

Shadows of Hodge theory in representation theory

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Thus it is surprising that several central conjectures (Kazhdan-Lusztig conjecture, Jantzen conjecture, Lusztig conjecture . . .) can be understood as saying that certain situations behave as though they were generic.

Example: a general bilinear form on a vector space is non-degenerate, however establishing that a specific form is non-degenerate might be very difficult.

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1. *Hard Lefschetz*: If $\lambda \in H^2$ is the class of an ample line bundle then for all $k \geq 0$, multiplication by λ^k gives an isomorphism

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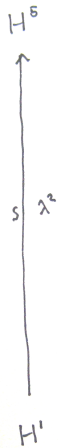
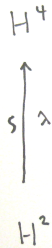
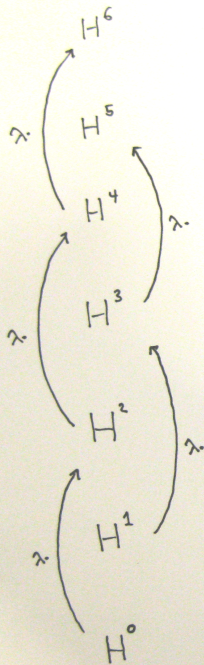
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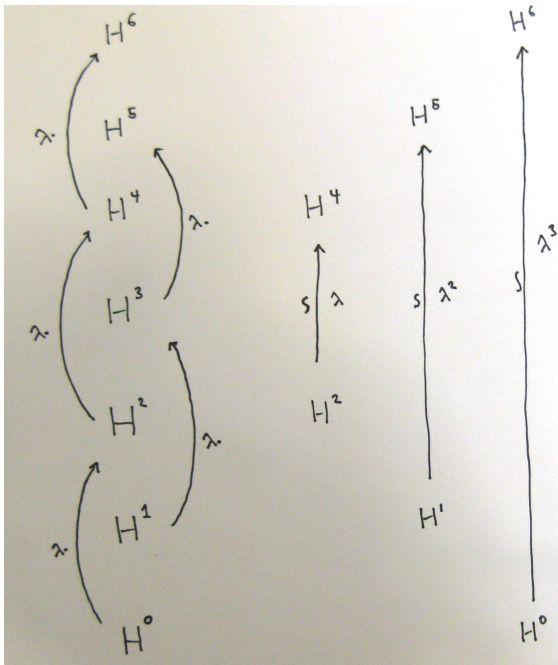
$$\lambda^k : H^{n-k}(X) \xrightarrow{\sim} H^{n+k}(X).$$

2. *Hodge-Riemann bilinear relations*: A formula (which we don't make explicit) for the signature of the forms

$$(a, b)_{\lambda} := \langle a, \lambda^k b \rangle$$

on H^{n-k} for all $k \geq 0$.





Hodge-Riemann:

$$\langle a, b \rangle_\lambda := \langle a, \lambda^i b \rangle.$$

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1. A finite-dimensional graded Λ -vector space $H = \bigoplus_{i \in \mathbb{Z}} H^i$ which vanishes in either even or odd degree and is equipped with a non-degenerate graded symmetric bilinear form $\langle -, - \rangle : H \times H \rightarrow \mathbb{R}$.

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2. A vector space V and an action of V on H via commuting degree two endomorphisms. We require compatibility with $\langle -, - \rangle$ in the sense that

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3. An open and non-empty convex cone $V_{\text{ample}} \subset V$ ("cone" means that V_{ample} is closed under multiplication by $\Lambda_{>0} := \mathbb{R}_{>0} \cap \Lambda$.)

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We require that all $\gamma \in V_{\text{ample}}$ satisfy hard Lefschetz and Hodge-Riemann in the sense of the following slide ...

We say that $\gamma \in V$ satisfies *hard Lefschetz* if for all $k \geq 0$ action by γ^k yields an isomorphism

$$\gamma^k : H^{-k} \xrightarrow{\sim} H^k.$$

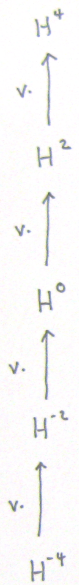
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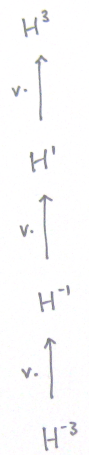
We say that $\gamma \in V$ satisfies the *Hodge-Riemann relations* if for all $k \geq 0$ the form

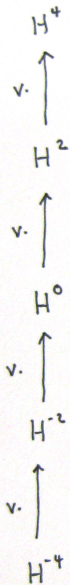
$$(a, b)_\gamma := \langle a, \gamma^k b \rangle$$

on H^{-k} is $(-1)^{(m-k)/2}$ -definite on $\ker(\gamma^{k+1} : H^{-k} \rightarrow H^{k+2})$. Here m denotes the minimal non-zero degree in H .

