Soergel bimodules and Kazhdan-Lusztig conjectures
QGM, Aarhus
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Tuesday problem sheet

Warming up:

1. Check that the one color relations hold in Soergel bimodules.

2. Describe all light leaves maps from $ss\dots s$ (m times).

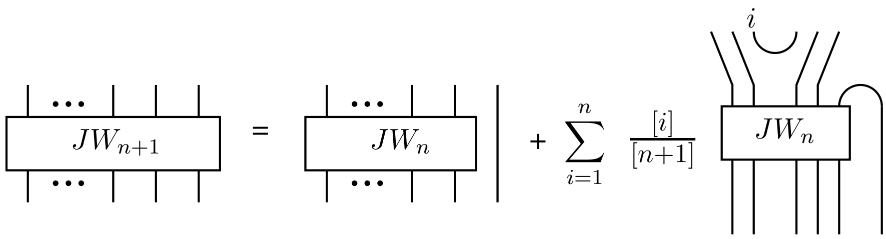
3. Let $m_{st} = m < \infty$. For $k > 0$, let $\underline{w} = stst\dots st$ of length $2(m+k)$. What is the dimension of $\mathrm{Hom}(BS(\underline{w}), R)$ in degree $-2k$? Draw several different graphs realizing the same morphism in this space.

4. Let $f \in \mathfrak{h}^* \in R$ be a linear polynomial. For a general expression $\underline{w}$, find a formula for $fc_{\underline{\varepsilon}}$ in the 01-sequence basis of $BS(\underline{w})$ as a right R-module.

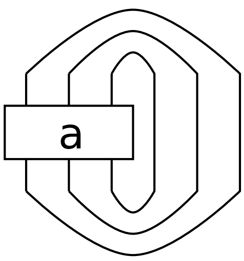
Longer exercises:

5. Let TL_n be the Temperley-Lieb algebra with n strands, where a circle evaluates to $-[2] = -(q+q^{-1}) \in \mathbb{Z}[q,q^{-1}]$. Show that the space of all elements killed by caps above (resp. cups below) is one-dimensional, and show that these spaces agree.

The Jones-Wenzl projector $JW_n \in TL_n$ is uniquely specified in this one-dimensional kernel by the fact that the coefficient of the identity is 1. Verify the following recursive formula.



The *trace* of an element $a \in TL_n$ is the evaluation in $\mathbb{Z}[q,q^{-1}]$ of the closed diagram below. Calculate the trace of JW_n (hint: use induction). In a specialization of $\mathbb{Z}[q,q^{-1}]$ where the trace of JW_n is zero, what do you get when you rotate JW_n by one strand?



6. a) Whenever $m_{st} = 2$ show that $B_sB_t \cong B_tB_s$.

b) Whenever $m_{st} = 3$ show that $B_sB_tB_s \cong B_{sts} \oplus B_s$ and $B_tB_sB_t \cong B_{tst} \oplus B_t$ in $\mathbb{S}Bim$, where $B_{sts} = B_{tst}$ is a common summand. (Harder, but very important.)

c) For any simply-laced Coxeter group (i.e. $m_{st} \in \{2,3\}$), show that the map $\mathbf{H} \to [\mathbb{S}Bim]$ sending $b_s \mapsto [B_s]$ is a homomorphism.

7. Let $S = \{s, t, u\}$ be type A_3. Let $\underline{w} = tstuts$ and let $\underline{y} = utstut$ be two expressions for the longest element $w_0 \in W$. There are (essentially) two paths from $\underline{w}$ to $\underline{y}$ in the reduced expression graph of w_0. Find a reasonably quick proof that the two corresponding morphisms of Bott-Samelson bimodules are not equal.

8. For a Soergel bimodule B, let $\overline{B}$ denote $B \otimes_R \mathbb{R}$ be the *right quotient*. For example, $\overline{BS(\underline{w})}$ has a basis over $\mathbb{R}$ given by 01-sequences. Just as $BS(\underline{w})$ has an intersection form valued in R, so too does $\overline{BS(\underline{w})}$ have an intersection form valued in $\mathbb{R}$.

The endomorphism  of B_sB_s gives a degree 2 endomorphism L of the vector space $\overline{B_sB_s}$. What is $\langle c_{\mathrm{bot}}, L^2(c_{\mathrm{bot}})\rangle$? What is $\langle L(c_{\mathrm{bot}}), L(c_{\mathrm{bot}})\rangle$? Find an element b of degree zero which is perpendicular to $L(c_{\mathrm{bot}})$. What is $\langle b, b\rangle$?

Now let L_0 be the degree 2 endomorphism of $\overline{B_sB_s}$ given by left multiplication by α_s. What is $L_0^2(c_{\mathrm{bot}})$?

9. In the previous question we defined the intersection form on $\overline{BS(\underline{w})}$. Now we repeat some of the same calculations with $B_sB_tB_s$ when $m_{st} = 3$. Let $\rho \in \mathfrak{h}^*$ satisfy $\partial_s(\rho) = \partial_t(\rho) = 1$. Let L be the degree 2 endomorphism of $\overline{B_sB_tB_s}$ given by left multiplication by ρ.

