Thursday problem sheet

Warming up:

1. Let $F_s$ and $F_s^{-1}$ denote the Rouquier complexes introduced in lectures. Check that $F_s F_s^{-1} \cong R$ in $K^b(R\text{-Bim})$ as sketched in lectures.

2. Write down the summands appearing in the minimal complex of $F_s F_u F_t F_s F_u$.

3. Consider $(B_s B_s)$, with the Lefschetz operator

$$L_{a,b} := (a \rho \cdot -) \text{id}_{B_s} + \text{id}_{B_s} (b \rho \cdot -)$$

for some $a, b \in \mathbb{R}$. For which $a, b$ does the hard Lefschetz property hold? For which $a, b$ do the Hodge-Riemann bilinear relations hold? For which $a, b$ does (HR) hold with the opposite signatures?

Longer exercises:

4. (...continuing Q1) Compute the minimal complex of $F_s^\otimes m$ for $m \geq 0$. Describe its perverse filtration explicitly.

5. Verify that $B_S(s t s)$ satisfies the Hodge-Riemann bilinear relations with respect to left multiplication by $\rho$.

6. In this exercise we prove an “easy” case of hard Lefschetz. Assume that $B_x$ is a Soergel bimodule such that hard Lefschetz holds on $B_x B_s$. Recall the operator

$$L_\zeta := (\rho \cdot -) \text{id}_{B_x} + \text{id}_{B_x} (\zeta \rho \cdot -)$$

on $B_x B_s$. It induces a Lefschetz operator $L_\zeta$ on $B_x B_s$. (You can equip $B_x$ with an invariant form if you wish, but it won’t be important for this exercise.)

a) Let $s \in S$ be such that $xs < s$. Show that $B_x B_s = B_x(1) \oplus B_x(-1)$. (You should be able to give an abstract argument, but in part b) the following fact is useful (see “Singular Soergel bimodules”: there exists an $(R, R)$-bimodule $B_x$ such that $B_x \otimes_{R^\text{op}} R \cong B_x$.)

b) Rewrite the Lefschetz operator $L_\zeta$ on $B_x B_s$ using a fixed choice of isomorphism $B_x B_s = B_x(1) \oplus B_x(-1)$. Conclude that in the right quotient $B_x B_s$, $L_\zeta$ has the form

$$
\begin{pmatrix}
\rho \cdot - & 0 \\
\zeta \gamma & \rho \cdot -
\end{pmatrix}
$$

for some non-zero scalar $\gamma$. (As above, $\rho \cdot -$ denotes the degree two endomorphism of left multiplication by $\rho$.)

c) Conclude that $L_\zeta$ satisfies hard Lefschetz on $B_x B_s$ if and only if $\zeta \neq 0$.

7. In this exercise we look at the effect of translation functors on category $O$, and see that they are easily understood on Verma modules.

i) Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight, and let $V$ be a finite dimensional representation of $\mathfrak{g}$. Show that $\Delta(\lambda) \otimes V$ has a Verma flag; that is, that there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \Delta(\lambda) \otimes V$$

such that $F_i/F_{i-1} \cong \Delta(\mu_i)$ for some $\mu_i \in \mathfrak{h}^*$. What can you say about the multiset $\{\mu_i\}$?

ii) Now suppose that $\lambda, \mu \in \mathfrak{h}^*$ are such that $\lambda + \rho, \mu + \rho$ are dominant, and such that $\lambda - \mu \in \mathbb{Z} R$. Show that $T_\lambda^\mu (\Delta(w \cdot \lambda)) \cong \Delta(w \cdot \mu)$. Conclude that $T_\lambda^\mu$ gives an equivalence $O_\lambda \to O_\mu$ if $\lambda + \rho$ and $\mu + \rho$ are strictly dominant.
iii) Now suppose that $\lambda$ is integral and that $\lambda + \rho$ is dominant. Show we have an isomorphism

$$[O_\lambda] \to ZW e_\lambda : [\Delta(w \cdot \lambda)] \mapsto e_\lambda \cdot w$$

where $e_\lambda = \sum_{x \in \text{Stab}_W(\lambda + \rho)} x$.

iv) Let $\lambda, \mu$ be as above. In addition, assume that $\lambda, \mu$ are integral, that $\lambda$ is regular (i.e. $\lambda + \rho$ is strictly dominant) and the $\mu$ is sub-regular (i.e. $e_\mu = (1 + s)$ for some $s \in S$). Show that we have a commutative diagram

$$\begin{array}{ccc}
[O_\lambda] & \xrightarrow{T^\mu_\lambda} & [O_\mu] \\
\sim & & \sim \\
ZW & \xrightarrow{(1+s)} & ZW(1+s)
\end{array}$$

(The vertical isomorphisms are those of the previous exercise).

v) (Optional) Can you give similar descriptions for more general weights? (I.e. non integral, or with $e_\lambda$ more complicated?)

8. Let $C$ denote the coinvariant ring, the (graded) quotient of $R$ by positive degree symmetric polynomials under the usual $W$ action. Let $R^W \subset R$ be the (non-graded) subring of invariants under the dot-action of $W$ on $h^*$, by $w \cdot \mu = w(\mu + \rho) - \rho$. Show that the composition $R^W \hookrightarrow R \twoheadrightarrow C$ is surjective. (This is a key point in the proof of the Endomorphismensatz.)

For fun?!

9. Here we give a torsion example that surprised a few people last decade! Consider $S_8$, a Weyl group of type $A_7$. Let $i = (i, i + 1)$ (we write $i$ instead of $s_i$ for reasons that should become clear).

a) Consider the reduced expression:

$$w = 1357246352461357.$$ 

Show that $e = 11110101100000$ is the unique subexpression of defect zero with endpoint $w_I = 13435437$.

(Note that $w_I$ is the maximal element in the standard parabolic subgroup generated by $I = \{1, 3, 4, 5, 7\}$.)

b) Hence calculate the local intersection form in degree zero of $w$ at $z$. What do you notice?

(We will see more about this when we discuss the $p$-canonical basis tomorrow.)

10. a) Let $C$ be a finite dimensional graded algebra, and $P$ a (non-graded) projective (resp. simple) module. Show that $P$ admits a graded lift.

b) Show that $B_x$ is indecomposable as a graded $R$-module if and only if it is indecomposable as an ungraded $R$-module.

Research level questions:

11. Formulate a good notion of hard Lefschetz and Hodge-Riemann bilinear relations in equivariant cohomology (i.e. without taking the quotient by $R^+$ on the right, so that one has infinite dimensional vector spaces), and use it to simplify our proofs of hard Lefschetz for Soergel bimodules.

12. When $\rho$ is not dominant (i.e. try $w(\rho)$ for $w \in W$), does the hard Lefschetz property hold? Determine the signatures of the Lefschetz forms.