1.4 Part I: Diagrammatic for Categories

We use planar diagrams to describe morphisms between (single) forgetful functors, but this is an accident. Planar diagrams are precisely the tool for the job.

**Baby case:** Linear Diagrams for (1-Categories)
- You're familiar with $P \xrightarrow{f} N \xrightarrow{M}$
- Objects fill a pt, morphisms a line. Let's take dual picture.
- Same data, but has some apparent positioning.

**In picture:** A (generic) pt is an object
- A (closed) interval is a morphism $f$ from RHS to LHS

**Composition:** $[a \xrightarrow{f} b] \circ [b \xrightarrow{g} c] = [a \xrightarrow{g \circ f} c]$ is $1_M$

**Axioms of a category $\iff$ Diagram** (up to linear isotopy) weakly/loosely represents a morphism (ie. could use positioning to keep track of parents but no need)

**Planar Diagrams for 2-Cats**
- **Old way** (2-cat of cats)
- **New way** (2-cat of 2-cats)

**pt \leftrightarrow object**
- **hori line $P \xrightarrow{f} C \xrightarrow{g}$ \leftrightarrow 1-mor $f$, some version 1-cat, $F \xrightarrow{1} = 1_C$
- **rectangle $P \xrightarrow{f} C \xrightarrow{g}$ \leftrightarrow 2-mor bottom to top:** $D \xrightarrow{f \circ g} F = 1_F$ $C \xrightarrow{1} = 1_A$

**Axioms of 2-cats $\iff$ Diagram** (up to rectilinear isotopy) weakly/loosely given as morphism

**Examples:**
1. What is an algebra in a monoidal cat? $\text{2-cat of one object}$ $\text{An object}$ $\text{a morphism}$ $\text{equipped with}$ $\text{st}$ $\text{and}$ $\text{and}$
2. What is a Frobenius object? 

\[ Y = Y \]  

and \[ \Rightarrow Y = Y = Y \]  

then \[ \Rightarrow N \Rightarrow Y = Y = Y \]  

(\text{Can view diagrams up to isotopy.})

3. Frobenius extension? \[ B \downarrow A \rightarrow B \]  

\[ \Rightarrow \text{graded: } \]  

[\text{To make an } \text{graded:}]

Satisfying \[ \Rightarrow B \downarrow A \Rightarrow A \Rightarrow B \]  

\[ B \Rightarrow A \Rightarrow B \]  

\[ B \Rightarrow B \Rightarrow B \]  

\[ B \Rightarrow B \Rightarrow B \]  

4. When \[ E \Rightarrow E \Rightarrow E \Rightarrow E \]  

\[ E \Rightarrow E \Rightarrow E \Rightarrow E \]  

If drawn \[ E \Rightarrow E \Rightarrow E \Rightarrow E \]  

then \[ E \Rightarrow E \Rightarrow E \Rightarrow E \]  

\[ E \Rightarrow E \Rightarrow E \Rightarrow E \]  

If biadjoint, also \[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

However, if \[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

If they are equal, \[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

is called \text{cyclic}. Can draw cycle as \[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

Axioms of biadjunction \[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]  

Diagram (up to true isotopy) \text{unsuspectingly represents} \text{a } 2\text{-morphism.}

Given such a category, you should use isotopy classes of diagrams.

\text{Remark: All 2-morphisms are cyclic when "taking biadjoints" is actually functional.}

Common situation in geometry + convolution categories.

In lectures to come, we'll show you how to draw morphism \[ E \Rightarrow E \] \text{Frobenius-Sweedler bimodule}.

You can already draw a lot for \text{Bimod} \Rightarrow \text{Bimod} \Rightarrow \text{Bimod}.
Let's draw another monoidal category.

Def: let $G$ be a group. The $2$-groupoid of $G$ is the monoidal category with objects $g \in G$ and $g \cdot h = gh$. Only morphisms are identity maps.

So, for instance, there is a map $\begin{array}{c} g \\ \downarrow \\ gh \\ \downarrow \\ g'h \\ \downarrow \\ gh' \end{array}$ satisfying $\begin{array}{c} 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \end{array}$ etc.

However, when $G$ has a presentation with generators, relations want to abuse that to simplify degree.

Ex: $G = \langle W, S \rangle$ a Coxeter gp. Generated by $s_0, s_1, s_2$. Since $s_i^2 = 1$ have maps $\begin{array}{c} s_0 \\ \downarrow \circ \downarrow \\ s_1 \\ \downarrow \circ \downarrow \\ s_2 \end{array}$ with $N = \{u \in W \}$ and $U = \{1, s_0, s_1, s_2, s_0s_1, s_0s_2, s_1s_2 \}$.

Since $s_0 s_1 s_0 = s_1 s_0 s_1$, have maps $\begin{array}{c} s_0 \\ \downarrow \circ \downarrow \\ s_1 \\ \downarrow \circ \downarrow \\ s_0 \end{array}$ s.t. $\begin{array}{c} s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \end{array}$ and other ones.

Are there any more relations? Sure!

Spoke $s_0 s_1 s_0 s_1 = 2$. Two maps $\begin{array}{c} s_0 \\ \downarrow \circ \downarrow \\ s_1 \\ \downarrow \circ \downarrow \\ s_0 \end{array}$.

Spoke $s_1 s_0 s_1 s_0 = 2$. Two maps $\begin{array}{c} s_1 \\ \downarrow \circ \downarrow \\ s_0 \\ \downarrow \circ \downarrow \\ s_1 \end{array}$.

Two maps $\begin{array}{c} s_0 \\ \downarrow \circ \downarrow \\ s_1 \\ \downarrow \circ \downarrow \\ s_0 \end{array}$.

Two maps $\begin{array}{c} s_1 \\ \downarrow \circ \downarrow \\ s_0 \\ \downarrow \circ \downarrow \\ s_1 \end{array}$.

$= \begin{array}{c} \circ \downarrow \\ \circ \downarrow \\ \circ \downarrow \\ \circ \downarrow \\ \circ \downarrow \end{array}$

but there can be only one, so relation $\begin{array}{c} s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \end{array}$.

"Zamolodchikov"

Thm (E-W): The following is a diagrammatic presentation for the $2$-groupoid of $(W, S)$ for any Coxeter gp.

Generators: $\begin{array}{c} e_0 \\ \downarrow \\ e_1 \end{array}$ Relation: $\begin{array}{c} e_0 \\ \downarrow \\ e_1 \end{array}$ $\begin{array}{c} e_1 \\ \downarrow \\ e_0 \end{array}$

3rs: One such relation for each finite rank $3$ Cox sub gp. Equality b/w distinct paths in boundary.

Idea: For any $w$, let $f_w$ be the reduced expression graph; vertices = reduced expressions, edges = braid relations.

Any path gives a morphism; any loop better is equal to identity. Draw each row in Zam above.

Try! Keep: $\begin{array}{c} 1 \\ \downarrow \\ 1 \end{array}$. Eliminate loops; all go by Zams.

What about non-reduced expressions? Toda? We proved using topology of Cox complexes.