...cture 1.4 Part I: Diagrammatics for Categories

We've defined monoidal SBim, 2-cat SSBim.

We use planar diagrams to describe morphisms b/w (Singular) Soergel bimods, but it's no accident! Planar diagrams are precisely the tool for the job.

Baby case: **Linear Diagrams for (1-)categories** You're familiar w/ $P \xleftarrow{g} N \xleftarrow{f} M$

objects fill a pt, morphisms a line. Let's take dual picture.

Same data, but has some apparent positioning. $P \xleftarrow{g} N \xleftarrow{\quad f \quad} M$

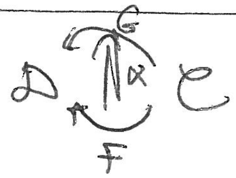
In picture: A (generic) pt is an object

A (stop saying it) interval is a morphism  from RHS to LHS

Composition  identity  is 1_M

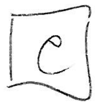
Axioms of a category $\longleftrightarrow$ Diagram (up to linear isotopy) unambiguously represents a morphism

(i.e. could use positioning to keep track of parens, but no need)

Planar Diags for 2-cats Old way (2-cat of cats)  New way 

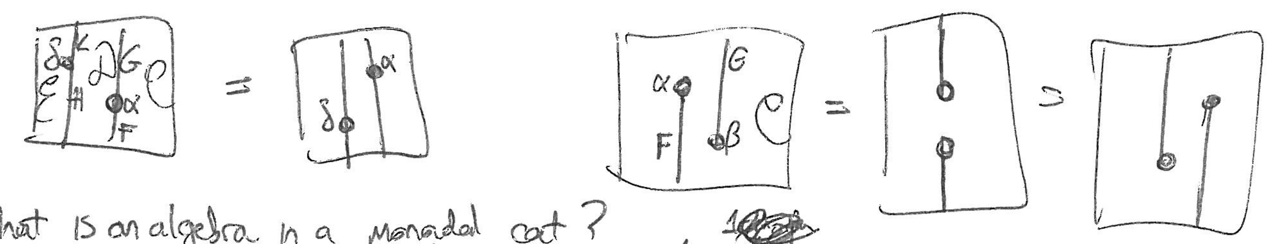
pt $\longleftrightarrow$ object

horiz line  $\longleftrightarrow$ 1-mor, same rules as above,  $= 1\!\!1_{\mathcal{C}}$

rectangle  $\longleftrightarrow$ 2-mor bottom to top.  $= 1_F$  $= 1_{1\!\!1_{\mathcal{C}}}$

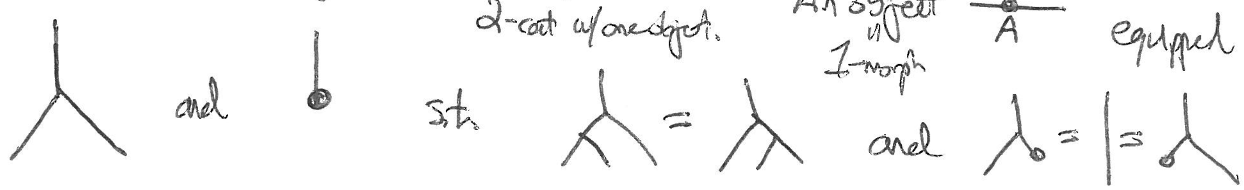
Compose horiz or vertically.

Axioms of 2-cats $\longleftrightarrow$ Diagram (up to rectilinear isotopy) unambiguously gives a 2-morphism

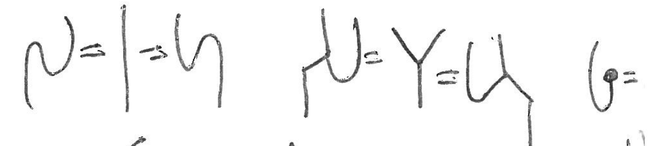


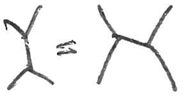
Examples: ① What is an algebra in a monoidal cat?
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ‖
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ 2-cat w/ one object.

