

Given weights λ, μ in X^+_ρ, let V be a simple module with extremal wt $\mu - \lambda$.

$$T_\lambda^\mu := \mathcal{O}_\lambda \xhookrightarrow{\text{inc}} \mathcal{O} \xrightarrow{-\otimes V} \mathcal{O} \twoheadrightarrow \mathcal{O}_\mu .$$

Then $(T_\lambda^\mu, T_\mu^\lambda)$ are exact biadjoint functors $\mathcal{O}_\lambda \underset{T_\lambda^\mu}{\overset{T_\mu^\lambda}{\rightleftarrows}} \mathcal{O}_\mu$

Exercise: T_λ^μ are equivalences if $\lambda, \mu \in X^+$. Hint: What does T_λ^μ do to Verma modules?

$\rightsquigarrow$ reduces study to $\mathcal{O}_0$.

Wall crossing functors $\mathcal{O}_0 \circlearrowleft \Theta_s$.

Take λ s.t. $\langle \lambda+\rho, \alpha_s^\vee \rangle = 0$ and $\langle \lambda+\rho, \alpha_t^\vee \rangle > 0$ for all $t \neq s$.

Define $\Theta_s : \mathcal{O}_0 \longrightarrow \mathcal{O}_0 : M \longmapsto T_\lambda^0 T_0^\lambda M := M\Theta_s$. (convenient to have the Θ_s act on the right.)

Exercise: Let $[\mathcal{O}_0]$ denote the Grothendieck group of $\mathcal{O}_0$.

Show that $x \mapsto [\Delta(x \cdot 0)]$ defines an isomorphism $[\mathcal{O}_0] \cong \mathbb{Z}W$.

Under this isomorphism show that we have a commutative diagram:

$$\begin{array}{ccc} [\mathcal{O}_0] & \xrightarrow{[\Theta_s]} & [\mathcal{O}_0] \\ \parallel & & \parallel \\ \mathbb{Z}W & \xrightarrow{\cdot(1+s)} & \mathbb{Z}W \end{array}$$

Facts about $\mathcal{O}_0$:

1) $\{\text{simple modules in } \mathcal{O}_0\}/\cong \; = \{L_x\}_{x\in W}$ $\quad L_x = $ simple quotient of $\Delta_w := \Delta(w \cdot 0)$.

2) $\mathcal{O}_0$ is of finite length (because Δ_w is). restricted dual and twist fixing simples

3) there is a contravariant duality $\mathbb{D} : \mathcal{O}_0 \to \mathcal{O}_0$. Set $\nabla_w := \mathbb{D}\Delta_w$. $\dim_{\mathbb{C}} \operatorname{Ext}^i(\Delta_x, \nabla_y) = \delta_{xy}\delta_{0i}$.

4) ~~every~~ enough projectives, have Δ-filtrations (e.g. by translation functors)

set $P_x :=$ projective cover of L_x.

5) BGG reciprocity:

$$(P_x : \Delta_y) \overset{3)}{=} \dim_{\mathbb{C}} \operatorname{Hom}(P_x, \nabla_y) \overset{4)\, P_x}{=} [\nabla_y : L_x] \overset{3)}{=} [\Delta_y : L_x] \overset{?}{=} h_{x,y}(1)$$

KL conjecture

Hence Kazhdan-Lusztig conjecture $\Longleftrightarrow$ $[P_x] = \sum_{y \leq x} h_{y,x}(1)\, y = \underline{h}_x$ (specialisation of KL basis at 1)

~~Exercise: Show that $\Delta_{id} = P_{id}$. Use this as the base of an induction to show that the KL conjecture is equivalent to $\sum (P_x : \Delta_y) = \sum h_{y,x}(1)$.~~

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<u>Soergel's approach:</u> Let $\overset{R}{\cancel{S}} = S(\mathfrak{h})$ denote the symmetric algebra on $\mathfrak{h}$

$= $ polynomials on $\mathfrak{h}^*$.

(~~Opposite~~ Dual to what we've been working with so far).

<u>Thm</u> (Soergel) 1) $\operatorname{End}(P_{w_0}) = C := S/(S_+^W)$. "coinvariant algebra"

2) The functor $\mathbb{V}: \mathcal{O}_0 \longrightarrow \text{mod-}C$

$M \longmapsto \operatorname{Hom}(P_{w_0}, M)$

is fully-faithful on morphisms of projectives.

3) $\mathbb{V}(-\Theta_s) \cong \mathbb{V}(-) \otimes_{C^s} C$.

(If time permits: comments on the proofs...)

From what we know about category $\mathcal{O}$ it follows that

$\mathbb{V}(P_x) \overset{\oplus}{\subseteq} \mathbb{C} \otimes_R BS(\underline{x})$ is the unique direct summand (as ungraded modules) which does not occur as a direct summand of $\mathbb{C} \otimes_R BS(\underline{y})$ for any shorter expression.

<u>Claim:</u> $\mathbb{V}(P_x) = \mathbb{C} \otimes_R B_x$.

Follows from the following claim: $\mathbb{C} \otimes_R B_x$ is indecomposable as an ungraded right C-module. (Exercise.)

<u>Why does Soergel conjecture $\Rightarrow$ KL conjecture?</u>

Let $\underline{h}_w = \sum h_{y,x}(1) y$. Suppose $[P_y] = \underline{h}_y$ for all $y \leq x$.

Write $\underline{h}_x \underline{h}_s = \underline{h}_{xs} + \sum_{y<x} a_y \underline{h}_y$.

Then KL for P_{xs} $\Leftrightarrow$ $P_x \Theta_s \cong P_{xs} \oplus \bigoplus P_y^{\oplus a_y}$

apply $\mathbb{V}$ $\Leftrightarrow$ $\mathbb{V}(P_x) \otimes_{C^s} C \cong \mathbb{V}(P_{xs}) \oplus \bigoplus \mathbb{V}(P_y)^{\oplus a_y}$

$\Uparrow$

$B_x \otimes_R B_s \cong B_{xs} \oplus \bigoplus B_y^{\oplus a_y}$ (Follows from Soergel cat theorem.)