Lecture 4.1

Rouquier Complexes are the Sergeï Bîm incarnation of many well-known constructions in other contexts - (twisting)shuffling functors, spherical functors, etc., that give back gp actions.

We've seen two SES of $R$-bin

$0 \rightarrow R(-1) \rightarrow B_S \rightarrow R(1) \rightarrow 0$

which yield

$0 \rightarrow R(0) \rightarrow B_S \rightarrow R(0) \rightarrow 0$

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$0 \rightarrow R(0) \rightarrow B_S \rightarrow R(0) \rightarrow 0$

Let $B_S^i$ be the usual euler-characteristic map,

$F_S^{-i} \rightarrow [B_S^i - R(-i)] \rightarrow H_S^{n-i} \rightarrow H_S^n$.

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$F_S^{-i} \rightarrow [B_S^i - R(-i)] \rightarrow H_S^{n-i} \rightarrow H_S^n$.

This is more useful, where our convention

Def: Let $k^b(S\text{-} BIN)$ denote the homotopy cat of $S\text{-} BIN$ (can do this for any additive cat).

Ob: Band of $S\text{-} BIN$ Mod: Clean maps modulo homotopy.

Let $D^r(R\text{-} BIN)$ be the derived cat of $R\text{-} BIN$ (can only do this for abelian cat).

Add inverses of quasi-isos.

Def: Rouquier Complexes are $F_\omega = F_S \otimes F_S^{-i} \otimes F_S \in k^b(S\text{-} BIN)$ (homo too).

Ex: $F_S \otimes F_S^{-1}$

$B_S(-1) \rightarrow B_S \rightarrow R \rightarrow R(1)$

There maps give you a hom eq.

$F_S \otimes F_S^{-1} = 1$ monoidal identity.

Ex: $F_S \otimes F_S$

Exercise

Ex: $m_{S^3} = 3$

$F_S \otimes F_S \otimes F_S$

Keep Examples on Board.
Thm (Poupa): $F_0, F_1$ give a strict categorification of the braided gp of $W$ in $K_{op}^{op}$. 

i.e. $F_0$ satisfy braided relations, $F_0, F_1$ are inverse functors, and $\text{End}(F_0) = R$. However, faithfulness is still an open problem! 

- $F_0 \cong F_1 \Rightarrow \omega \cong 1$ in braided gp

Also, they give a strict categorification of $W$ on $D^b(R-Bim)$.

Since $F_0 \cong 0 \to R_0(-1) \to 0$, $F_1 \cong 0 \to R_1(1) \to 0$, $F_0F_1 \cong 0 \to R \to 0$.

Rmk: (E-Krause) For you topological folk - any braided bordism gives chain map, get action of $BGL$.

Let's look at the examples we've seen. Whenever $B_k(n)$ appeared in two adjacent degrees, there was a homotopy contracting the two summands away. What's with that?

For Homological Alg: Let $A$ be a (graded) local ring. Then inside $K^b(A-mod)$, any complex $C^\bullet$ is isomorphic to a minimal complex $C^\bullet_{\text{min}}$, for which all differentials lie in the maximal ideal. Any two such are (non-canonical) isomorphic. Why? Any differential not in the max ideal gives an isom to too summands, can contract it.

Exercise: $\text{End}(\bigoplus B_{k\ell})$ is a local ring. Modify the above to deduce that minimal complexes exist in $K^b(S^2Bim)$. Let $F_{k\ell} = F_{k\ell, \text{min}}$ for any real exp, only $F_{k\ell}$.

Examples you've seen. However, we can't deduce that adjacent $B_k$'s can be eliminated, since we don't know that $\text{End}(B_k) = R$, there might be deg 0 maps in max ideal. If $S^2$ congibiliy any nonzero diff $B_k(n) \to B_k(n)$ can be cancelled.