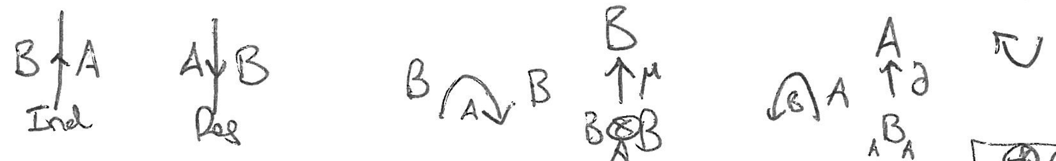
An object ($=$ 1-morph) A equipped with



2) What is a Frob alg object?  s.t. algebra, coalg,

and  then 

Assoc $\Longleftrightarrow$  (can view diagrams up to isotopy!)

3) Frobenius extension? 

satisfying  (again, isotopy!)

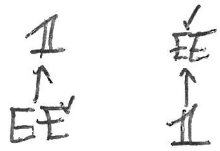
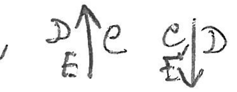
+ a bit more

To make graded: deg ∩ ∪ = ... deg ∩ ∪ = ...

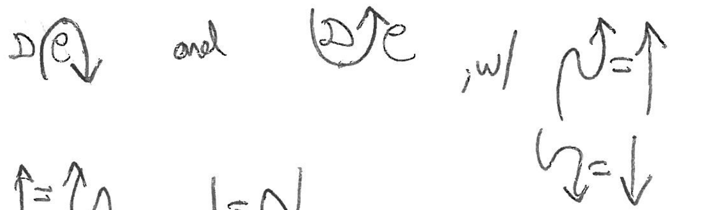
Examples I said $B \otimes_A B$ a Frob alg object in B-bimod

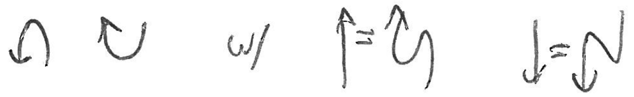
$\Updownarrow$

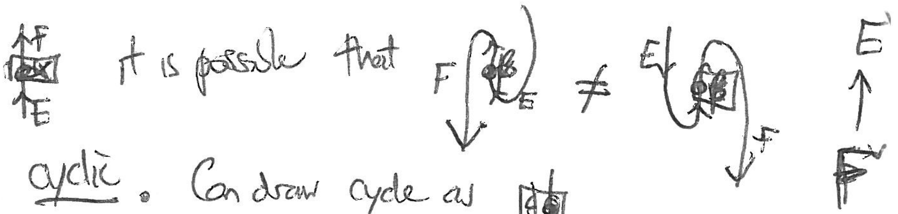
 etc.

4) When $E \dashv E^\vee$ we have  If draw  then

$\mathcal{D} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{C}$



If biadjoint, also 

However, if E biadjoint to $F^\vee$, $F \xrightarrow{\;f\;} F^\vee$ it is possible that 

If they are equal, β is called cyclic. Can draw cycle as 

Axioms of biadjunction + cyclicity $\longleftrightarrow$ Diagram (up to true isotopy) unambiguously represents a 2-morphism.

(If all 1-morphisms have duals and all 2-morphisms cyclic)

Given such a category, you should use isotopy classes of diagrams.

Rmk: All 2-morphisms are cyclic when "taking biadjoints" is actually functorial. Common situation in geometry + convolution categories.

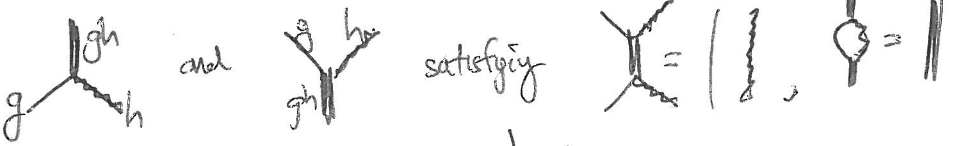
In lectures to come we'll show you how to draw morphisms b/w Bott-Samelson bimodules. You can already draw a bit for $B_s \otimes B_t \otimes \ldots \otimes B_s$

Lecture 14 Part II

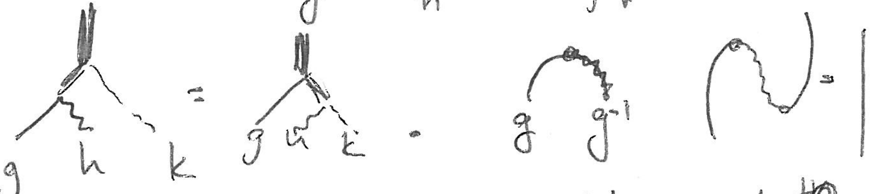
Let's draw another monoidal category.

