Last time: $D$ has a basis given by Clifford leaves $\mathfrak{g} \otimes \mathfrak{g}$.

For $w \in W$, let $R = \mathbb{C}[D]$. Then $\text{Hom}_D(w, x) = \mathbb{C}[D]$. For $v \in W$, $g \in \mathfrak{g}$, subrep for $v$.

Moreover, $D$ is filtered. For any ideal $I \subseteq W$, $\bigoplus_{v \in I} R$, $\mathfrak{g}$ is an ideal in the cat. This is because

$$\text{comp} \quad \text{factor-thru} \quad \Delta_{v'} = \bigoplus_{u \leq v} \Delta_{uv} = \sum_{u \leq v} \Delta_{uv} \quad \text{for} \quad u \leq v \Rightarrow u \in I$$

Usual ideals: $\mathfrak{g} \mathfrak{w}$ for $w \in W$. Let $w$ a rep for $w$. Then $\text{End}(w) = \mathbb{C}[D]$. For $\alpha \in \mathfrak{g}$, $\frac{d}{d\alpha} = 1_{\mathfrak{g}} \otimes R$.

This is FANTASTIC.

Given a complete control over $D / D \mathfrak{w} 1$,

$$\Psi \quad \varphi = ?$$

$$\Psi \quad \varphi = 1_{\mathfrak{g}} \otimes f + \text{lower terms}, \text{so} \quad \Psi \quad \varphi = \frac{d}{d\alpha}$$

This is a pairing on $M(\Psi, \varphi) = \text{subrep of } \mathfrak{g}$ for $w$.

The assignment is controlled by $\varphi$ for all $\varphi$. This kind of filtered category is known as an (object adapted) cellular category, has lovely properties. Let's use them!

Claim: For any rep $w \in \mathfrak{g}$ indecomp $Bw \subseteq w$ sit. $Bw \otimes \mathfrak{g}$ for $\mathfrak{g}w$. (In $\text{Kar}(D)$)

Pf: $1_w = e_{-2} = \cdots$ ortho. decp. If two had nonzero coeff of $1_w \otimes e_{-1}$, then $e_{-1} \in \mathfrak{g}$.

So $1$ has nonzero coeff (in fact, coeff 1). Call it image $Bw$.

Rmk: Any indecomposable $R$-linear rep $w$ of $\mathfrak{g}$ Hom spaces (in each degree) is Krull-Schmidt, things split into indecomposable in unique way, and $\text{End}(\text{Indecomps})$ is a local ring.
Claim 2: Sparse \( x \) any \( \exists a, b, c, M(x, w) \) s.t. \( y^a(b, c) \in \mathcal{E} \) \( \text{deg } a = d, \text{deg } b = -2 \)

Then \( B_a(-d) \in \mathcal{E} \).

**Pf:** \( b \frac{\omega}{\omega} x = \frac{1}{k} 1 + \text{lower terms} \Rightarrow \frac{e}{e} \frac{a}{a} = \frac{1}{k} 1 + \text{lower terms} \)

Now \( \frac{a}{a} \frac{b}{b} \in \mathcal{E} \) \( \Rightarrow \frac{b}{b} \frac{a}{a} \in \mathcal{E} \) \( \text{comp. is invertible in } \text{End}(B_a) \)

Why? \( B_u \text{ indecomp} \Rightarrow \text{End}(B_u) \text{ is local, if it were in max. ideal, would still be in max. ideal in } \mathcal{A} \text{, but its invertible there.} \)

**Thus** \( B_a(-d) \in \mathcal{E} \) \( \Rightarrow \text{Recursively, also get } B_a(+d) \in \mathcal{E} \) !

**SCF:** These \( B_u \) don't depend on \( \text{rex chain} \), get all indecomps up to shift.

**Pf:** Let \( x \) arbitrary, \( E \in \text{End}(A) \) a primitive indecomp, \( E = \sum \frac{a}{a} \frac{b}{b} \frac{c}{c} \frac{d}{d} \)

choose \( w \) with \( f_{a,b} \neq 0 \) (not nec. unique). \( \text{ETS } \text{Im}(E) \ni B_a(-d) \)

work in \( A \frac{\omega}{\omega} \text{ (natter / } 2 \text{ \neq } w) \)

\( E^2 = \sum \frac{a}{a} \frac{b}{b} \frac{c}{c} f_{a,b} f_{c,d} = \sum \frac{a}{a} f_{a,b} f_{c,d} \)

so \( \sum \frac{a}{a} f_{a,b} f_{c,d} = f_{a,b} \).

If all \( \psi(c, d) \in R+ \text{ (or 0) } \)

then so are all \( f_{a,b} \), but no indecomp is in max. ideal. So some \( \psi(c, d) \in R^+ \).

But this only says \( B_u \in \mathcal{E} \). Want \( B_u \in \text{Im}(E) \), i.e., want

\( \frac{e}{e} \frac{a}{a} \frac{b}{b} \frac{c}{c} \)

with \( \frac{e}{e} \frac{a}{a} = \frac{1}{k} 1 + \text{lower terms} \)

but \( E^3 = \sum \frac{a}{a} \frac{b}{b} \frac{c}{c} \frac{d}{d} \)

\( \Rightarrow \frac{e}{e} \frac{a}{a} \frac{b}{b} \frac{c}{c} \frac{d}{d} \), use same argument.
So how do we find $B_w^e$?

$e = \mathbf{1}_w - \sum \text{other idempotents}$

Find other idempotents by finding nondegenerate parts of $\mathcal{P}_{\text{hom}}$.

**Ex:** $x = st5$. All $LL$ are nondegree except $LL_{111}$ and $LL_{100}$

Look at $LL_{5,5} = \{ \text{something} \}$

$$\mathcal{P}(\psi, \psi) = \frac{\psi(\psi)}{\psi} = \frac{\psi(\psi)}{\psi} = \frac{\psi(\psi)}{\psi} = a_{st}$$

So when $a_{st} \neq 0$, get an idempotent

$$\frac{1}{a_{st}}$$

(over $\mathbb{R}$, this is full $\text{Mat}(2)$)

over $\mathbb{F}_2$ in type $B_2$, interesting.

**Ex:** $x = tsut$. No other $LL$ maps have degree 50, indecomposable.

**Ex:** $x = tuv$ or $tuv$ consider $\mathcal{P}_{\text{hom}}$

8 maps: degree -2

degree 0

degree 2

degree 4

pair degree -2 against +2:

pair degree 0

pair $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

$\det = -2$

So $B_{tuv} \subset C_B x$ (but in characteristic 2, $B_{tuv} \subset C_B x$ so $B_x$ is bigger!)