SCHUBERT CALCULUS AND TORSION

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ABSTRACT. We observe that certain numbers occuring in Schubert calculus for SL_n also occur as entries in intersection forms controlling decompositions of Soergel bimodules and parity sheaves in higher rank. These numbers grow exponentially in the rank. This observation gives many counterexamples to Lusztig's conjecture on the characters of simple rational modules for SL_n over a field of positive characteristic.

Dedicated to Meg and Gong.

1. INTRODUCTION

Let G be a connected algebraic group over an algebraically closed field. A basic question in representation theory asks for the dimensions and characters of the simple rational G-modules. Structure theory of algebraic groups allows one to assume that G is reductive. If the ground field is of characteristic zero, then the theory runs parallel to the well-understood theory for compact Lie groups. In positive characteristic p, Steinberg's tensor product theorem, the linkage principle and Jantzen's translation principle reduce this to a question about finitely many modules which occur in the same block as the trivial module (the "principal block"). For these modules Lusztig has proposed a conjecture if $p \geq h$, where h denotes the Coxeter number of the root system of G [Lus80].[1] He conjectures an expression for the characters of the simple modules in terms of affine Kazhdan-Lusztig polynomials and the (known) characters of standard modules.

Lusztig's conjecture has been shown to hold for p large (without an explicit bound) thanks to work of Andersen, Jantzen and Soergel [AJS94], Kashiwara and Tanisaki [KT95, KT96], Kazhdan and Lusztig [KL93, KL94a, KL94b] and Lusztig [Lus94]. Alternative proofs for large p have been given by Arkhipov, Bezrukvanikov and Ginzburg [BG04], Bezrukvanikov, Mirkovic and Rumynin [BMR08] and Fiebig [Fie11]. Fiebig also gives an explicit (enormous) bound [Fie12], which is exponential in the rank, and establishes the multiplicity one case [Fie10]. For any fixed G and "reasonable" p very little is known: the case of rank 2 groups can be deduced from Jantzen's sum formula, and intensive computational efforts have checked the conjecture for small p and certain groups, all of rank ≤ 5. There is no conjecture as to what happens if p is smaller than the Coxeter number.

In [Soe00] Soergel introduced a subquotient of the category of rational representations, dubbed the "subquotient around the Steinberg weight", as a toy model for the study of Lusztig's conjecture. Whilst the full version of Lusztig's conjecture is

[1] Lusztig first proposed his conjecture under the restriction $p \geq 2h - 3$. This is not the orginal formulation, see [Jan08, §4] and [Jan03, §8.22] for a discussion. The statement of the conjecture for $p \geq h$ seems to have first been made by Kato [Kat85], who also showed that Lusztig's conjecture is compatible with Steinberg's tensor product theorem.

based on the combinatorics of alcoves and the affine Weyl group, the subquotient around the Steinberg weight is controlled by the finite Weyl group, and behaves like a modular version of category $\mathcal{O}$. Lusztig's conjecture implies that the multiplicities in the subquotient around the Steinberg weight are given by finite Kazhdan-Lusztig polynomials. Thus Lusztig's conjecture implies that "the subquotient around the Steinberg weight satisfies the Kazhdan-Lusztig conjecture".

In [Soe00] Soergel also explains how the subquotient around the Steinberg weight is controlled by Soergel bimodules. This allows him to relate this category to the category of constructible sheaves on the Langlands dual flag variety, with coefficients in the field of definition of G. Using Soergel's results and the theory of parity sheaves [JMW09], one can see that a part of Lusztig's conjecture is equivalent to absence of $p \geq h$ torsion in the stalks and costalks of integral intersection cohomology complexes of Schubert varieties in the flag variety. It has been known since the birth of the theory that 2-torsion occurs in type B_2, and 2 and 3 torsion occurs in type G_2. For over a decade no other examples of torsion were known. In 2002 Braden discovered 2-torsion in the stalks of integral intersection cohomology complexes on flag varieties of types D_4 and A_7. Much more recently Polo and Riche discovered 3-torsion in the cohomology of the flag variety of type E_6, and Polo discovered n-torsion in a flag variety of type A_{4n-1}. Polo's (as yet unpublished) results are significant, as they emphasize how little we understand in high rank (see the final lines of [Wil12]).

