Shadows of Hodge theory in representation theory

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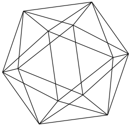

ECM, July 2016.

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Example: a general bilinear form on a vector space is non-degenerate, however establishing that a specific form is non-degenerate might be very difficult.

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1. *Hard Lefschetz:* If $\lambda \in H^2$ is the class of an ample line bundle then for all $k \geqslant 0$, multiplication by λ^k gives an isomorphism

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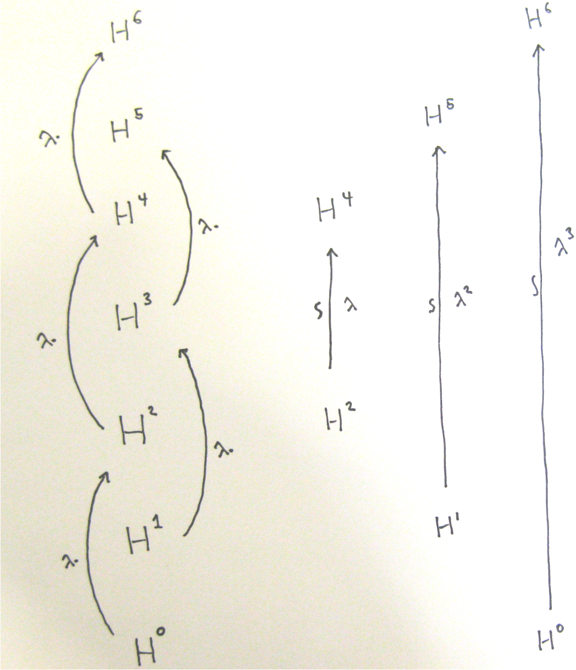
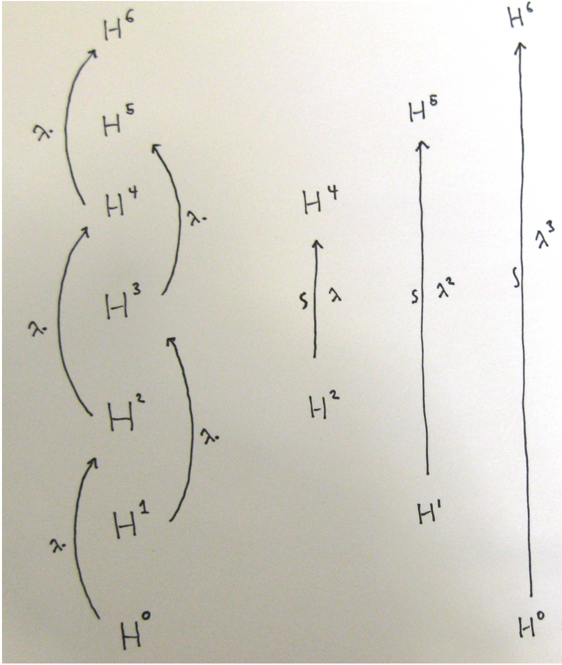
2. *Hodge-Riemann bilinear relations:* A formula (which we don't make explicit) for the signature of the forms
$$(a, b)_\lambda := \langle a, \lambda^k b \rangle$$
on H^{n-k} for all $k \geqslant 0$.



H^6
H^5
H^4
H^3
H^2
H^1
H^0
$\lambda\cdot$
λ
λ^2
λ^3



<u>Hodge-Riemann:</u>

$(a,b)_\lambda := \langle a, \lambda^i b\rangle.$

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1. A finite-dimensional graded Λ-vector space $H = \bigoplus_{i \in \mathbb{Z}} H^i$ which vanishes in either even or odd degree and is equipped with a non-degenerate graded symmetric bilinear form $\langle -, - \rangle : H \times H \to \mathbb{R}$.

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2. A vector space V and an action of V on H via commuting degree two endomorphisms. We require compatibility with $\langle -, - \rangle$ in the sense that

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3. An open and non-empty convex cone $V_{\mathrm{ample}} \subset V$ ("cone" means that V_{ample} is closed under multiplication by $\Lambda_{>0} := \mathbb{R}_{>0} \cap \Lambda$.)

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We require that all $\gamma \in V_{\mathrm{ample}}$ satisfy hard Lefschetz and Hodge-Riemann in the sense of the following slide . . .

