Soergel bimodules and Kazhdan-Lusztig conjectures

QGM, Aarhus

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Monday problem sheet

Warming up:

1. To practice with Coxeter groups, we play with some embeddings and foldings.

a) Let $\{s, t, u\}$ be the simple reflections inside the Coxeter group of type A_3. Show that the subgroup generated by (su) and t is a Coxeter group of type $B_2 = I_2(4)$, with simple reflections $\{su, t\}$.

b) Let $\{s, t, u, v\}$ be the simple reflections inside the Coxeter group of type A_4. Show that the subgroup generated by (su) and (tv) is a Coxeter group of type $H_2 = I_2(5)$, with simple reflections $\{su, tv\}$.

c) Embed the Coxeter group of type $I_2(m)$ inside the Coxeter group of type A_{m-1} for $m \geq 3$, using products of distinct simple reflections.

2. Draw the reduced expression graph for a few elements in S_4 including the longest element. Do the same for B_3. If you're not warm enough yet do the same for H_3.

3. Suppose (W, S) is a finite dihedral group, with $a_{s,t} = a_{t,s} = -2\cos(\frac{\pi}{m})$.

a) Find a formula for a quadratic polynomial $z \in R$ for which $s(z) = t(z) = z$.

b) Enumerate the roots, i.e. the W-orbit of α_s and α_t. Devise a reasonable notion of positive roots. Let $\mathbb{L}$ be the product of the positive roots. Show that $s(\mathbb{L}) = t(\mathbb{L}) = -\mathbb{L}$.

c) Assume $m \leq 3$ and let w_0 be the longest element. What is $\partial_{w_0}(\mathbb{L})$?

d) Let ω_s be the unique element satisfying $\partial_s(\omega_s) = 1$ and $\partial_t(\omega_s) = 0$. Let $Z := \prod_{x \in W/\langle t \rangle} x(\omega_s)$. Show that $s(Z) = t(Z) = Z$. In fact, $R^{s,t} = \mathbb{R}[z, Z]$.

e) Suppose that $m = 2$. Find dual bases $\{a_i\}$ and $\{b_i\}$ for R over $R^{s,t}$, under the pairing $(f, g) \mapsto \partial_{w_0}(fg)$. Show that $\sum a_i b_i = \mathbb{L}$.

4. For any Frobenius extension $A \subset B$, show diagrammatically that $B \otimes_A B$ is a Frobenius algebra object in the category of B-bimodules.

5. (A last tricky warm-up...) Construct a map $B_s \otimes_R B_t \to B_{s,t}$ sending $1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1$, when $m_{s,t} = 2$. Why is there no such map when $m_{s,t} > 2$?

Longer exercises:

6. Let (W, S) be a dihedral Coxeter group. That is

$$W = \langle s, t \mid s^2 = t^2 = (st)^{m_{st}} = e \rangle$$

where $e \in W$ is the identity, and $m_{st} \in \{2, 3, 4, \ldots, \infty\}$. Given $0 \leq m \leq m_{st}$ write $st(m)$ for the product $stst\ldots$ where m terms appear and similarly for $ts(m)$. For example $st(0) = e$, $ts(1) = t$, $st(2) = st$, $ts(3) = tst$ etc.

a) Give explict descriptions of all elements of W, and hence describe the Bruhat order on W explicitly.

b) For $1 \leq m < m_{st}$ find an explicit formula for the products

$$\underline{H}_s \underline{H}_{st(m)},\ \underline{H}_s \underline{H}_{ts(m)},\ \underline{H}_t \underline{H}_{ts(m)} \qquad \text{and} \qquad \underline{H}_t \underline{H}_{st(m)}$$

in terms of the Kazhdan-Lusztig basis. (*Hint:* Calculate the first few cases and then use induction. Use caution with small m.)

c) Conclude that $h_{x,y} = v^{\ell(y)-\ell(x)}$ for all $x \leq y \in W$.

7. Let $W = S_4$, the symmetric group on $\{1,2,3,4\}$. Then W has the structure of a Coxeter group with $S = \{s_1, s_2, s_3\}$ where s_i denotes the transposition $(i, i+1)$.

a) Compute reduced expressions for all elements of W.

b) Calculate the Kazhdan-Lusztig basis $\{\underline{H}_x \mid x \in W\}$. How many non-trivial Kazhdan-Lusztig polynomials are there?

c) (*) Use the Kazhdan-Lusztig basis to describe the irreducible representations of the Hecke algebra of W over $\mathbb{Q}(v)$.

8. Let W be a Weyl group of type D_4 with generating reflections s, t, u, v such that s, u, v all commute. Let $\underline{w} = suvtsuv$.

a) Use the defect formula to write the element $\underline{H}_{\underline{w}}$ in terms of the standard basis.

b) Write the element $\underline{H}_{\underline{w}}$ in terms of the Kazhdan-Lusztig basis.

c) Hence compute the Kazhdan-Lusztig polynomial $h_{suv,suvtsuv}$.