In general these topological calculations appear extremely difficult (for a sample computation see Braden's appendix to [WB12]). Recently Ben Elias and the author have found a presentation for the monoidal category of Soergel bimodules by generators and relations, completing work initiated by Libedinsky [Lib10] and Elias-Khovanov [EK]. One of the applications of this theory is that one can decide whether a given intersection cohomology complex has p-torsion in its stalks or costalks (the bridge between intersection cohomology and Soergel bimodules is provided by the theory of parity sheaves).[2] The basic idea is as follows: given any pair $(\underline{w}, x)$, where $x, w \in W$ and $\underline{w}$ is a reduced expression for $w \in W$, one has an "intersection form", an integral matrix. Then the stalks of the intersection cohomology complex w are free of p-torsion if no elementary divisors of the intersection forms associated to all elements $x \leq w$ are divisible by p.[3] In principle, this gives an algorithm to decide whether Lusztig's conjecture is correct around the Steinberg weight.[4] This algorithm (in a slightly different form) was discovered independently by Libedinsky [Lib].

The generators and relations approach certainly makes calculations easier. However this approach still has its difficulties: the diagrammatic calculations remain

[2] One can also perform this calculation using the theory of moment graphs [FW]. However the computations using generators and relations are generally much simpler.

[3] The terminology "intersection form" comes from geometry: in de Cataldo and Migliorini's Hodge theoretic proof of the decomposition theorem, a key role is played by certain intersection forms associated to the fibres of proper maps [dCM02, dCM05]. In our setting, these intersection forms are associated to the fibres of Bott-Samelson resolutions of Schubert varieties. The relevance of these forms for the study of torsion in intersection cohomology was pointed out in [JMW09].

[4] One can extend this to the full version of Lusztig's conjecture by using a certain subset of the affine Weyl group, thanks to the work of Fiebig [Fie11]. Although it seems likely that the converse holds, at present one only knows one implication: the absence of $p \geq h$ torsion implies the truth of Lusztig's conjecture in characteristic p.

extremely subtle, and the "light leaves" basis in which the intersection form is calculated depends on additional choices which seem difficult to make canonical. Recent progress in this direction has been made by Xuhua He and the author, who discovered that certain entries in the intersection form (which in some important examples are all entries) are canonical and may be evaluated in terms of expressions in the nil Hecke ring.

The main result of this paper is the observation that one may embed certain structure constants of Schubert calculus for SL_n as the entries of 1×1 intersection forms associated to pairs $(\underline{w}, x)$ in higher rank groups. In this way one can produce many new examples of torsion which grow exponentially in the rank. For example, using Schubert calculus for the flag variety of SL_4 we observe that the Fibonacci numbers F_n and F_{n+1} occur as torsion in SL_{4n+7}. We deduce that there is no linear function $f(h)$ of h such that Lusztig's conjecture holds for all $p \geq f(h)$. It seems likely that the prime factors occuring grow exponentially[5], which would imply the non-existence of any polynomial bound.

1.1. **Main result.** Let $R = \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables graded such that $\deg x_i = 2$ and let $W = S_n$ the symmetric group on n-letters. Then S_n acts by permutation of variables on R. Let $s_1, \dots, s_{n-1}$ denote the simple transpositions of S_n and let ℓ denote the corresponding length function. Let ∂_i denote the i^{th} divided difference operator:

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}} \in R.$$

For any element $w \in S_n$ we set $\partial_w = \partial_{i_1} \dots \partial_{i_m}$ where $w = s_{i_1} \dots s_{i_m}$ is a reduced expression for w.

