Warming up:

1. Check that the one color relations hold in Soergel bimodules.

2. Describe all light leaves maps from ss . . . s (m times).

3. Let $m_{st} = m < \infty$. For $k > 0$, let $w = stst . . . st$ of length $2(m + k)$. What is the dimension of $\text{Hom}(BS(w), R)$ in degree $-2k$? Draw several different graphs realizing the same morphism in this space.

4. Let $f \in b^* \in R$ be a linear polynomial. For a general expression $w$, find a formula for $f e_w$ in the 01-sequence basis of $BS(w)$ as a right $R$-module.

Longer exercises:

5. Let $TL_n$ be the Temperley-Lieb algebra with $n$ strands, where a circle evaluates to $-[2] = -(q + q^{-1}) \in \mathbb{Z}[q, q^{-1}]$. Show that the space of all elements killed by caps above (resp. cups below) is one-dimensional, and show that these spaces agree. The Jones-Wenzl projector $JW_n \in TL_n$ is uniquely specified in this one-dimensional kernel by the fact that the coefficient of the identity is 1. Verify the following recursive formula.

\[
JW_{n+1} = JW_n + \sum_{i=1}^{n} \frac{[i]}{[n+1]} JW_n
\]

The trace of an element $a \in TL_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the closed diagram below. Calculate the trace of $JW_n$ (hint: use induction). In a specialization of $\mathbb{Z}[q, q^{-1}]$ where the trace of $JW_n$ is zero, what do you get when you rotate $JW_n$ by one strand?

6. a) Whenever $m_{st} = 2$ show that $B_s B_t \cong B_t B_s$.

b) Whenever $m_{st} = 3$ show that $B_s B_t B_s \cong B_{sts} \oplus B_s$ and $B_t B_s B_t \cong B_{tst} \oplus B_t$ in $SBim$, where $B_{sts} = B_{tst}$ is a common summand. (Harder, but very important.)

c) For any simply-laced Coxeter group (i.e. $m_{st} \in \{2, 3\}$), show that the map $H \rightarrow [SBim]$ sending $b_s \mapsto [B_s]$ is a homomorphism.
7. Let \( S = \{s, t, u\} \) be type \( A_3 \). Let \( w = tsuts \) and let \( y = utstut \) be two expressions for the longest element \( w_0 \in W \). There are (essentially) two paths from \( w \) to \( y \) in the reduced expression graph of \( w_0 \). Find a reasonably quick proof that the two corresponding morphisms of Bott-Samelson bimodules are not equal.

8. For a Soergel bimodule \( B \), let \( \overline{B} \) denote \( B \otimes_R R \) be the right quotient. For example, \( \overline{BS(w)} \) has a basis over \( R \) given by 01-sequences. Just as \( BS(w) \) has an intersection form valued in \( R \), so too does \( BS(w) \) have an intersection form valued in \( R \).

The endomorphism \( L \) of \( B \) gives a degree 2 endomorphism \( L \) of the vector space \( \overline{B} \). What is \( \langle c_{\text{bot}}, L^2(c_{\text{bot}}) \rangle \)? What is \( \langle L(c_{\text{bot}}), L(c_{\text{bot}}) \rangle \)? Find an element \( b \) of degree zero which is perpendicular to \( L(c_{\text{bot}}) \). What is \( \langle b, b \rangle \)?

Now let \( L_0 \) be the degree 2 endomorphism of \( \overline{B} \) given by left multiplication by \( \alpha_s \). What is \( L^2_0(c_{\text{bot}}) \)?

9. In the previous question we defined the intersection form on \( \overline{BS(w)} \). Now repeat some of the same calculations with \( B_sB_tB_s \). What is \( \langle c_{\text{bot}}, L^2(c_{\text{bot}}) \rangle \)? What is \( \langle L(c_{\text{bot}}), L(c_{\text{bot}}) \rangle \)? Find a basis for \( B_sB_tB_s^{-1} \) (i.e. the elements in degree \(-1\)) in the kernel of \( L^2 \). Are they orthogonal to \( L^2(c_{\text{bot}}) \) under the intersection form? Show that the form \( \langle v, w \rangle = \langle v, Lw \rangle \) on this orthogonal subspace of \( \overline{B_sB_tB_s}^{-1} \) is negative definite.

Bonus problem: what does the picture look like when restricted to the summand \( B_s \subset B_sB_tB_s \)? What does it look like when restricted to the summand \( B_{sts} \subset B_sB_tB_s \)?

