Lecture 2.1: The classical approach to Soergel bimodules

Def: "classical" = known to more than two people for more than two years.

Recall: \( \mathcal{L} \) is a realization of \( W \) (e.g. geometrical rep)
\[ R = S(\mathcal{L}^*) = \text{polynomial functions on } \mathcal{L} \otimes W. \]

Standard bimodules: For any \( x \in W \) consider \( R_x \in R\text{-Bim} \) defined as follows
\[ R_x \cong R \text{ as left } R\text{-module}, \quad m \cdot r = x(r)m \text{ for } m \in R_x, \quad r \in R. \]

\( R_x \) can be viewed as a completely porous wall!

Obviously \( R_x \otimes_R R_y = R_x R_y \in R\text{-Bim}. \) Also
\[ \text{Hom}(R_x, R_y) = \begin{cases} R & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \]

\( \text{Std.Bim} = \text{full } \otimes_R (n) \text{ subcat of } R\text{-Bim} \) generated by \( \{ R_x | x \in W \}. \)

Then \( \text{Std.Bim} \cong R\text{-groupoid of } W \text{ over } R. \)

Remark: One can draw \( \text{Std.Bim} \) as in Ben's lecture. Only difference now is that \( \text{End}(R_x) = R \), hence one has polynomials in boxes.

Soergel bimodules: \( \text{S.Bim} \subset R\text{-Bim} \) is the full \( \otimes_R (n) \text{ Karoubian subcat} \)
\[ \text{generated by } B_S := R \otimes_R R^1. \]

Hence objects of \( \text{S.Bim} \) are isomorphic to sums of summands of \( B_S(w) = R \otimes_R R \ldots \otimes_R R^m \).

Verbal: associated to each direct sum simple reflection one has a Frobenius object, one looks at the full subcategory generated by these!

Example: a) If \( W = S_2 \), we have seen \( B_3 B_3 \cong B_5(1) \otimes B_3(-1). \) Hence
\[ \text{ind. Soergel bimodules } / \cong 0 = \{ R, B_5 \}. \]

b) if \( W = S_3 \), \( R, B_3, B_3, B_3 = B_3 B_3, B_3 = B_3 B_3 \) are indecomposable. (Ben's lecture)

Exercise: \( B_3 B_3 B_3 \cong B_3 B_3 \otimes B_3, \quad B_3 B_3 B_3 \cong B_3 \otimes B_3. \) Hence (give me argument verbally)
\[ \text{ind. Soergel bimod. } = \{ B_3 \text{id} = R, B_3, B_3, B_3, B_3 \}. \]
$\mathbb{B}im$ is not abelian. For example one has exact sequences:

$$\begin{align*}
\mathbb{Bim} & \to A, & & \mathbb{B}im \to \mathbb{B}im
\end{align*}$$

If we tensor these exact sequences together we see that any Böhm-Schöbel bimodule has a filtration $0 \subset \mathbb{B}^{2} \subset \cdots \subset \mathbb{B}^{m} = \mathcal{B}(w)$ s.t. $\mathbb{B}^{m}/\mathbb{B}^{m-1} \cong \bigoplus \mathcal{R}_{x}(?)$'s.

In general the order in which summands $\mathcal{R}_{x}$ appear has nothing to do with the Bruhat order.

Fix an enumeration $w_{0}, w_{1}, \ldots$ of $W$ s.t. $w_{i} \leq w_{j} \Rightarrow i \leq j$.

Def: A standard filtration on a Soergel bimodule is a filtration $0 = \mathbb{B}^{0} \subset \mathbb{B}^{1} \subset \cdots \subset \mathbb{B}^{m} = \mathcal{B}$ s.t. $\mathbb{B}^{i}/\mathbb{B}^{i-1} \cong \bigoplus \mathcal{R}_{x}^{h_{x_{i}}}$ for some $h_{x} \in \mathbb{N}[v^{\pm 1}]$.

