Given any Coxeter group \((W, S)\) we can produce a coloured simplicial complex whose automorphisms are precisely \(W\). This complex is called the \textit{Coxeter complex} and will be denoted \(|(W, S)|\).

Let \(n = |S|\) denote the rank of \(W\). Its construction is as follows:

- colour the \(n\) faces of the \(n - 1\)-simplex \(\Delta\) by the set \(S\),
- take one such simplex \(\Delta_w\) for each element \(w \in W\),
- glue \(\Delta_w\) to \(\Delta_{ws}\) along the wall coloured by \(s\).
For example, consider the symmetric group on three letters:

\[ W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{ e, s, t, st, ts, sts \}. \]
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The Coxeter complex of $S_4 = \bullet \quad \bullet \quad \bullet$:

(barycentric subdivision of the tetrahedron).
\[ s \quad 4 \quad t \quad 4 \quad u \]
Let $\ell : W \to \mathbb{N}$ denote the length function on $W$. It is easy to describe the length function using the Coxeter complex:

$$\ell(w) = \text{length of a minimal expression for } w \text{ in the generators } s$$

$$= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W, S)|.$$
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The Bruhat order is trickier...
By construction $|(W, S)|$ has a left action of $W$.

$W$ also acts on the alcoves of $|(W, S)|$ on the right by

$$\Delta_w \cdot s = \Delta_{ws}.$$ 

This action is \textit{not} simplicial, but is “local”: cross the wall coloured by $s$. 
Using the Coxeter complex makes it easy to visualize elements of the Hecke algebra $H$.

We view an element $f = \sum f_x H_x$ as the assignment of $f_x \in \mathbb{Z}[\nu^{\pm 1}]$ to the alcove indexed by $x \in W$. 
Recall the Kazhdan-Lusztig generator $H_s := H_s + νH_{id}$. The formulas for the action of $H_s$ on the standard basis can be rewritten

$$H_x H_s = \begin{cases} 
H_{xs} + νH_x & \text{if } ℓ(xs) > ℓ(x), \\
H_{xs} + ν^{-1}H_x & \text{if } ℓ(xs) < ℓ(x).
\end{cases}$$
Recall the Kazhdan-Lusztig generator \( \underline{H}_s := H_s + vH_{id} \). The formulas for the action of \( \underline{H}_s \) on the standard basis can be rewritten

\[
H_x \underline{H}_s = \begin{cases} 
H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\
H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). 
\end{cases}
\]

We can visualise this as follows: ("quantized averaging operator")
Recall that the Kazhdan and Lusztig basis has the form

\[
H_x := H_x + \sum_{y < x} h_{y,x} H_y
\]

with \( h_{y,x} \in \nu \mathbb{Z}[\nu] \) and satisfies \( \overline{H_x} = H_x \).

The polynomials \( h_{y,x} \) are the Kazhdan-Lusztig polynomials.
We want to use the Coxeter complex to understand how to calculate the Kazhdan-Lusztig basis. The first few Kazhdan-Lusztig basis elements are easily defined:

\[ H_{id} := H_{id}, \quad H_s := H_s + vH_{id} \quad \text{for } s \in S. \]

Now the work begins. Suppose that we have calculated \( H_y \) for all \( y \) with \( \ell(y) \leq \ell(x) \). Choose \( s \in S \) with \( \ell(xs) > \ell(x) \) and write

\[ H_x H_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y. \]

The formula for the action of \( H_s \) shows that \( g_y \in \mathbb{Z}[v] \) for all \( y < \ell(xs) \). If all \( g_y \in v\mathbb{Z}[v] \) then \( H_{xs} := H_x H_s \). Otherwise we set

\[ H_{xs} = H_x H_s - \sum_{y} g_y(0) H_y. \]
\[ H_{id} = \begin{array}{c}
\end{array} \]

\[ H_{t} = \begin{array}{c}
\end{array} \]

\[ H_{s} = \begin{array}{c}
\end{array} \]
\[ H_{id} = \] 

\[ H_t = \] 

\[ H_s = \] 

\[ H_t H_s = \] 

\[ H_s = \] 

\[ = H_{ts} \]
\[ H_{id} = \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \quad H_t = \begin{array}{c}
\begin{array}{c}
1 \quad v
\end{array}
\end{array} \quad H_s = \begin{array}{c}
\begin{array}{c}
v \quad 1
\end{array}
\end{array} \]