We say that $\gamma \in V$ *satisfies hard Lefschetz* if for all $k \geqslant 0$ action by γ^k yields an isomorphism

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$$(a, b)_\gamma := \langle a, \gamma^k b \rangle$$

on H^{-k} is $(-1)^{(m-k)/2}$-definite on $\ker(\gamma^{k+1} : H^{-k} \to H^{k+2})$. Here m denotes the minimal non-zero degree in H.

$$
\begin{array}{ccc}
H^{4} & & \\
\big\uparrow{\scriptstyle v.} & & H^{3} \\
H^{2} & & \big\uparrow{\scriptstyle v.} \\
\big\uparrow{\scriptstyle v.} & & H^{1} \\
H^{0} & \text{or} & \big\uparrow{\scriptstyle v.} \\
\big\uparrow{\scriptstyle v.} & & H^{-1} \\
H^{-2} & & \big\uparrow{\scriptstyle v.} \\
\big\uparrow{\scriptstyle v.} & & H^{-3} \\
H^{-4} & &
\end{array}
$$

$$\begin{tikzcd} H^4 \\ H^2 \arrow[u, "v\cdot"] \\ H^0 \arrow[u, "v\cdot"] \\ H^{-2} \arrow[u, "v\cdot"] \\ H^{-4} \arrow[u, "v\cdot"] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} H^3 \\ H^1 \arrow[u, "v\cdot"] \\ H^{-1} \arrow[u, "v\cdot"] \\ H^{-3} \arrow[u, "v\cdot"] \end{tikzcd}$$

$$P^{-m} = \ker(H^{-m} \longrightarrow H^{m+2} = 0) = H^{-m}$$

$$P^{-m+2} = \ker(H^{-m+2} \longrightarrow H^{m}) \subset H^{-m+2}$$

$$\vdots$$

$$P^0 = \ker(H^0 \longrightarrow H^2).$$

Hodge-Riemann relations:

$\langle -,- \rangle$ positive definite on P^{-m}

$\langle -,- \rangle$ negative definite on P^{-m+2}

$\vdots$

$\langle -,- \rangle$ $(-1)^{m/2}$ definite on P^0.

(m even)

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The cohomology ring of X yields Lefschetz data:

$$H := H^*(X, \mathbb{R})[n] \quad \text{i.e. } H^i := H^{n+i}(X, \mathbb{R})$$

$$\langle -, - \rangle = \text{intersection form}$$

$$V = H^2(X, \mathbb{R})$$

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In fact, everything above is defined over $\mathbb{Q}$. Thus algebraic varieties give rise to Lefschetz data over $\mathbb{Q}$.

Remarkably there are three other sources of Lefschetz data:

1. Polytopes (Stanley, McMullen, Bressler-Lunts, Karu, Barthel-Brasselet-Fieseler-Kaup, Braden 1980 - 2005)
2. Coxeter groups (Elias-W., W., 2014 - 2016)
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It is a fascinating question whether there are other examples, a unifying principle, or more general theories.

Polytopes: To any polytope in $P \subset \mathbb{R}^n$ one may associate Lefschetz data (McMullen, Stanley, Bressler-Lunts, Karu). The hard Lefschetz theorem implies the necessity of McMullen's conditions on the face numbers of simplicial polytopes. The Hodge-Riemann relations imply generalisations of the Aleksandrov-Fenchel inequalities in convex geometry. If the polytope has rational vertices then the hard Lefschetz and Hodge-Riemann relations follow from classical Hodge theory. The existence of non-rational polytopes gives rise to Lefschetz data which is not defined over $\mathbb{Q}$.

Matroids: To any matroid M one may associate Lefschetz data (Adiprasito-Huh-Katz). The Hodge-Riemann relations imply the longstanding conjecture as to the log concavity of the absolute value of the coefficients of the characteristic polynomial of M. (This generalises an earlier proof by Huh of the log concavity of the absolute value of the coefficients of the chromatographic polynomial of a graph.) If the matroid is realisable over a field k then the hard Lefschetz and Hodge-Riemann relations are implied by Grothendieck's standard conjectures.

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In part the motivation for doing this is to understand positivity conjectures in Kazhdan-Lusztig theory.

Let (W, S) denote a *Coxeter system*:

$$W = \langle s \in S \mid (st)^{m_{st}} = \mathrm{id} \text{ for all } s, t \in S \rangle$$

for certain $m_{st} \in \mathbb{Z}_{\geqslant 0}$ such that $m_{ss} = 1$ for all $s \in S$ and $m_{st} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for $s \neq t$.

