Lecture 1.3: Soergel Bimodules

The cat. $\text{SBim}$ of Soergel Bimodules is an algebraic, "combinatorial" cat of $\mathcal{H}$. Fairly easy to define & play with - it amounts to the $\infty$-depth study of the reflection reps $\mathcal{H}$. Fix $(W,S)$. Well define $\Lambda^+$ using the symmetric (generalized) Cartan matrix $A$ of $(W,S)$.

$$A = \begin{pmatrix} 2 & a_{st} \\ a_{ts} & 2 \\ a_{tt} & a_{tt} \end{pmatrix} \quad \text{with } a_{st} - a_{ts} = -2 \cos \frac{\pi}{M_{st}} \quad \text{for } M_{st} \neq 0$$

\[\text{or for } M_{st} = 0, \quad a_{st} - a_{ts} = \pm 2 \text{ is an option, as is anything suitably generic.}\]

See exercises.

Let $\Lambda^+$ be the $1\text{SL-dim } W/\mathbb{R}$ spanned by $\langle \alpha_j \rangle_{j \in S}$, called simple roots.

$W\text{C}(\Lambda^+)$ by $s(\alpha_t) = \alpha_t - a_{st} \alpha_s$ so $s_j : \Lambda^+ \to \Lambda^+$.

\[s_j : s(\alpha_j) = \alpha_j + 2 \cos \frac{\pi}{M_{st}} \alpha_s \quad \text{is } \text{a technically part - in fact, many other versions of } \Lambda^+ \text{ will}\]

do (not-symmetric matrices, extension to $\times 1\text{SL-dim, etc.)}. \text{This will suffice for our purposes.}\]

From $\Lambda^+$, we will extract a bunch of interesting graded commutative rings.

Def: Let $R = \text{Sym}(\Lambda^+) = \mathbb{R}[\alpha_j]$ (a poly ring), graded w/ depth $= 2$. WGB.

For $J \subseteq S$, $W_J \subseteq W$ consider $R^J = R^{\Lambda_+^J}$ invariants. (Defined for all $J$, but we're really only interested when $W_J$ is finite, i.e. $J$ is frozen, for reasons you'll see.)

Ex: $W = S_p, C = \text{TR}[x_1, \ldots, x_n]/(x_i - 0) \quad \text{as in Remark, can ignore this.}$

$W_1 = S_3 \times S_2 \times S_1$ \quad $R_1^2 = \mathbb{R}[x_i, x_j, x_k, x_l, x_m, \alpha_i, \alpha_j, \alpha_k, \alpha_l, \alpha_m, \alpha_n]$. \quad \text{is still a poly ring}

Key Ex: $R_1^S$, $R_1^R$ for fixed $s \in S$. We have $\alpha_i \in \mathcal{R}^S$. Since $\alpha_i \leftrightarrow \alpha_i + 2 \cos \frac{\pi}{M_{it}} \alpha_s$, we see $\alpha_i + 2 \cos \frac{\pi}{M_{it}} \alpha_s \in \mathcal{R}_1^S$.

In fact, $R_1^S = \text{TR}[\mathcal{R}_1^S, \alpha_i + 2 \cos \frac{\pi}{M_{it}} \alpha_s]$. \quad $R_2^1 = R_1^S \otimes R_1^{S_2}$, but here $R_2 = R_1^S \otimes R_1^{S_2}$. Any $f \in g + h \alpha_s, \quad g \in R^S, \quad h \in R_1^{S_2}$, so in fact, $R$ is free over $R^S$ w/ basis $\{ 1, \alpha_1^\perp \}$, $R^1 = R^S \otimes R^{1_2}$ as graded $R^S$-mod.