Def: Let G be a group. The 2-groupoid of G is the monoidal category w/ objects $g \in G$ and $g \otimes h = gh$. Only morphisms are identity maps.

So, for instance, there is a map

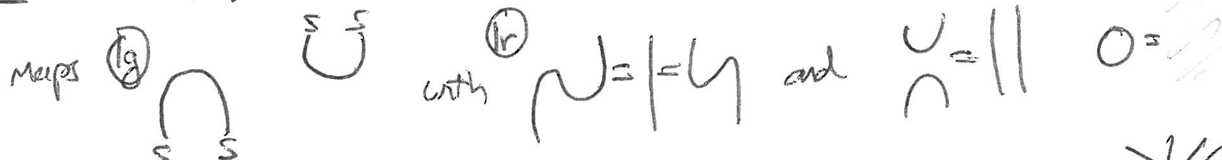


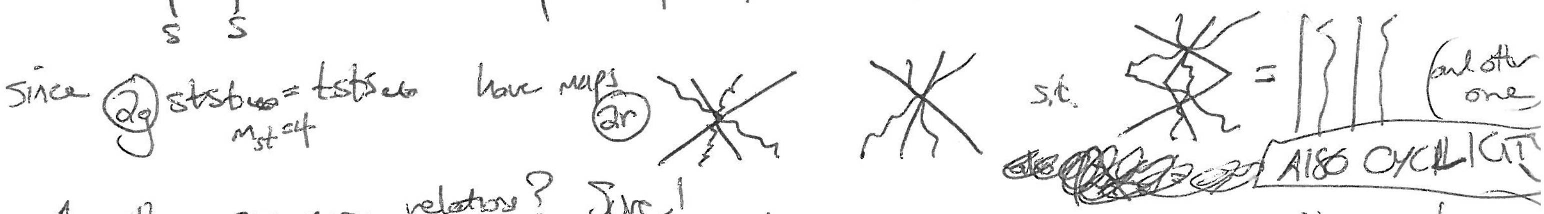
and



However, when G has a presentation w/ gens + relations want to abuse that to simplify diagrams

Ex: $G = (W,S)$ a Coxeter gp. Generated by $s \in S$. Since $s^2 = 1$ have





Are there any more relations? Sure!

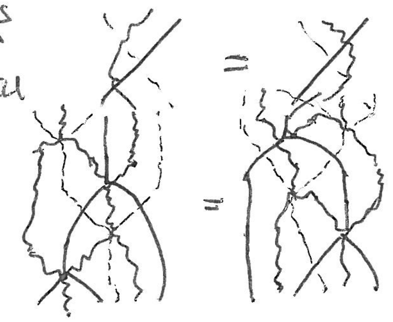
Spose $m_{st} = m_{tu} = m_{su} = 2$. Two maps

uts
↑
stu

but there can be only one, so relation

Spose  . Two maps

utu stu = w_0
↑
stsuts = w_0



"Zamolodchikov"

Thm (E-W): The following is a diagrammatic presentation for the 2-groupoid of (W,S) for any Coxeter gp

Generators: 1g 2g

Relations: 1r 2r

3r: One such relation for each finite rank 3 Cox subgp

Equality b/w distinct paths in longest element

Idea: For any w, let Γ_w be the reduced expression graph; vertices – reduced expressions, edges – braid relations

Ex: each row in Zam above.

Any path gives a morphism any loop better be equal to identity.

Trivial loops:  $=$ || ; Fact: Nontrivial loops all gen by Zams.

What about non-reduced expressions. Trickier. We proved using topology of Coxeter complexes.