3.3 The Hodge theory of Soergel bimodules

<u>Goal:</u> statement of the results and sketch of the overall strategy.

For any $x \in W$ we choose an embedding $B_x \overset{\oplus}{\subset} BS(\underline{x})$. This equips B_x with an invariant form $\langle -,- \rangle_{B_x}$. If Soergel's conjecture holds (i.e. $S(x)$) then $\langle -,- \rangle$ is unique up to a scalar, easily seen to be non-zero, and hence is non-degenerate. (cf Ben's lecture).

Now B_x is free of finite rank as an ~~R~~ right R-module and $\mathbb{D}B_x = B_x$.

Hence $\overline{B_x} := B_x \otimes_R \mathbb{R}$ is finite dimensional, $\langle -,- \rangle_{\overline{B_x}}$ non-degenerate form.

For any $f \in R^2 = \mathfrak{h}^*$, multiplication by f yields a Lefschetz operator $\overline{B_x} \to \overline{B_x}(2)$.

<u>Assumption:</u> realization satisfies $\langle w(\rho), \alpha_s^\vee \rangle > 0 \iff sw > w$. True for geometric representation.
$\exists\, \rho \in \mathfrak{h}^*$ s.t.

<u>hL + HR Thm:</u> For $i \geq 0$, left multiplication by ρ yields an isomorphism

$$\rho : (\overline{B_x})^{-i} \to (\overline{B_x})^{i}.$$

Moreover, if we normalize $\langle -,- \rangle_{\overline{B_x}}$ s.t. $\langle \rho^{\ell(x)} c_{bot}, c_{bot} \rangle > 0$ then the Hodge-Riemann bilinear relations are satisfied.

<u>Strategy of proof:</u> Fix $x \in W$, $s \in S$ with $sx > x$.

Assume "everything" for all $y < xs$. (what "everything" is will become clear.)

Write: $\underline{H}_x \underline{H}_s = \sum f_z H_z \implies \underline{H}_x \underline{H}_s = \underline{H}_{xs} + \sum_{z<x} f_z(0) \underline{H}_z$.

$$S(xs) \iff B_x B_s \cong B_{xs} \oplus \bigoplus_{z<x} B_z^{\oplus f_z(0)} \quad \text{(all in degree 0)}.$$

Hence we want to show that B_z occurs with multiplicity $f_z(0)$ in $B_x B_s$ $\forall z < x$.

$\Updownarrow$ Exercises

rank of $(-,-)^{x,s}_y : \mathrm{Hom}(B_z, B_x B_s) \times \mathrm{Hom}(B_{xs}, B_z) \to \mathrm{End}(B_z) \overset{S(z)}{=} \mathbb{R}$

is equal to $f_z(0)$.

Soergel's Hom formula + asymptotic orthogonality $\implies f_z(0) = \dim \mathrm{Hom}^0(B_z, B_x B_s)$.

want to show non-degeneracy.

<u>Exercise:</u> By induction for any inclusion $B_x \overset{\oplus}{\subset} BS(\underline{x})$ the induced form on B_x is non-degenerate (and unique up to a scalar).

$\longrightarrow$ The induced form $\langle -,- \rangle_{B_x B_s}$ on $B_x B_s \overset{\oplus}{\subset} BS(\underline{x}) B_s = BS(\underline{xs})$ is non-degenerate.

Hence B_z and $B_x B_s$ have non-degenerate forms $\langle -,- \rangle_{B_z}$, $\langle -,- \rangle_{B_x B_s}$.

Given $\varphi : B_z \to B_x B_s$, $\varphi^* : B_x B_s \to B_z$ adjoint, i.e.

$$\langle \varphi(b), b' \rangle_{B_x B_s} = \langle b, \varphi^*(b') \rangle_{B_z}.$$

Gives identification $\mathrm{Hom}(B_z, B_x B_s) = \mathrm{Hom}(B_x B_s, B_z)$.

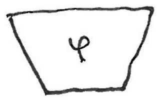
$(-,-)_z^{x,s}$ form on $\mathrm{Hom}(B_z, B_x B_s)$ "local intersection form".