Easy way to find coeff: g1h: Demazure operator: $d_\alpha(f) = \frac{f - sf}{\alpha_s}$, kills $R^S$. $d^2 = 0$, exact. On $\Lambda^+$, get $S$ column of $A$.

\[d_\alpha : R \to R^S, \quad d_\alpha \text{ is } R^S\text{-linear, and kills } R^S. \text{ } d^2 = 0. \text{exact.} \quad \text{On } \Lambda^+ \text{ get } S \text{ column of } A\]
\[ f = g + h \mu s \]
\[ h = \frac{1}{a} \partial_s(f) \]
\[ g = \frac{1}{a} \partial_s(f) \mu s \]

- **Twisted Leibniz rule**
  \[ \partial_s(fg) = \partial_s(f)g + sf(fg) \]

- The pairing \((fg) \rightarrow ds(fg)\)
  is perfect. I.e. the bases \[ \{1, \frac{a}{s}, \frac{a}{s} 1\} \] are dual.
  \[ ds(a; b) = \delta_{ab} \] \[ \Rightarrow R^s \otimes R \text{ is a graded Frobenius} \]

What can we do with this ring ext?

**Def.** Let \( B_\omega = R \otimes R(1) \) be an \( R \)-bimodule.

- as a porous well
- \[ \sum f \frac{g}{g} \]
- that only \( s \)-symmetric poly can osmose through.
- well make this notation more precise soon.

If \( f = g + h \mu s \) then \( \text{leaf} = \text{gol} + 1 \text{hool} \). As left \( R \)-mod, \( B_\omega \) has basis:

\( \{1, \frac{a}{s} 1\} \)

- **Def.** A **Bott-Samelson bimodule** is
  \[ B_\omega = B_\omega \otimes R \otimes \cdots \otimes B_\omega = R \otimes R \otimes \cdots \otimes R \]

- **Exerc.** a) **Generalize** argument above. \( B_\omega \) has basis as left \( R \)-mod given

- by \( \{1 \otimes a \otimes s^t \otimes \cdots \otimes s^{u} \} \) for \( s \in E_\omega \)

b) \( B_\omega \otimes B_\omega \) can slide anything out of middle, since \( \mu_s c \mu^w + (s)^t \) (except \( a_{st} \neq 2 \))

\[ \begin{align*}
\begin{cases}
\text{as an } R\text{-bimodule, generated by } 1 \otimes 1 \\
\text{So } \exists \text{ surjective map } R \otimes R(2) \rightarrow B_\omega \otimes B_\omega.
\end{cases}
\end{align*} \]

- Inverse map?
  - When \( m_s = 2 \), yes!
  - When \( m_s + 2 \), no!

- How to slide at \( 1 \otimes 1 \) ?

- **Exerc.** c) Make this decom explicit:

- **Def.** A **Sergel bimodule** is a \((\oplus, \odot, \omega)\) of a summand of a **Bott-Samelson bimodule**.

Forms a full monoidal subset of \( R\text{-bimodules.} \)
The theory of Frobenius extensions:

**Def:** A (commutative) ring ext \( A \rightarrow B \) is Frobenius if \( E : B \rightarrow A \) is \( A \)-linear \( \text{trace map} \) and if \( B \) is free finite rank \( A \)-module with dual bases \( E_x \)'s \( E_y \)'s s.t. \( \varphi(x, y) = \delta_y \).

Frobenius extension \( \Rightarrow \) Frobenius reciprocity holds.

The bimodule \( \text{Ind}_A B \) gives functor \( \text{Ind}_A B : \text{Mod}_A \rightarrow \text{Mod}_B \) \( \text{"Induction"} \)

\[ \text{Res}^B_A : \text{Mod}_B \rightarrow \text{Mod}_A \] \( \text{"Restriction"} \)

For any ring ext \( \text{Ind} \rightarrow \text{Res} \), i.e.