Consider elements of the form

$$C = \partial_{w_m}(x_1^{a_m} x_n^{b_m} \partial_{w_{m-1}}(x_1^{a_{m-1}} x_n^{b_{m-1}} \dots \partial_{w_1}(x_1^{a_1} x_n^{b_1}) \dots))$$

where $w_i \in S_n$ are arbitrary. We assume that $\sum \ell(w_i) = a + b$ where $a = \sum a_i$ and $b = \sum b_i$ so that $C \in \mathbb{Z}$ for degree reasons. Given a subset $I \subset \{1, \dots, n-1\}$ let w_I denote the longest element in the parabolic subgroup $\langle s_j \rangle_{j \in I}$. Our main theorem is the following:

Theorem 1.1. *Suppose that $C \neq 0$. Then there exists a (reduced) expression $\underline{w}$ for an element of S_{a+n+b} such that the intersection form of $\underline{w}$ at w_I, where $I = \{1, 2, \dots, a+n+b-1\} \setminus \{a, a+n\}$, is the matrix $(\pm C)$.*

The construction of the expression $\underline{w}$ is explicit and combinatorial based on $w_1, \dots, w_m$, $a_1, \dots, a_m$ and $b_1, \dots, b_m$. In §5 we will see that the numbers C (and probably their prime factors) grow exponentially in $h = n + a + b$.

1.2. **Schubert calculus.** Let us briefly explain why "Schubert calculus" occurs in the title. In some sense this explains the meaning of the above constants C. Consider the coinvariant ring H for the action of $W = S_n$ on R. That is, H is equal to R modulo the ideal generated by W-invariant polynomials of positive degree. The Borel isomorphism gives a canonical identification of H with the integral cohomology of the complex flag variety of $G = SL_n$.

[5] I expect to be able to prove this in the near future. This statement would already be a theorem if one knew that infinitely many Fibonacci numbers are prime, which numbers theorists seem to expect.

The divided difference operators ∂_w act on H, as do elements of R. The coinvariant ring H has a graded $\mathbb{Z}$-basis given by the Schubert classes $\{X_w \mid w \in S_n\}$ (normalized with $X_{w_0} = x_1^{n-1}x_2^{n-2}\dots x_{n-1}$ and $X_w = \partial_{ww_0}X_{w_0}$). We have:

$$\partial_i X_w = \begin{cases} X_{s_i w} & \text{if } s_i w < w, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The action of multiplication by $f \in R$ of degree two is given as follows (the Chevalley formula):

$$f \cdot X_w = \sum_{\substack{t \in T \\ \ell(tw)=\ell(w)+1}} \langle f, \alpha_t^\vee \rangle X_{tw}. \tag{1.2}$$

(Here T denotes the set of reflections (transpositions) in S_n, ℓ denotes the length function and if $t = (i,j)$ with $i < j$ then $\alpha_t^\vee = \varepsilon_i - \varepsilon_j$ where $\{\varepsilon_i\}$ is a dual basis to $x_1, \dots, x_n$.)

Now consider the numbers one may obtain as coefficients in the basis of Schubert classes by multiplication by x_1 and x_n and by applying Demazure operators, starting with X_{id}. Then Theorem 1.1 says that any such number occurs as torsion in SL_{n+a+b} where a (resp. b) counts the number of times that one has applied the operator of multiplication by x_1 (resp. x_n).

1.3. **Note to the reader.** This paper is not yet in its final form and should be considered as an announcement, together with a sketch of the argument. It should be possible to replace the diagrammatic proof of the main theorem with a geometric argument, using techniques explained to me by Patrick Polo. I hope to provide both arguments in the final version.

1.4. **Acknowledgements.** The ideas of this paper crystallized after long discussions with Xuhua He. I would like to thank him warmly for the invitation to Hong Kong and the many interesting discussions that resulted from the visit. I would also like to thank Ben Elias for countless hours (often productive, always enjoyable) getting to know Soergel bimodules. His influence is omnipresent in this paper.

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2. Soergel bimodules

In this section and the next we give a brief overview of Soergel bimodules and intersection forms. This paper is not self-contained. The main references are [Soe90, Soe92, Soe07, JMW09, EK, EWa].[6] The purpose of these two sections is to orient the reader in the literature and give some examples of intersection forms.