Notation: $p = \sum a_{i}v^{i} \in \mathbb{N}[v^{\pm 1}]$, $M \otimes p = \bigoplus (M(-i) \otimes i)$.

Soergel: any Soergel bimodule admits a unique standard filtration.

Exercise! (it's a support calculation)

$$\begin{align*}
\text{ch} : \mathbb{B}im & \to \mathcal{Q} \text{ : } B & \to \sum_{x \in W} v^{\ell(x)}h_{x} \mathcal{R}_{x}.
\end{align*}$$

Eg: $\text{ch}(\mathcal{B}) = H_{S}$.

Localization: Let $Q$ denote the fraction field of $R$.

Lemma: $\mathbb{B}(w) \otimes Q = \mathbb{R} \otimes R_{S} \otimes_{R_{S}} \mathbb{R} \otimes R_{S} \otimes Q \cong Q \otimes Q \otimes \cdots \otimes Q$

Hence $\mathbb{B}(w) \otimes Q$ is actually a $Q$-bimodule.

Proof: Consider the inclusion $\mathbb{R} \otimes R_{S} \otimes Q \lhd \mathbb{R} \otimes Q$. Enough to show it is an iso.

We have $\frac{1}{r_{s}^{t}} \otimes 1 = s(t) \otimes \frac{1}{f_{s}(t)} \in \text{im } i$. Hence $i$ is an isomorphism. $\square$
It follows that localization gives a monoidal functor
\[ \mathcal{B} \text{im} \to \mathcal{Q} \text{-} \mathcal{B} \text{im}. \]

Also, both sequences split (a) and (v). (In fact they split each other!) In fact, the standard filtration on any Soergel bimodule splits after localization.

**Remark:** Localization categories have specialization \( v \mapsto 1 \).
\[ \mathcal{B} \text{im} \xrightarrow{ch} \mathcal{Q} \]
\[ \downarrow \]
\[ \mathcal{B} \text{im}_v \xrightarrow{ch} \mathcal{Q}_v \]

**Szegedy's categorification theorem:**

\[ [\mathcal{B} \text{im}] \] Split Grothendieck group of \( \mathcal{B} \text{im} \): generated by symbols \( [M] \) for \( M \in \mathcal{B} \text{im} \).

- **Ring:** \[ [M][N] = [M \otimes N] = [M \otimes N] \]
- **\( \mathbb{Z} [v, v^{-1}] \) algebra:** \[ v[M] = [M(-1)] \]

Thus a) \( \exists! \) homomorphism \( H \xrightarrow{c} [\mathcal{B} \text{im}]: H_\mathcal{B} \to [\mathcal{B} \mathcal{S}] \).

b) One has a bijection:
\[
\left\{ \text{indecomposable } \begin{array}{l}
\{ \text{Soergel bimodules} \} \\
\{ \text{ch} \}
\end{array}
\right\}_{(w), \sim} \xleftrightarrow{c} W
\]

Moreover \( B_w \) is the unique summand of \( BS(y) \) which does not occur as a summand \( BS(z) \) for any \( z \) with \( l(z) < l(w) \).

c) \( c \) is an isomorphism with inverse \( ch \). For any \( B, B' \in \mathcal{B} \text{im} \) \( \text{Hom}(B, B') \) is a free left (or right) \( R \)-module of graded rank.

Soergel's conjecture: \( \text{ch}(B) = H_{B} : s(x) \) (implies KL positivity + KL conjecture).

Assume Soergel's conjecture. Check:
\[
(H_x, H_y) \in \left\{ \begin{array}{l}
1 + vZ \mathbb{N} \quad \text{if } x = y \\
vZ \mathbb{N} \quad \text{if } x \neq y.
\end{array}
\right.
\]

"asymptotic orthogonality"

Hence if Soergel's conjecture implies that
\[
\text{Hom}^0(B, B) = \left\{ \begin{array}{l}
R \quad \text{if } x = y, \\
0 \quad \text{otherwise}.
\end{array}
\right.
\]

Exercise: This is if and only if.