10. Fix a Soergel bimodule \( B \) and consider the two maps \( \alpha, \beta : B \to B \otimes_R B \) given by \( \alpha(b) := bc_{\text{id}} \) and \( \beta(b) := bc_s \).

Together, \( \alpha(B) \) and \( \beta(B) \) span \( BB_s \). Show that \( \beta \) is a morphism of bimodules, whilst \( \alpha \) is a morphism of left modules. Find a formula for \( \alpha(br) \) for \( b \in B \) and \( r \in R \).

Suppose that \( B \) is equipped with an invariant form \( \langle \cdot, \cdot \rangle_B \). Prove that there is a unique invariant form \( \langle \cdot, \cdot \rangle_{BB_s} \) on \( BB_s \), which we call the induced form, satisfying

\[
\langle \alpha(b), \alpha(b') \rangle_{BB_s} = \varDelta_s(\langle b, b' \rangle_B) \quad (1)
\]

\[
\langle \alpha(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \quad (2)
\]

\[
\langle \beta(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \alpha_s \quad (3)
\]

for all \( b, b' \in B \). Show that the intersection form on a Bott-Samelson bimodule agrees with the form induced many times from the canonical form on \( R \).

Now consider \( \overline{BB_s} \), with its induced form valued in \( R \). Calculate a matrix for this form in some basis. Prove that the induced form is non-degenerate whenever the original form on \( \overline{B} \) is non-degenerate.

11. After localization to \( Q \), the fraction field of \( R \), the Bott-Samelson bimodule \( B_s \otimes_R Q \) splits as a direct sum of \( Q_s \) and \( Q \) (when using localization we ignore the grading). Therefore, for any subsequence \( e \subset w \), there is a summand \( Q_e \subset BS(w) \otimes_R Q \), a tensor product of either \( Q_{w} \) or \( Q \) depending on whether \( e_i \) is 1 or 0. Obviously \( Q_e \cong Q_{x} \) when \( e \) expresses the element \( x \).

Use localization and the Bruhat path dominance order to prove that the images in \( \mathbb{E}BS\text{Bim} \) of the light leaves maps in \( \Xi_{w,x} \) are all linearly independent.

12. Show that the functor from \( D \) to \( \mathbb{E}BS\text{Bim} \) is an equivalence of categories, assuming that double leaves form a basis for morphisms in \( D \).
13. (Assumes knowledge of the support of a coherent sheaf.) For \( w \in W \), let \( \text{Gr}_w = \{(w(v), v) \subset \mathfrak{h} \times \mathfrak{h} \} \). Let \( w_1, w_2, \ldots \) be an enumeration of the elements of \( W \), and let \( B \) be an \( R \)-bimodule. Suppose there exists a filtration \( 0 \subset B^1 \subset \ldots \subset B^m = B \) such that \( B^i/B^{i-1} \cong \oplus R_{w_i}^{\oplus n_w} \). Show that \( B^i \) is equal to the submodule of \( B \) consisting of sections with support on the subvariety \( \bigcup_{j=1}^i \text{Gr}_{w_j} \). Deduce that a standard filtration on a Soergel bimodule is unique and is preserved by all morphisms. (Hint: the support of any nonzero element of \( R_x \) is \( \text{Gr}_x \).)

**For fun?:**

14. Find the appropriate notion of the Jones-Wenzl relation in type \( B_2 \), with the usual non-symmetric Cartan matrix. Find the orthogonal idempotents giving the direct sum decomposition \( B_sB_tB_sB_t \cong B_{stat} \oplus B_{st} \oplus B_{st} \). (Warning: Computationally intensive.)

**Research level questions:**

15. Consider the space \( \text{Hom}(BS(w), BS(y)) \), and let \( s \) be a color which does not appear in either sequence (i.e. does not appear on the boundary). Exercise: Soergel’s Hom formula implies that this Hom space is spanned by diagrams which do not involve the color \( s \). Exercise: Similarly, if \( s \) only appears on the boundary once, show that the Hom space is spanned by diagrams for which the \( s \)-colored strand ends immediately in a dot, and \( s \) is otherwise nonexistent. This phenomenon is called color elimination.

Is there a diagrammatic algorithm to take a graph with extraneous colors, and rewrite it as a linear combination of graphs only involving the colors on the boundary? Is there a simple, graph-theoretic proof of color elimination (without deducing it from double leaves, for instance)? Such a proof was given for “extremal colors” in type \( A \) in Elias-Khovanov, and in dihedral type by Elias.

16. Is there a way to make light leaves canonical? Is there a way to make them adapted to intersection forms?