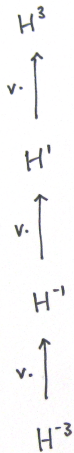


or





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$$P^{-m} = \ker(H^{-m} \rightarrow H^{m+2} = 0) = H^{-m}$$

$$P^{-m+2} = \ker(H^{-m+2} \rightarrow H^{m+2}) \subset H^{-m+2}$$

\vdots

$$P^0 = \ker(H^0 \rightarrow H^2).$$

Hodge-Riemann relations:

$(-, -)$ positive definite on P^m

$(-, -)$ negative definite on P^{m+2}

\vdots

$(-, -)$ $(-1)^{m/2}$ definite on P^0 .

(m even)

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The cohomology ring of X yields Lefschetz data:

$$H := H^*(X, \mathbb{R})[n] \quad \text{i.e. } H^i := H^{n+i}(X, \mathbb{R})$$

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In fact, everything above is defined over \mathbb{Q} . Thus algebraic varieties give rise to Lefschetz data over \mathbb{Q} .

Remarkably there are three other sources of Lefschetz data:

1. Polytopes (Stanley, McMullen, Bressler-Lunts, Karu, Barthel-Brasselet-Fieseler-Kaup, Braden 1980 - 2005)
2. Coxeter groups (Elias-W., W., 2014 - 2016)
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It is a fascinating question whether there are other examples, a unifying principle, or more general theories.

Polytopes: To any polytope in $P \subset \mathbb{R}^n$ one may associate Lefschetz data (McMullen, Stanley, Bressler-Lunts, Karu). The hard Lefschetz theorem implies the necessity of McMullen's conditions on the face numbers of simplicial polytopes. The Hodge-Riemann relations imply generalisations of the Aleksandrov-Fenchel inequalities in convex geometry. If the polytope has rational vertices then the hard Lefschetz and Hodge-Riemann relations follow from classical Hodge theory. The existence of non-rational polytopes gives rise to Lefschetz data which is not defined over \mathbb{Q} .

Matroids: To any matroid M one may associate Lefschetz data (Adiprasito-Huh-Katz). The Hodge-Riemann relations imply the longstanding conjecture as to the log concavity of the absolute value of the coefficients of the characteristic polynomial of M . (This generalises an earlier proof by Huh of the log concavity of the absolute value of the coefficients of the chromatographic polynomial of a graph.) If the matroid is realisable over a field k then the hard Lefschetz and Hodge-Riemann relations are implied by Grothendieck's standard conjectures.

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In part the motivation for doing this is to understand positivity conjectures in Kazhdan-Lusztig theory.

Let (W, S) denote a *Coxeter system*:

$$W = \langle s \in S \mid (st)^{m_{st}} = \text{id for all } s, t \in S \rangle$$

for certain $m_{st} \in \mathbb{Z}_{\geq 0}$ such that $m_{ss} = 1$ for all $s \in S$ and
 $m_{st} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for $s \neq t$.

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To (W, S) we can associate its *Hecke algebra*. It is a free $\mathbb{Z}[v^{\pm 1}]$ -algebra H with basis $\{h_x\}_{x \in W}$ and multiplication determined by the rules (for $s \in S$ and $x \in W$)

$$h_s h_x = \begin{cases} h_{sx} & \text{if } \ell(sx) > \ell(x), \\ (v^{-1} - v)h_x + h_{sx} & \text{if } \ell(sx) < \ell(x). \end{cases}$$

The basis $\{h_x \mid x \in W\}$ is the *standard basis* of H .

The algebra H possesses an involution $h \mapsto \bar{h}$ determined by $v \mapsto v^{-1}$ and $h_x \mapsto h_{x^{-1}}^{-1}$. The Kazhdan-Lusztig basis is the unique basis $\{b_x\}$ for H such that:

$$\bar{b}_x = b_x \text{ ("self-duality")} \quad \text{and} \quad b_x \in h_x + \sum_{\ell(y) < \ell(x)} v\mathbb{Z}[v]h_y.$$

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There are efficient inductive methods to calculate Kazhdan-Lusztig polynomials. The basis seems to enjoy deep positivity properties.

