Soergel bimodules and Kazhdan-Lusztig conjectures
QGM, Aarhus
March 2013

Wednesday problem sheet

Warming up:

1. Let $H = \bigoplus H^i$ be a finite dimensional graded $\mathbb{R}$-vector space and $L : H^\bullet \to H^{\bullet+2}$ an operator of degree 2. Show that H admits a representation of $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ with $e = L$ and $hx = mx$ for all $x \in H^m$ if and only if L satisfies the hard Lefschetz theorem (i.e. $L^m : H^{-m} \to H^m$ is an isomorphism for all $m \geq 0$).

Longer exercises:

2. Suppose that $H = \oplus H^i$ and $W = \oplus W^j$ are finite dimensional graded real vector spaces with forms $\langle -,- \rangle$ and Lefschetz operators L_H and L_W. Suppose that $H^{\text{odd}} = 0$ or $H^{\text{even}} = 0$, that L satisfies the hard Lefschetz theorem on H and that

$$\underline{\dim} W := \sum \dim W^i v^i = (v + v^{-1}) \underline{\dim} H.$$

Show that W satisfies (HR) if and only if the signature of the Lefschetz form $(-,-)^{-i}_{L_W}$ on W^{-i} is equal to the dimension of the primitive subspace $P^{-i+1}_{L_H} \subset H^{-i+1}$ (by convention $P^1_{L_H} = 0$).

3. (If you know a little Hodge theory). Show that the hard Lefschetz theorem for complex algebraic varieties of (complex) dimension n is a formal consequence of the Hodge-Riemann bilinear relations for varieties of dimension $n-1$ and the weak Lefschetz theorem. Why is this not a proof of the hard Lefschetz theorem? (This exercise is intended to explain where the terminology “weak Lefschetz substitute” comes from. We will see tomorrow that although one does not have an analogue of the weak Lefschetz theorem for Soergel bimodules, the first differential on a Rouquier complex provides a substitute.)

4.
a) In type B_2, write $\underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t$ as a sum of KL basis elements.

b) Calculate the local intersection form of $B_s B_t B_s B_t$ at $B_s B_t$, and hence compute the decomposition of $B_s B_t B_s B_t$ into indecomposables. Any observations on the definiteness of the form?

c) In type H_2 (i.e. $m_{st} = 5$), write $\underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t \underline{H}_s$ as a sum of KL basis elements.

d) Calculate the appropriate local intersection forms, and decompose $B_s B_t B_s B_t B_s$. Again, you should make observations about definiteness.

e) Do your observations agree with the definiteness expected from the embedding theorem?

5. In lectures we saw that for any expression $\underline{w}$, $BS(\underline{w})$ has a basis as a right R-module given by 01-sequences. It contains two canonical elements c_{bot} and c_{top} which project to elements of minimal and maximal degree in $\overline{BS(\underline{w})}$. In this exercise we find a recursive formula for

$$N_{\underline{w}}(f) := \langle f^{\ell(\underline{w})} c_{\text{bot}}, c_{\text{bot}} \rangle.$$

for any degree two element $f \in R$.

a) Find a formula for $N_{\underline{w}}(f)$ in terms of $N_{\underline{w}'}(f)$, over all subexpressions $\underline{w}'$ obtained by omitting a simple reflection from $\underline{w}$.

b) Show that $N_{\underline{w}}(f) = 0$ unless $\underline{w}$ is reduced. (*Hint:* It might help to use the light leaves description of $BS(\underline{w})$ or the decomposition of $BS(\underline{w})$ into indecomposable Soergel bimodules.) Use this to simplify your formula in part (a).

c) Suppose that $\partial_s(f) > 0$ for all $s \in S$. Show that $N_{\underline{w}}(f) > 0$ for $\underline{w}$ reduced. (First prove that $sw > w$ if and only if $\partial_s(wf) > 0$.)

d) (*) What is $N_{\underline{w}}(f)$ in terms of Schubert calculus?

Recall that a *Krull-Schmidt category* is an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local.

6. Some exercises to get used to Krull-Schmidt categories:

a) Show that the Krull-Schmidt theorem holds in Krull-Schmidt categories: any object can be written as a direct sum of indecomposable objects, and this decomposition is unique up to permutation of the factors.

b) *(Idempotent lifting)* Let A be an algebra and $\mathfrak{m} \subset A$ an ideal such that $\mathfrak{m}^2 = 0$. Show that given an idempotent $e \in A/\mathfrak{m}$ there exists an idempotent $\widetilde{e} \in A$ such that $e = \widetilde{e}$ in $A/\mathfrak{m}$. Now prove the same statement assuming only that A is complete with respect to the topology defined by $\mathfrak{m}$.