\[ \text{Hom}_B(\text{Ind} M, N) \cong \text{Hom}_A(M, \text{Res} N) \]

determined by unit + counit of adjunction, \( \text{co-unit} \) \( \text{Hom}_B(\text{Ind} \text{ Res} M, N) \cong \text{Hom}_A(\text{Res} M, \text{Res} N) \Rightarrow \text{unit} \)

get natural terms \( \text{Ind} \text{ Res} \rightarrow \text{Id}_{\text{Mod}_B} \)

\[ \& \text{Res} \text{ Ind} \rightarrow \text{Id}_{\text{Mod}_A} \]

just multiplication, \( \text{Res} \rightarrow \text{Ind} \)

Unit: Similarly get map \( \text{Res} \rightarrow \text{Id}_{\text{Mod}_B} \)

For Frobenius ext, also get \( \text{Res} \rightarrow \text{Ind} \). Defined by \( \text{Res} A \rightarrow A \)

Said another way, \( B \otimes_A B \) is a Frobenius object in \( \text{Mod}_B \)

Our example: \( A = \mathbb{Z} \), \( B = R \), \( R \otimes_A R = \mathbb{Z} \).

**Rmk:** Every graded version of all this business: \( \text{Ind}, \text{Res} \) commute up to shift by \( l \).

deg \( \varphi = -2l \) then call ext degree \( l \).

Next lecture and defn.

**Thm:** If \( I \) is finitary, \( R \otimes_I \) is a Frobenius object, \( R \otimes IC \) \( R \) is a Frobenius ext. of degree \( l \).

To get the trace \( R \otimes_I : \text{Claim:} \) satisfy braid relation.

\[ \varphi \Rightarrow \text{defn for any } w \in W. \text{ Gysin's trick} \]

Clearly \( \text{Im}(\varphi) \cap \text{Im}(\psi) = \varphi \)

\( b/c \) can choose rel. exp with \( s \) on left.

**Rmk:** For \( I \) finitary, \( R \otimes IC \) is Frobenius of degree \( l(a_y) - l(a_z) \), \( \varphi \Rightarrow \text{associativity} \)

**Def:** Singular Seung-ki Bimodules are \( \otimes, \otimes_{(n)}, \otimes_{(1)} \) of \( \text{Ind} \rightarrow \text{Res} \) for finitary \( I \)

More precisely, 2-Act: \( \text{Ob: } I \text{-finitary} \)

\[ \text{1-Meil: } K^{(n)} : \otimes K^{(n)} \otimes K^{(n)} \]

2-Meil: Bimodule maps.

and similar, etc.
If time

Ex. \( S = \frac{3}{2}, 3 \) \( m \leftrightarrow \lambda \cdot M. \)

\( Z = \alpha_s^2 + \alpha_s \alpha_t^2 + \alpha_t^2 = \alpha_s^2 + \alpha_t^2(\alpha_s \neq 0 \alpha_t) \in R^{st} \)

a) If \( m = \infty \), \( \alpha_s \neq \pm 2 \) (i.e., \( z \) is not a square) then \( R^{st} = R[z] \)

b) If \( m = \infty \), \( \alpha_s = \pm 2 \) then \( R^{st} = R[\alpha_s + \alpha_t] \)

either way, wrong transcendene degree, \( R^{st} \nsubseteq R \) is NOT a finite extension.

c) \( m < \infty \). Can define positive roots \( \Omega^+ \): Note that \( \omega_{st}(w) = w \), the collection of \( \{ \cdot \omega_{st}(\alpha_i) \}_{\alpha_i \in \Omega^+} \) = \( \Omega^+ \), full \( W \) orbit is \( \pm \Omega^+ \).

\( \Pi = \Pi \Omega^+ \) the \( m \)-fold, \( s(\Pi) = \{ s(\Pi) \} = -1 \) b/c permute roots except one, \( R^{st} = R^{st} \).

Similarly, let \( Z = \Pi (\Omega^+)^1 \), \( Z \in R^{st} \) and \( R^{st} = R[\Omega, Z] \).

More exercises on roots, Demazure operators, etc on exercise sheet.

Finding dual basis explicitly is very annoying. Want root-theoretic description.