What is $L^3(c_{\mathrm{bot}})$? What is $\langle c_{\mathrm{bot}}, L^3(c_{\mathrm{bot}})\rangle$? Find a basis for $\overline{B_sB_tB_s}^{-1}$ (i.e. the elements in degree -1) in the kernel of L^2. Are they orthogonal to $L^2(c_{\mathrm{bot}})$ under the intersection form? Show that the form $(v, w) = \langle v, Lw\rangle$ on this orthogonal subspace of $\overline{B_sB_tB_s}^{-1}$ is negative definite.

Bonus problem: what does the picture look like when restricted to the summand $B_s \overset{\oplus}{\subset} B_sB_tB_s$? What does it look like when restricted to the summand $B_{sts} \overset{\oplus}{\subset} B_sB_tB_s$?

10. Fix a Soergel bimodule B and consider the two maps $\alpha, \beta : B \to BB_s = B \otimes_R B_s$ given by

$$\alpha(b) := bc_{\mathrm{id}} \quad \text{and} \quad \beta(b) := bc_s.$$

Together, $\alpha(B)$ and $\beta(B)$ span BB_s. Show that β is a morphism of bimodules, whilst α is a morphism of left modules. Find a formula for $\alpha(br)$ for $b \in B$ and $r \in R$.

Suppose that B is equipped with an invariant form $\langle -, -\rangle_B$. Prove that there is a unique invariant form $\langle -, -\rangle_{BB_s}$ on BB_s, which we call the *induced form*, satisfying

$$\langle \alpha(b), \alpha(b')\rangle_{BB_s} = \partial_s(\langle b, b'\rangle_B) \tag{1}$$
$$\langle \alpha(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \text{ and } \langle \beta(b), \alpha(b')\rangle_{BB_s} = \langle b, b'\rangle_B \tag{2}$$
$$\langle \beta(b), \beta(b')\rangle_{BB_s} = \langle b, b'\rangle_B \alpha_s \tag{3}$$

for all $b, b' \in B$. Show that the intersection form on a Bott-Samelson bimodule agrees with the form induced many times from the canonical form on R.

Now consider $\overline{BB_s}$, with its induced form valued in $\mathbb{R}$. Calculate a matrix for this form in some basis. Prove that the induced form is non-degenerate whenever the original form on $\overline{B}$ is non-degenerate.

11. After localization to Q, the fraction field of R, the Bott-Samelson bimodule $B_s \otimes_R Q$ splits as a direct sum of Q_s and Q (when using localization we ignore the grading). Therefore, for any subsequence $\mathbf{e} \subset \underline{w}$, there is a summand $Q_{\mathbf{e}} \overset{\oplus}{\subset} BS(\underline{w}) \otimes_R Q$, a tensor product of either $Q_{\underline{w}_i}$ or Q depending on whether $\mathbf{e}_i$ is 1 or 0. Obviously $Q_{\mathbf{e}} \cong Q_x$ when $\mathbf{e}$ expresses the element x.

Use localization and the Bruhat path dominance order to prove that the images in $\mathbb{BS}Bim$ of the light leaves maps in $\mathbb{LL}_{\underline{w},x}$ are all linearly independent.

12. Show that the functor from $\mathcal{D}$ to $\mathbb{BS}Bim$ is an equivalence of categories, assuming that double leaves form a basis for morphisms in $\mathcal{D}$.

13. (Assumes knowledge of the support of a coherent sheaf.) For $w \in W$, let $\mathrm{Gr}_w = \{(w(v), v\} \subset \mathfrak{h} \times \mathfrak{h}$. Let $w_1, w_2, \ldots$ be an enumeration of the elements of W, and let B be an R-bimodule. Suppose there exists a filtration $0 \subset B^1 \subset \ldots \subset B^m = B$ such that $B^i/B^{i-1} \cong \oplus R_{w_i}^{\oplus n_i}$. Show that B^i is equal to the submodule of B consisting of sections with support on the subvariety $\cup_{j=1}^{i} \mathrm{Gr}_{w_j}$. Deduce that a standard filtration on a Soergel bimodule is unique and is preserved by all morphisms. (Hint: the support of any nonzero element of R_x is Gr_x.)

For fun?:

14. Find the appropriate notion of the Jones-Wenzl relation in type B_2, with the usual non-symmetric Cartan matrix. Find the orthogonal idempotents giving the direct sum decomposition $B_s B_t B_s B_t \cong B_{stst} \oplus B_{st} \oplus B_{st}$. (Warning: Computationally intensive.)

Research level questions:

15. Consider the space $\mathrm{Hom}(BS(\underline{w}), BS(\underline{y}))$, and let s be a color which does not appear in either sequence (i.e. does not appear on the boundary). Exercise: Soergel's Hom formula implies that this Hom space is spanned by diagrams which do not involve the color s. Exercise: Similarly, if s only appears on the boundary once, show that the Hom space is spanned by diagrams for which the s-colored strand ends immediately in a dot, and s is otherwise non-existent. This phenomenon is called *color elimination*.

Is there a diagrammatic algorithm to take a graph with extraneous colors, and rewrite it as a linear combination of graphs only involving the colors on the boundary? Is there a simple, graph-theoretic proof of color elimination (without deducing it from double leaves, for instance)? Such a proof was given for "extremal colors" in type A in Elias-Khovanov, and in dihedral type by Elias.

16. Is there a way to make light leaves canonical? Is there a way to make them adapted to intersection forms?