c) Let $(\mathbb{O}, \mathfrak{m})$ be a complete local ring. Let $\mathcal{C}$ be a Karoubian $\mathbb{O}$-linear additive category such that all hom spaces are finitely generated. Show that $\mathcal{C}$ is Krull-Schmidt. (*Hint:* It might help to first consider the case when $\mathbb{O}$ is a field.)

d) Show that the category of graded modules over a polynomial ring is a Krull-Schmidt category. Conclude that the category of Soergel bimodules is Krull-Schmidt.

e) (*) Let X be an affine variety. When does the Krull-Schmidt theorem hold for vector bundles on X? (Answer: almost never.) Conclude that the Krull-Schmidt theorem fails for ungraded modules over a polynomial ring. (Optional: show that the Krull-Schmidt theorem holds for vector bundles on a projective algebraic variety.)

7. Let $\mathcal{C}$ be a Krull-Schmidt category over an algebraically closed field $\mathbb{k}$. Show that the multiplicity of B as summand of X is given by the rank of the form

$$\mathrm{Hom}(B, X) \times \mathrm{Hom}(X, B) \to \mathrm{End}(B)/\mathfrak{m}_B.$$

where $\mathfrak{m}_B$ denotes the maximal ideal of $\mathrm{End}(B)$. What is the correct statement for general fields or local rings $\mathbb{k}$?

8. Let Fl_n denote the complex flag variety G/B in type A_{n-1}. In other words, $\mathrm{Fl}_n = \{V^\bullet = (0 \subset V^1 \subset \ldots \subset V^n = \mathbb{C}_n) \mid \dim V^i = i\}$ is the set of flags in a fixed vector space $\mathbb{C}^n$ with basis $\{e_i\}_{1 \le i \le n}$. There is an action of GL_n on Fl_n, and thus an action of the subgroup $S_n \subset GL_n$. The *standard flag* V_{std} is given by $V^k_{\mathrm{std}} = \mathbb{C} \cdot \langle e_i \rangle_{1 \le i \le k}$; its stabilizer is a Borel subgroup B. For any $w \in S_n$, the dimension of $V^k_{\mathrm{std}} \cap w(V_{\mathrm{std}})^l$ is equal to the size of the intersection $\{1, 2, \ldots, k\} \cap \{w(1), w(2), \ldots, w(l)\}$. For any two flags $V^\bullet$ and $W^\bullet$, we say that they are in *relative position* w if $\dim(V^k \cap W^l) = \dim(V^k_{\mathrm{std}} \cap w(V_{\mathrm{std}})^l)$.

a) Show that Fl_n splits into B orbits based on the relative position of a flag with the standard flag, and that this agrees with the usual Bruhat decomposition of G/B. Show that $\mathrm{Fl}_n \times \mathrm{Fl}_n$ splits into G orbits based on the relative position of the two flags. Show that the orbit closure relation agrees with the Bruhat order, in either setting.

Clearly $V^\bullet$ and $W^\bullet$ are in relative position $s_i \in S \subset S_n$ if and only if $V^i \neq W^i$ and $V^k = W^k$ for all $k \neq i$. We say that $V^\bullet$ and $W^\bullet$ are in *relative position* $\overline{s_i}$ if $V^k = W^k$ for all $k \neq i$ (with no condition on V^i and W^i). Let $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_d}$ be a sequence of simple reflections. The *Bott-Samelson resolution* $BS(\underline{w})$ is the space consisting of sequences of flags, ending in the standard flag, and successively in relative position determined by $\underline{w}$:

$$\{(V_i^\bullet)_{i=0}^d \mid V_d^\bullet = V_{\text{std}}^\bullet, \text{ and the pair } (V_{k-1}^\bullet, V_k^\bullet) \text{ is in relative position } \overline{s_{i_k}} \text{ for each } 1 \leq k \leq d\}.$$

It is equipped with a map $\mu\colon BS(\underline{w}) \to \mathrm{Fl}_n$, $\mu((V_i^\bullet)) = V_0^\bullet$.

b) Show that this description of the Bott-Samelson resolution agrees with the one given in lecture.

Set $n = 4$, and let s, t, u denote the simple reflections in S_4 with $su = us$. For an arbitrary flag $W^\bullet$ in each orbit, calculate the fiber $\mu^{-1}(W^\bullet)$ when:

c) $\underline{w} = tt$.

d) $\underline{w} = sts$.

e) $\underline{w} = tsut$.

f) $\underline{w} = sutsu$.

Now, for each of the above cases, construct the table for $\mu_*(\mathbb{C}[\ell(\underline{w})])$. Use these tables (and possibly other calculations) to decompose this pushforward into **IC** sheaves.

Research level questions:

9. What is the geometric meaning of the weak Lefschetz substitute? Can it be applied usefully in other settings? Geordie has a rough idea, and would LOVE to discuss.