Exercise: $F_{k\ell}$ suit. In $K^b(S^2Bim)$, $B_{k\ell}$ suit $B_k$ suit $B_{k\ell}$ suit $B_{k+1}$ suit $B_k$ suit $B_{k+1}$ suit $B_{k+2}$ suit $B_{k+2}$ suit $B_{k+3}$ suit $B_{k+4}$ suit $B_{k+5}$ suit $B_{k+6}$ suit $B_{k+6}$ suit $B_{k+7}$ suit $B_{k+7}$ suit $B_{k+8}$ suit $B_{k+8}$ suit $B_{k+9}$ suit $B_{k+9}$ suit $B_{k+10}$ suit $B_{k+10}$ suit $B_{k+11}$ suit $B_{k+11}$ suit $B_{k+12}$ suit $B_{k+12}$ suit $B_{k+13}$ suit $B_{k+13}$ suit $B_{k+14}$ suit $B_{k+14}$ suit $B_{k+15}$ suit $B_{k+15}$ suit $B_{k+16}$ suit $B_{k+16}$ suit $B_{k+17}$ suit $B_{k+17}$ suit $B_{k+18}$ suit $B_{k+18}$ suit $B_{k+19}$ suit $B_{k+19}$ suit $B_{k+20}$ suit $B_{k+20}$ suit $B_{k+21}$ suit $B_{k+21}$ suit $B_{k+22}$ suit $B_{k+22}$ suit $B_{k+23}$ suit $B_{k+23}$ suit $B_{k+24}$ suit $B_{k+24}$ suit $B_{k+25}$ suit $B_{k+25}$ suit $B_{k+26}$ suit $B_{k+26}$ suit $B_{k+27}$ suit $B_{k+27}$ suit $B_{k+28}$ suit $B_{k+28}$ suit $B_{k+29}$ suit $B_{k+29}$ suit $B_{k+30}$ suit $B_{k+30}$ suit $B_{k+31}$ suit $B_{k+31}$ suit $B_{k+32}$ suit $B_{k+32}$ suit $B_{k+33}$ suit $B_{k+33}$ suit $B_{k+34}$ suit $B_{k+34}$ suit $B_{k+35}$ suit $B_{k+35}$ suit $B_{k+36}$ suit $B_{k+36}$ suit $B_{k+37}$ suit $B_{k+37}$ suit $B_{k+38}$ suit $B_{k+38}$ suit $B_{k+39}$ suit $B_{k+39}$ suit $B_{k+40}$ suit $B_{k+40}$ suit $B_{k+41}$ suit $B_{k+41}$ suit $B_{k+42}$ suit $B_{k+42}$ suit $B_{k+43}$ suit. 

They're not so obvious.
Now for the key properties of Rqrr complexes:

**Exercise**: \( \text{Def: } K^{\infty} = \text{Complex associated to those when degree i has all shifts } i \geq 0 \)

\[ K^{\infty} = \{ \text{shifts } i \} \quad (\text{SColy } \Rightarrow i\text{-structure}) \]

**Exercise**: Most of what we've seen is in the case \( K^{\infty} \cap K^{\infty} \)

But \( F^\infty \subset K^{\infty} \cap K^{\infty} \quad F^\infty \subset K^{\infty} \cap K^{\infty} \).

**Exercise**: Prove that if \( \text{Hom} \) is a positive twist, then \( F^\infty \in K^{\infty} \) (shifts are \( i \)).

**Hint**: Show that whenever \( Bx \) makes the shift go up, it is cancelled by \( \rightarrow R^1 \).

Shall assume SColy for this exercise - that way \( Bx \oplus y \) is \( \oplus y^{\text{reg}} \) without shifts.

**Theorem (Dragged Miracle)**: \( F^\infty \in K^{\infty} \cap K^{\infty} \), it is \( R^w \) in degree 0.

**Assume SColy**

Assuming this, we get nice families for inverse KL polynomials! Go back to \( F^\infty \) and count the appearance of \( Bx \) for instance. Give formula for \( H^{-1} \).

The proof uses local filtrations and a result of W-Lubinsky stating that \( R^w \) is not required on the associated graded, slightly technical.

We won't truly need \( F^\infty \) to prove SColy, but it helps speed things up.

**Homology**: \( H^*(F^\infty) = H^*(F^\infty) = R^w(-llw) \) in degree 0, nothing else.

\( i.e. \) free \( R^w \)-module generated in degree \( llw \).

Thus the map \( BS(\infty) \oplus BS(\infty) \) is injective below degree \( llw \).

What could we possibly use \( F^\infty \)?...