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To (W, S) we can associate its *Hecke algebra*. It is a free $\mathbb{Z}[v^{\pm 1}]$-algebra H with basis $\{h_x\}_{x \in W}$ and multiplication determined by the rules (for $s \in S$ and $x \in W$)

$$h_s h_x = \begin{cases} h_{sx} & \text{if } \ell(sx) > \ell(x), \\ (v^{-1} - v) h_x + h_{sx} & \text{if } \ell(sx) < \ell(x). \end{cases}$$

The basis $\{h_x \mid x \in W\}$ is the *standard basis* of H.

The algebra H posesses an involution $h \mapsto \overline{h}$ determined by $v \mapsto v^{-1}$ and $h_x \mapsto h_{x^{-1}}^{-1}$. The Kazhdan-Lusztig basis is the unique basis $\{b_x\}$ for H such that:

$$\overline{b_x} = b_x \ \ (\text{“self-duality”}) \quad \text{and} \quad b_x \in h_x + \sum_{\ell(y)<\ell(x)} v\mathbb{Z}[v]h_y.$$

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There are efficient inductive methods to calculate Kazhdan-Lusztig polynomials. The basis seems to enjoy deep positivity properties.

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4. *Unimodality of structure constants:* If we set

$$[m] := \frac{v^m - v^{-m}}{v - v^{-1}} = v^{-m+1} + v^{-m+3} + \cdots + v^{m-3} + v^{m-1}$$

   and, for all $x, y, z \in W$, write

$$\mu_{x,y}^z = \sum_{m \geqslant 1} a_{x,y}^{z,m} [m] \quad \text{then} \quad a_{x,y}^{z,m} \in \mathbb{Z}_{\geqslant 0}. \tag{4}$$

   (In other words, $\mu_{x,y}^z$ is the character of a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module.)

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Properties (1) and (3) were conjectured in 1979 by Kazhdan-Lusztig. They proved their conjecture a year later for Weyl and affine Weyl groups via intersection cohomology methods. (2) and (4) have also been known for some time for Weyl and affine Weyl groups, and have become folklore conjectures for arbitrary Coxeter systems.

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Our proof proceeds by uncovering Lefschetz data in Soergel bimodules, a monoidal category which categorifies the Hecke algebra. Lefschetz data turns out to be a powerful and flexible tool to carry out certain inductive arguments.

Let $\mathfrak{h}$ denote a finite dimensional real vector space together with subsets $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ and $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ of *coroots* and *roots* satisfying the following two conditions:

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Then $s \mapsto \phi_s^\vee \in GL(\mathfrak{h})$ (resp. $s \mapsto \phi_s \in GL(\mathfrak{h}^*)$) where

$$\phi_s^\vee(v) := v - \langle \alpha_s, v \rangle \alpha_s^\vee \qquad (\text{resp.} \quad \phi_s(\lambda) := \lambda - \langle \lambda, \alpha_s^\vee \rangle \alpha_s)$$

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Basically we are imitating how the Weyl group of a complex semi-simple Lie algebra acts on the Cartan subalgebra.

Let R denote the regular functions on $\mathfrak{h}$. (After choosing a basis $x_1, \dots, x_n$ for $\mathfrak{h}^*$, R is simply the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$.)

By our assumptions above the intersections of half-spaces

$$\mathfrak{h}^+_{\mathrm{reg}} := \bigcap_{s \in S} \{ v \in \mathfrak{h} \mid \langle \alpha_s, v \rangle > 0 \}$$
$$\mathfrak{h}^{*+}_{\mathrm{reg}} := \bigcap_{s \in S} \{ \lambda \in \mathfrak{h} \mid \langle \lambda, \alpha_s^\vee \rangle > 0 \}$$

are non-empty. Borrowing terminology from Lie theory we refer to elements in either set as *dominant regular*.

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Inductive step: Suppose we have constructed H_y for all $y \in W$ with $\ell(y) < k$ and suppose $\ell(x) = k$. Choose $s \in S$ such that $\ell(xs) = k - 1$ (this is possible) and consider

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This is naturally a graded R-module and comes equipped with a graded symmetric form $\langle -, - \rangle$.

Miracle: H_{pre} is Lefschetz data.

This miracle means that the situation is "essentially semi-simple". We define H_x to be the indecomposable R-module direct summand which is non-zero in degree $\ell(x)$. It turns out that H_x does not depend on the choice of s.

Suppose that W is the Weyl group of a complex semi-simple algebraic group G with maximal torus and Borel subgroup $T \subset B \subset G$.

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Then, for any $w \in W$ we have (Soergel)

$$H_x \cong IH^*(\overline{BwB/B}).$$

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(Intersection cohomology of a Schubert variety.)

In this case the "miracle" follows from Saito's theory of mixed Hodge modules or de Cataldo and Migliorini's proof of the decomposition theorem.