<u>Embedding theorem:</u> The map $\varphi \mapsto \overline{\varphi(c_{bot})}$ defines an embedding

$$\mathrm{Hom}^\circ(B_z, B_x B_s) \overset{i}{\hookrightarrow} p_s^{-\ell(z)} \subset \left(\overline{B_x B_s}\right)^{-\ell(z)}.$$

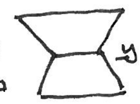
<u>Proof:</u>

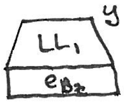
$$B_x B_s \overset{\oplus}{\subset} BS(\underline{xs})$$
$$\uparrow \varphi$$
$$B_z \overset{\oplus}{\subset} BS(\underline{z})$$

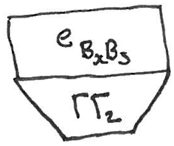


Assume $\varphi(\overset{c_{bot}}{\cancel{\ }})=0$. Then

<u>Ben's lecture:</u>

$$\varphi = \sum$$  $g_y \quad y < z$

But $S(z) \Rightarrow$  $=0$ if $\deg LL_1 \leq 0$.

and $\underline{H}_x \underline{H}_s = \sum f_z(0) \underline{H}_z \Rightarrow$  $=0$ if $\deg \Pi_2 < 0$.

$\Rightarrow \varphi = 0$ if φ is of degree 0.

$\Rightarrow \varphi(c_{bot}) = 0 \Rightarrow \varphi = 0.$

Similarly, $\varphi(c_{bot}) \in B_x B_s R^+ \Rightarrow \varphi = 0.$

Hence i is injective.

Now $\overline{B_z}^{-\ell(z)}$

$\ell(z)+2$	0
$\ell(z)$	$*$
$\vdots$	$**$
$\vdots$	$**$
$-\ell(z)$	$*$

$g^{\ell(z)+1} c_{bot} \in B_z R^+.$

Hence $g^{\ell(z)+1} \varphi(c_{bot}) = \cancel{\varphi} B_z R^+.$

$\quad \|$

$\Rightarrow \varphi(g^{\ell(z)+1} c_{bot}) \in B_z B_s R^+.$

$\Rightarrow g^{\ell(z)+1} c_{bot} = 0.$

Hence $\varphi(c_{bot}) \subset p_s^{-\ell(z)} \subset \left(\overline{B_x B_s}\right)^{-\ell(z)}.$

<u>Exercise:</u> $\langle \rho^{\ell(z)} c_{bot}, c_{bot} \rangle = N > 0$ "degree of Schubert variety"

Given $\alpha, \beta \in \mathrm{Hom}(B_z, B_{xs})$ we have $\langle c_{bot}, c_{top} \rangle_{B_z} = 1$. Hence

$$
\begin{aligned}
(\alpha, \beta)_z^{x,s} &= \langle \beta^* \circ \alpha (c_{bot}), c_{top} \rangle = \frac{1}{N} \langle \beta^* \circ \alpha (c_{bot}), \rho^{\ell(z)} c_{bot} \rangle \\
&= \frac{1}{N} \langle \alpha(c_{bot}), \rho^{\ell(z)} \beta(c_{bot}) \rangle = \frac{1}{N} (i(\alpha), i(\beta))_L^{-\ell(z)} .
\end{aligned}
$$

$\square$

Hence i is an isometry up to a scalar.

Because the restriction of a pos. def. form to a subspace stays definite we have

(HR) for $\overline{B_x B_s}$ $\Longrightarrow$ Soergels conjecture for B_{xs} !

Remember dCM: use some limiting argument.

<u>Proposition:</u> Let $B \overset{\oplus}{\subset} BS(\underline{w})$ be s.t. $\overline{B}$ satisfies (HR) wrt ρ.

Consider the Lefschetz operator on $B_x B_s \subset BS(\underline{w}s)$ given as follows

$$L_y := \rho \,\Big|\, \overbrace{\xi \cdots}^{\underline{w}} \,\Big|\, \overbrace{y\rho}^{s} \,\Big| \qquad \text{for } y \geq 0.$$

Then $\overline{BB_s} \subset \overline{BS(\underline{w}s)}$ satisfies (HR) for $y \gg 0$.

<u>Proof:</u> Explicit calculation. In the limit Lefschetz form on

$$\overline{B_x B_s} \rightsquigarrow \overline{B_x} \otimes V$$

$\uparrow$

$H^*(\mathbb{P}^1)(1)$ aka natural rep of $\mathfrak{sl}_2(\mathbb{R})$.

See exercises!
and our paper

$\square$