\[ H_{ts} = \begin{array}{c}
\begin{array}{c}
1 \quad v \quad v^2 \quad v
\end{array}
\end{array} \quad H_{st} = \begin{array}{c}
\begin{array}{c}
v \quad v^2 \quad 1
\end{array}
\end{array} \]
\[
\begin{align*}
H_{id} &= \begin{array}{c}
\text{Diagram 1}
\end{array} \\
H_t &= \begin{array}{c}
\text{Diagram 2}
\end{array} \\
H_s &= \begin{array}{c}
\text{Diagram 3}
\end{array} \\
H_{ts} &= \begin{array}{c}
\text{Diagram 4}
\end{array} \\
H_{st} &= \begin{array}{c}
\text{Diagram 5}
\end{array}
\end{align*}
\]
\[
\begin{align*}
H_{id} &= \begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array} & H_t &= \begin{array}{c}
\begin{array}{c}
1 \\
V \\
V^2 \\
V \\
\end{array}
\end{array} & H_s &= \begin{array}{c}
\begin{array}{c}
V \\
V^2 \\
1 \\
V \\
\end{array}
\end{array} \\
H_{ts} &= \begin{array}{c}
\begin{array}{c}
1 \\
V \\
V^2 \\
V \\
\end{array}
\end{array} & H_{st} &= \begin{array}{c}
\begin{array}{c}
V \\
V^2 \\
1 \\
V \\
\end{array}
\end{array} & H_{ts}H_t &= \begin{array}{c}
\begin{array}{c}
1 \\
V \\
V^2 \\
V \\
\end{array}
\end{array} \cdot H_t &= \begin{array}{c}
\begin{array}{c}
1 + V^2 \\
V \\
V + V^3 \\
V^2 \\
V \\
\end{array}
\end{array}
\end{align*}
\]
\[ H_{id} = \begin{array}{c}
1 \\
\end{array} \quad H_t = \begin{array}{c}
1 \\
\end{array} \quad H_s = \begin{array}{c}
1 \\
\end{array} \]

\[ H_{ts} = \begin{array}{c}
1 \\
\end{array} \quad H_{st} = \begin{array}{c}
1 \\
\end{array} \]

\[ H_{ts} H_t = \begin{array}{c}
1 \\
\end{array} \quad \cdot H_t = \begin{array}{c}
1 + \nu^2 \\
\end{array} \]

Hence: \[ H_{tst} = H_{ts} H_t - H_t = \begin{array}{c}
\nu^2 \\
\end{array} \]
For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x) - \ell(y)}$ (Kazhdan-Lusztig basis elements are smooth.)

However in higher rank the situation quickly becomes more interesting...
Kazhdan-Lusztig positivity conjecture (1979):

\[ h_{x,y} \in \mathbb{Z}_{\geq 0}[v] \]
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\[ h_{x,y} \in \mathbb{Z}_{\geq 0}[v] \]

Established for crystallographic \( W \) by Kazhdan and Lusztig in 1980, using Deligne’s proof of the Weil conjectures.

Crystallographic: \( m_{st} \in \{2, 3, 4, 6, \infty\} \).
Why are Kazhdan-Lusztig polynomials hard?
Why are Kazhdan-Lusztig polynomials hard?

*Polo’s Theorem (1999)*

For any $P \in 1 + q\mathbb{Z}_{\geq 0}[q]$ there exists an $m$ such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.
Why are Kazhdan-Lusztig polynomials hard?

Polo’s Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geq 0}[q]$ there exists an $m$ such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

*Roughly*: all positive polynomials are Kazhdan-Lusztig polynomials!
The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

\[ 152q^{22} + 3 472q^{21} + 38 791q^{20} + 293 021q^{19} + 1 370 892q^{18} + 
\]
\[ + 4 067 059q^{17} + 7 964 012q^{16} + 11 159 003q^{15} + 
\]
\[ + 11 808 808q^{14} + 9 859 915q^{13} + 6 778 956q^{12} + 
\]
\[ + 3 964 369q^{11} + 2 015 441q^{10} + 906 567q^{9} + 
\]
\[ + 363 611q^{8} + 129 820q^{7} + 41 239q^{6} + 
\]
\[ + 11 426q^{5} + 2 677q^{4} + 492q^{3} + 61q^{2} + 3q 
\]

(This polynomial is associated to the reflection group of type \( E_8 \). See [www.liegroups.org](http://www.liegroups.org).)