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4. *Unimodality of structure constants:* If we set

$$[m] := \frac{v^m - v^{-m}}{v - v^{-1}} = v^{-m+1} + v^{-m+3} + \dots + v^{m-3} + v^{m-1}$$

and, for all $x, y, z \in W$, write

$$\mu_{x,y}^z = \sum_{m \geq 1} a_{x,y}^{z,m} [m] \quad \text{then} \quad a_{x,y}^{z,m} \in \mathbb{Z}_{\geq 0}. \quad (4)$$

(In other words, $\mu_{x,y}^z$ is the character of a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$ -module.)

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Properties (1) and (3) were conjectured in 1979 by Kazhdan-Lusztig. They proved their conjecture a year later for Weyl and affine Weyl groups via intersection cohomology methods. (2) and (4) have also been known for some time for Weyl and affine Weyl groups, and have become folklore conjectures for arbitrary Coxeter systems.

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Our proof proceeds by uncovering Lefschetz data in Soergel bimodules, a monoidal category which categorifies the Hecke algebra. Lefschetz data turns out to be a powerful and flexible tool to carry out certain inductive arguments.

Let \mathfrak{h} denote a finite dimensional real vector space together with subsets $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ of *coroots* and *roots* satisfying the following two conditions:

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Then $s \mapsto \phi_s^\vee \in GL(\mathfrak{h})$ (resp. $s \mapsto \phi_s \in GL(\mathfrak{h}^*)$) where

$$\phi_s^\vee(v) := v - \langle \alpha_s, v \rangle \alpha_s^\vee \quad (\text{resp. } \phi_s(\lambda) := \lambda - \langle \lambda, \alpha_s^\vee \rangle \alpha_s)$$

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Basically we are imitating how the Weyl group of a complex semi-simple Lie algebra acts on the Cartan subalgebra.

Let R denote the regular functions on \mathfrak{h} . (After choosing a basis x_1, \dots, x_n for \mathfrak{h}^* , R is simply the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$.)

By our assumptions above the intersections of half-spaces

$$\mathfrak{h}_{\text{reg}}^+ := \bigcap_{s \in S} \{v \in \mathfrak{h} \mid \langle \alpha_s, v \rangle > 0\}$$

$$\mathfrak{h}_{\text{reg}}^{*+} := \bigcap_{s \in S} \{\lambda \in \mathfrak{h} \mid \langle \lambda, \alpha_s^\vee \rangle > 0\}$$

are non-empty. Borrowing terminology from Lie theory we refer to elements in either set as *dominant regular*.

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This miracle means that the situation is “essentially semi-simple”. We define H_x to be the indecomposable R -module direct summand which is non-zero in degree $\ell(x)$. It turns out that H_x does not depend on the choice of s .

Suppose that W is the Weyl group of a complex semi-simple algebraic group G with maximal torus and Borel subgroup
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In this case the “miracle” follows from Saito’s theory of mixed Hodge modules or de Cataldo and Migliorini’s proof of the decomposition theorem.

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The techniques used to prove the “miracle” on the previous slide also yield the positivity properties (2) and (3).

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Remark: Outside of the geometric setting H_{w_0} is usually not defined over \mathbb{Q} . (Remember $\cos(2\pi/m_{st})!$)


Some examples of Betti numbers . . .

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$I_2(5)$: symmetries of the pentagon:

$\Gamma \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$

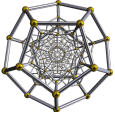


H_3 : symmetries of  :

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

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H_4 : symmetries of 120 cell  $\cup \mathbb{R}^4$:

1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 256 289 324 361 396 441 484 529 576 625 676 729 784 841 900 961 1024 1093 1164 1236 1309 1384 1461 1540 1621 1704 1789 1876 1965 2056 2149 2244 2341 2440 2541 2644 2749 2856 2965 3076 3189 3304 3421 3540 3661 3784 3909 4036 4165 4296 4429 4564 4701 4840 4981 5124 5269 5416 5565 5716 5869 6024 6181 6340 6501 6664 6829 6996 7165 7336 7509 7684 7861 8040 8221 8404 8589 8776 8965 9156 9349 9544 9741 9940 10141 10344 10549 10756 10965 11176 11389 11604 11821 12040 12261 12484 12709 12936 13165 13396 13629 13864 14101 14340 14581 14824 15069 15316 15565 15816 16069 16324 16581 16840 17101 17364 17629 17896 18165 18436 18709 18984 19261 19540 19821 20104 20389 20676 20965 21256 21549 21844 22141 22440 22741 23044 23349 23656 23965 24276 24589 24904 25221 25540 25861 26184 26509 26836 27165 27496 27829 28164 28501 28840 29181 29524 29869 30216 30565 30916 31269 31624 31981 32340 32701 33064 33429 33796 34165 34536 34909 35284 35661 36040 36421 36804 37189 37576 37965 38356 38749 39144 39541 39940 40341 40744 41149 41556 41965 42376 42789 43204 43621 44040 44461 44884 45309 45736 46165 46596 47029 47464 47901 48340 48781 49224 49669 50116 50565 51016 51469 51924 52381 52840 53301 53764 54229 54696 55165 55636 56109 56584 57061 57540 58021 58504 58989 59476 59965 60456 60949 61444 61941 62440 62941 63444 63949 64456 64965 65476 65989 66504 67021 67540 68061 68584 69109 69636 70165 70696 71229 71764 72301 72840 73381 73924 74469 75016 75565 76116 76669 77224 77781 78340 78901 79464 80029 80596 81165 81736 82309 82884 83461 84040 84621 85204 85789 86376 86965 87556 88149 88744 89341 89940 90541 91144 91749 92356 92965 93576 94189 94804 95421 96040 96661 97284 97909 98536 99165 99796 100429 101064 101701 102340 102981 103624 104269 104916 105565 106216 106869 107524 108181 108840 109501 110164 110829 111496 112165 112836 113509 114184 114861 115540 116221 116904 117589 118276 118965 119656 120349 121044 121741 122440 123141 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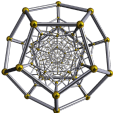
H_3 : symmetries of



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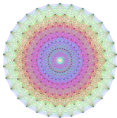
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1 4 9 16 25 36 49 64 81 100 100 121 144 165 180 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 475 475 471 464 455 444 431 416 399 380 359 336 312 294 264 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

E_8 :



1 36 36 21 21 672 2600 4710 405 336 3024 12360 21870 25200 21600 18144 15120 12096 9450 7560 5040 4032 3240 2520 2160 1814 1458 1134 945 756 648 540 450 378 324 270 225 180 162 135 108 90 72 63 54 45 36 30 25 20 18 15 12 10 9 8 6 5 4 3 2 1

Recall the unimodality property (4) above: if we write $x, y \in W$

$$b_x b_y = \sum \mu_{x,y}^z b_z$$

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i.e. we can write

$$\mu_{x,y}^z = \sum_{m \geq 1} a_{x,y}^{z,m} [m] \quad \text{with } a_{x,y}^{z,m} \geq 0$$

where $[m] := v^{-m+1} + v^{-m+3} + \dots + v^{m-3} + v^{m-1}$.

Theorem (Elias-W. 2016)

For all $x, y, z \in W$ there exists a graded R -module $V_{x,y}^z$ equipped with a symmetric form $\langle -, - \rangle$ such that:

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This immediately implies the unimodality property (4) for the structure constants of the Kazhdan-Lusztig basis.

(Unimodality has been checked by Fokko du Cloux for a finite reflection group of type H_4 by computer. Here almost three trillion polynomials $\mu_{x,y}^z$ were computed!)

Soergel's conjecture implies the Kazhdan-Lusztig conjecture on the formal characters of simple highest weight modules for a complex semi-simple Lie algebra. (Indeed, this was his initial motivation for the introduction of Soergel modules.)

We thus obtain an algebraic proof of the Kazhdan-Lusztig conjecture. (The conjecture was first proved by Brylinski-Kashiwara and Beilinson-Bernstein in 1981 via D -module techniques.)

There is a third (“local”) way to associate Lefschetz data to Soergel bimodules (W. 2015).

The local Hodge theory of Soergel bimodules gives an algebraic proof of the Jantzen conjectures (1979) on the Jantzen filtration on Verma modules, via a bridge built by Soergel (2008) and Kübel (2012).

(The Jantzen conjectures were first proved by Beilinson-Bernstein in 1990 again via D -module techniques.)

The hard Lefschetz theorem for the “multiplicity spaces” $V_{x,y}^z$ allows one to construct many “exotic” modular tensor categories associated to any Coxeter systems.

Thank you!

Slides: people.mpim-bonn.mpg.de/geordie/talks.html