9. Let $\underline{w} = s_1 \dots s_m$ denote an expression. We write $x \leq \underline{w}$ if there exists a subexpression $\mathbf{e}$ of $\underline{w}$ with $x = \underline{w}^{\mathbf{e}}$ (for example $\{x \in W \mid x \leq \underline{w}\} = \{x \in W \mid x \leq w\}$ if $\underline{w}$ is reduced). Given two subexpressions $\mathbf{e}, \mathbf{e}'$ of $\underline{w}$ let $x_0, x_1, \dots$ and $x_0', x_1', \dots$ be their Bruhat strolls (e.g. $x_i := s_1^{e_1} \dots s_i^{e_i}$). We define the *path dominance order* on subexpressions by saying that $\mathbf{e} \leq \mathbf{e}'$ if $x_i \leq x_i'$ for $1 \leq i \leq \ell(\underline{w})$. Show that for any $x \leq \underline{w}$ there is a unique subexpression $\mathbf{e}$ of $\underline{w}$, the *canonical subexpression*, which is uniquely characterised by the following equivalent conditions:

a) $\mathbf{e} \leq \mathbf{e}'$ for any subexpression $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$ (i.e. $\mathbf{e}$ is the unique minimal element in the path dominance order).

b) $\mathbf{e}$ has no D's in its UD labelling.

c) $\mathbf{e}$ is of maximal defect amongst all subexpressions $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$.

(If you know about Bott-Samelson resolutions: What geometric fact does the existence of $\mathbf{e}$ correspond to?)

10. Suppose that W is a dihedral group, with $S = \{s,t\}$ and $m = m_{s,t}$. Instead of writing $a_{s,t} = -2\cos(\frac{\pi}{m_{s,t}})$, let us just write $a_{s,t} = -(q + q^{-1})$. After all, when $q = e^{\frac{\pi}{m_{s,t}}}$, the two formula agree. This will allow us to write formulae which work simultaneously for all dihedral groups, using quantum numbers.

a) Consider the quantum number

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{3-n} + q^{1-n}.$$

One has $[1] = 1$ and $[0] = 0$. Find a formula for $[2][n]$ in terms of quantum numbers. Does this remind you of any formulas in previous exercises?

b) The statement that q^2 is a primitive m-th root of unity is equivalent to what statement about quantum numbers? The statement that q is a primitive $2m$-th root of unity is equivalent to what statement about quantum numbers? What about when q is a primitive m-th root of unity for m odd? Compare $[m-k]$ and $[k]$. Compare $[m+k]$ and $[m-k]$.

c) Compute the matrix for the action of $(st)^k$ on the 2-dimensional space spanned by α_s and α_t, in terms of quantum numbers. When does (st) have finite order m? When $m = 2k+1$ is the order of (st), what is $(st)^k(\alpha_s)$?

11. We will now use the term "Cartan matrix" to refer to any matrix indexed by S, satisfying $a_{s,s} = 2$ and $a_{s,t} = 0 \iff a_{t,s} = 0$, with coefficients in a base ring $\Bbbk$ (not necessarily integers). A Cartan matrix need not be symmetric, or even symmetrizable (i.e. conjugate by a diagonal matrix to a symmetric matrix).

a) Given a Cartan matrix, one can still construct a vector space $\mathfrak{h}^*$ with involutions $s \in S$ acting upon it. Show that (st) has order m if and only if $a_{s,t}a_{t,s}$ is algebraically equivalent to $[2]^2$ for q a primitive $2m$-th root of unity.

b) Show that any Cartan matrix admitting a representation of a Weyl group is symmetrizable.

c) Show that the following matrix admits a representation of the affine Weyl group $\tilde{A}_4$, for any $q \in \mathbb{C}^*$. When is it symmetrizable? When is it conjugate, by a diagonal matrix, to a representation defined over $\mathbb{R}$?

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -q^{-1} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -q \\ -q & 0 & 0 & -q^{-1} & 2 \end{pmatrix}.$$

Just for fun:

12. ... continuing Q1.

a) Embed H_3 inside D_6. Embed H_4 inside E_8. Generalize this.

b) Look at star-shaped Coxeter groups: A_2, A_3, D_4, $\tilde{D}_4$, and so forth. Consider the subgroup generated by the hub and by the product of the spokes. What subgroups do you get?

13. Now we do the previous exercise "in reverse." Let (W, S) be a Coxeter group, and fix $s \in S$. Consider the set Γ_s of elements of W which have a unique reduced expression, and which have s in their right descent set. Γ_s has the structure of a labeled graph, where each element $w \in \Gamma_s$ is labeled by the (unique!) element $t \in S$ in its left descent set, and where w, v are connected by an edge if and only if $w = uv$ for some $u \in S$.

a) Let $\{s, t\}$ be the simple reflections in type B_2. Compute that Γ_s is A_3, with the labelings corresponding to the embedding of B_2 inside A_3 from Q1.

b) Do the same for $I_2(m)$ and A_{m-1}.

c) Let (W, S) be the Coxeter group of type H_4. For $s \in S$, compute the labeled graph Γ_s.

d) Repeat the exercise for $I_2(\infty)$. What labeled graph do you obtain?

(If you know about such things, Γ_s is the W-graph of the left cell containing s. See Lusztig "Some examples of square integrable functions on a p-adic group".)

14. Continuing Q3... (These exercises are a bit more computational.)

a) Show that $R^{s,t} = \mathbb{R}[z, Z]$.

b) Suppose that $m = 3$. Find dual bases for R over $R^{s,t}$. Show that $\sum a_i b_i = \mathbb{L}$.

c) Suppose that $m = 3$. Find dual bases for R^s over $R^{s,t}$, under the pairing using $\partial_s \partial_t$. Show that $\sum a_i b_i = \frac{\mathbb{L}}{\alpha_s}$.

Research level questions:

15. Let $W \subset GL(\mathfrak{h})$ be a finite reflection group and let R denote the regular functions on $\mathfrak{h}$. Let ∂_{w_0} denote the Frobenius trace for the inclusion $R^W \subset R$. Find a good description of dual bases for the form $(f, g) \mapsto \partial_{w_0}(fg)$.