Soergel bimodules and Kazhdan-Lusztig conjectures

QGM, Aarhus
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Thursday problem sheet

Warming up:

1. Let F_s and F_s^{-1} denote the Rouquier complexes introduced in lectures. Check that $F_sF_s^{-1} \cong R$ in $K^b(R\text{-Bim})$ as sketched in lectures.

2. Write down the summands appearing in the minimal complex of $F_sF_uF_tF_sF_u$.

3. Consider $\overline{(B_sB_s)}$, with the Lefschetz operator

$$L_{a,b} := (a\rho \cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_s}(b\rho \cdot -)$$

for some $a, b \in \mathbb{R}$. For which a, b does the hard Lefschetz property hold? For which a, b do the Hodge-Riemann bilinear relations hold? For which a, b does (HR) hold with the opposite signatures?

Longer exercises:

4. (...continuing Q1) Compute the minimal complex of $F_s^{\otimes m}$ for $m \geq 0$. Describe its perverse filtration explicitly.

5. Verify that $\overline{BS(sts)}$ satisfies the Hodge-Riemann bilinear relations with respect to left multiplication by ρ.

6. In this exercise we prove an "easy" case of hard Lefschetz. Assume that B_x is a Soergel bimodule such that hard Lefschetz holds on $\overline{B_x}$. Recall the operator

$$L_\zeta := (\rho \cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho \cdot -)$$

on B_xB_s. It induces a Lefschetz operator L_ζ on $\overline{B_xB_s}$. (You can equip B_x with an invariant form if you wish, but it won't be important for this exercise.)

a) Let $s \in S$ be such that $xs < s$. Show that $B_xB_s = B_x(1) \oplus B_x(-1)$. (You should be able to give an abstract argument, but in part b) the following fact is useful (see "Singular Soergel bimodules"): there exists an (R, R^s)-bimodule $B_{\overline{x}}$ such that $B_{\overline{x}} \otimes_{R^s} R \cong B_x$.)

b) Rewrite the Lefschetz operator L_z on B_xB_s using a fixed choice of isomorphism $B_xB_s = B_x(1) \oplus B_x(-1)$. Conclude that in the right quotient $\overline{B_xB_s}$, L_ζ has the form

$$\begin{pmatrix} \rho \cdot - & 0 \\ \zeta\gamma & \rho \cdot - \end{pmatrix}.$$

for some non-zero scalar γ. (As above, $\rho \cdot -$ denotes the degree two endomorphism of left multiplication by ρ.)

c) Conclude that L_ζ satisfies hard Lefschetz on $\overline{B_xB_s}$ if and only if $\zeta \neq 0$.

7. In this exercise we look at the effect of translation functors on category $\mathcal{O}$, and see that they are easily understood on Verma modules.

i) Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight, and let V be a finite dimensional representation of $\mathfrak{g}$. Show that $\Delta(\lambda) \otimes V$ has a Verma flag; that is, that there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \Delta(\lambda) \otimes V$$

such that $F_i/F_{i-1} \cong \Delta(\mu_i)$ for some $\mu_i \in \mathfrak{h}^*$. What can you say about the multiset $\{\mu_i\}$?

ii) Now suppose that $\lambda, \mu \in \mathfrak{h}^*$ are such that $\lambda + \rho, \mu + \rho$ are dominant, and such that $\lambda - \mu \in \mathbb{Z}R$. Show that $T_\lambda^\mu(\Delta(w \cdot \lambda)) \cong \Delta(w \cdot \mu)$. Conclude that T_λ^μ gives an equivalence $\mathcal{O}_\lambda \xrightarrow{\sim} \mathcal{O}_\mu$ if $\lambda + \rho$ and $\mu + \rho$ are strictly dominant.

iii) Now suppose that λ is integral and that $\lambda + \rho$ is dominant. Show we have an isomorphism

$$[\mathcal{O}_\lambda] \to \mathbb{Z}W e_\lambda : [\Delta(w \cdot \lambda)] \mapsto e_\lambda \cdot w$$

where $e_\lambda = \sum_{x \in \mathrm{Stab}_W(\lambda+\rho)} x$.

iv) Let λ, μ be as above. In addition, assume that λ, μ are integral, that λ is regular (i.e. $\lambda + \rho$ is strictly dominant) and the μ is sub-regular (i.e. $e_\mu = (1 + s)$ for some $s \in S$). Show that we have a commutative diagram

$$\begin{array}{ccccc} [\mathcal{O}_\lambda] & \xrightarrow{T_\lambda^\mu} & [\mathcal{O}_\mu] & \xrightarrow{T_\mu^\lambda} & [\mathcal{O}_\lambda] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathbb{Z}W & \xrightarrow{\cdot(1+s)} & \mathbb{Z}W(1+s) & \xrightarrow{\text{inclusion}} & \mathbb{Z}W \end{array}$$

(the vertical isomorphisms are those of the previous exercise).

v) (Optional) Can you give similar descriptions for more general weights? (I.e. non integral, or with e_λ more complicated?)

8. Let C denote the coinvariant ring, the (graded) quotient of R by positive degree symmetric polynomials under the usual W action. Let $R^{W\cdot} \subset R$ be the (non-graded) subring of invariants under the dot-action of W on $\mathfrak{h}^*$, by $w \cdot \mu = w(\mu + \rho) - \rho$. Show that the composition $R^{W\cdot} \hookrightarrow R \twoheadrightarrow C$ is surjective. (This is a key point in the proof of the Endomorphismensatz.)

For fun?!

9. Here we give a torsion example that surprised a few people last decade! Consider S_8, a Weyl group of type A_7. Let $i = (i, i+1)$ (we write i instead of s_i for reasons that should become clear).

a) Consider the reduced expression:

$$\underline{w} = 1357246352461357.$$

Show that $\mathbf{e} = 1111010110100000$ is the unique subexpression of defect zero with endpoint

$$w_I = 13435437.$$

(Note that w_I is the maximal element in the standard parabolic subgroup generated by $I = \{1, 3, 4, 5, 7\}$.)

b) Hence calculate the local intersection form in degree zero of $\underline{w}$ at z. What do you notice?

(We will see more about this when we discuss the p-canonical basis tomorrow.)

10. a) Let C be a finite dimensional graded algebra, and P a (non-graded) projective (resp. simple) module. Show that P admits a graded lift.

b) Show that $\overline{B_x}$ is indecomposable as a graded R-module if and only if it is indecomposable as an ungraded R-module.

Research level questions:

11. Formulate a good notion of hard Lefschetz and Hodge-Riemann bilinear relations in equivariant cohomology (i.e. without taking the quotient by R^+ on the right, so that one has infinite dimensional vector spaces), and use it to simplify our proofs of hard Lefschetz for Soergel bimodules.

12. When ρ is not dominant (i.e. try $w(\rho)$ for $w \in W$), does the hard Lefschetz property hold? Determine the signatures of the Lefschetz forms.