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The techniques used to prove the "miracle" on the previous slide also yield the positivity properties (2) and (3).

Recall the Borel isomorphism:

$$H^*(G/B) = R/(R_+^W).$$

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Remark: Outside of the geometric setting H_{w_0} is usually not defined over $\mathbb{Q}$. (Remember $\cos(2\pi/m_{st})$!)

Some examples of Betti numbers . . .

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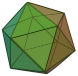
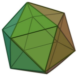
$I_2(5)$: symmetries of the pentagon:

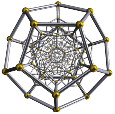
1 2 2 2 2 1

H_3: symmetries of  :

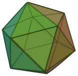
1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_3: symmetries of  :

1 3 5 7 9 11 12 12 12 12 11 9 7 5 3 1

H_4: symmetries of 120 cell  $\subset \mathbb{R}^4$:

1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

H_3: symmetries of  :

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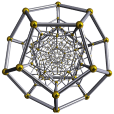
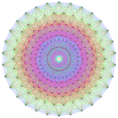
H_4: symmetries of 120 cell  $\subset \mathbb{R}^4$:

1 4 9 16 25 36 49 64 81 100 121 144 168 192 216 240 264 288 312 336 359 380 399 416 431 444 455 464 471 476 478 476 471 464 455 444 431 416 399 380 359 336 312 288 264 240 216 192 168 144 121 100 81 64 49 36 25 16 9 4 1

E_8: 

1 8 35 112 294 672 1386 2640 4718 8000 12978 20272 30645 45016 64470 90264 123829 166768 220849 287992 370250 469784 588833 729680 894613 1085880 1305640 1555912 1838523 2155056 2506798 2894688 3319268 3780640 4278429 4811752 5379194 5978792 6608029 7263840 7942628 8640288 9352240 10073472 10798593 11521896 12237428 12939064 13620586 14275768 14898464 15482696 16022740 16513208 16949127 17326016 17639957 17887656 18066494 18174568 18210722 18174568 18066494 17887656 17639957 17326016 16949127 16513208 16022740 15482696 14898464 14275768 13620586 12939064 12237428 11521896 10798593 10073472 9352240 8640288 7942628 7263840 6608029 5978792 5379194 4811752 4278429 3780640 3319268 2894688 2506798 2155056 1838523 1555912 1305640 1085880 894613 729680 588833 469784 370250 287992 220849 166768 123829 90264 64470 45016 30645 20272 12978 8000 4718 2640 1386 672 294 112 35 8 1

Recall the unimodality property (4) above: if we write $x, y \in W$

$$b_x b_y = \sum \mu_{x,y}^z b_z$$

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then $\mu_{x,y}^z$ is the character of a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, i.e. we can write

$$\mu_{x,y}^z = \sum_{m \geqslant 1} a_{x,y}^{z,m}[m] \quad \text{with } a_{x,y}^{z,m} \geqslant 0$$

where $[m] := v^{-m+1} + v^{-m+3} + \cdots + v^{m-3} + v^{m-1}$.

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For all $x, y, z \in W$ there exists a graded R-module $V_{x,y}^z$ equipped with a symmetric form $\langle -, - \rangle$ such that:

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This immediately implies the unimodality property (4) for the structure constants of the Kazhdan-Lusztig basis.

(Unimodality has been checked by Fokko du Cloux for a finite reflection group of type H_4 by computer. Here almost three trillion polynomials $\mu_{x,y}^z$ were computed!)

Soergel's conjecture implies the Kazhdan-Lusztig conjecture on the formal characters of simple highest weight modules for a complex semi-simple Lie algebra. (Indeed, this was his initial motivation for the introduction of Soergel modules.)

We thus obtain an algebraic proof of the Kazhdan-Lusztig conjecture. (The conjecture was first proved by Brylinski-Kashiwara and Beilinson-Bernstein in 1981 via D-module techniques.)

There is a third ("local") way to associate Lefschetz data to Soergel bimodules (W. 2015).

The local Hodge theory of Soergel bimodules gives an algebraic proof of the Jantzen conjectures (1979) on the Jantzen filtration on Verma modules, via a bridge built by Soergel (2008) and Kübel (2012).

(The Jantzen conjectures were first proved by Beilinson-Bernstein in 1990 again via D-module techniques.)

The hard Lefschetz theorem for the "multiplicity spaces" $V_{x,y}^z$ allows one to construct many "exotic" modular tensor categories associated to any Coxeter systems.

Thank you!

Slides: people.mpim-bonn.mpg.de/geordie/talks.html