Let $(X, R, X^\vee, R^\vee)$ denote a reduced root datum, W its Weyl group, $S \subset W$ a fixed choice of simple reflections and $\{\alpha_s\}_{s\in S}$, $\{\alpha_s^\vee\}_{s\in S}$ the corresponding simple roots and coroots. We denote by $\ell : W \to \mathbb{N}$ the length function and $\leq$ the Bruhat order.

Fix a field $\Bbbk$ of coefficients and let $R = S^\bullet(X \otimes_{\mathbb{Z}} \Bbbk)$ denote the symmetric algebra on the $\Bbbk$-vector space $X \otimes_{\mathbb{Z}} \Bbbk$, graded so that $X \otimes_{\mathbb{Z}} \Bbbk$ is homogenous of degree 2.

[6] There are also a week's worth of lectures, notes and exercises available at http://qgm.au.dk/video/ and http://people.mpim-bonn.mpg.de/geordie/aarhus/.

Then R is a graded W-algebra, and given $s \in S$ we denote by R^s the invariant subring.

The category of *Soergel bimodules* $\mathcal{B}$ is the full additive monoidal graded Karoubian subcategory of graded R-bimodules generated by $B_s = R \otimes_{R^s} R(1)$ for all $s \in S$. (Here $R \otimes_{R^s} R(1)$ denotes the R-bimodule $R \otimes_{R^s} R$ shifted so that $1 \otimes 1$ is in degree -1.) In other words, the indecomposable Soergel bimodules are the shifts of the indecomposable direct summands of the *Bott-Samelson bimodules*

$$BS(\underline{w}) = B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_m}$$

for all words $\underline{w} = s_1 s_2 \dots s_m$ in S. In [Soe07] Soergel proves that the category of Soergel bimodules categorifies the Hecke algebra, as long the characteristic of $\Bbbk$ is not 2 and the representation of W on $X^\vee \otimes_{\mathbb{Z}} \Bbbk$ satisfies a non-degeneracy condition (it should be "reflection faithful" [Soe07, Def. 1.5]). The most difficult part of Soergel's proof is a classification of the indecomposable Soergel bimodules: for all $w \in W$ there exists an indecomposable Soergel bimodule B_w which occurs with multiplicity one as a summand of $BS(\underline{w})$ for any reduced expression $\underline{w}$ for w, and the set $\{B_w\}$ coincides with the set of all indecomposable Soergel bimodules, up to shifts in the grading.

Remark 2.1. Under mild and explicit restrictions on the characteristic of $\Bbbk$, B_w may be realized as the equivariant intersection cohomology of the indecomposable parity sheaf [JMW09] of the Schubert variety labelled by w in the flag variety of the complex group with root data $(X, R, X^\vee, R^\vee)$ [Fie08, FW]. In particular, if $\Bbbk$ is of characteristic zero, then B_w is the equivariant intersection cohomology of a Schubert variety.

In [EWa] the category of Soergel bimodules is presented by generators and relations, following earlier work of Libedinsky [Lib10] and Elias-Khovanov [EK]. More precisely, we define a graded R-linear monoidal category $\mathcal{D}$ by generators and relations and prove that it is equivalent to the category of Soergel bimodules as an additive graded monoidal category (under the same assumptions on $\Bbbk$ and $X^\vee \otimes_{\mathbb{Z}} \Bbbk$ as above). We will not repeat the rather complicated list of generators and relations here, see [EWa, §1.4].

Remark 2.2. The theory of Soergel bimodules seems to break down if $\Bbbk$ is of characteristic 2 or if the representation of W on $X^\vee \otimes_{\mathbb{Z}} \Bbbk$ is not reflection faithful. On the other hand, the category defined by generators and relations still makes sense and appears to be the correct substitute [EWa, §3.2].

An important tool in the study of Soergel bimodules is Soergel's hom formula [Soe07, Theorem 5.15] which describes the graded rank of homomorphisms between Soergel bimodules in terms of the canonical form on the Hecke algebra. This formula was categorified by Libedinsky [Lib08] who introduced a recursively defined set of morphisms between Bott-Samelson bimodules, which he called *light leaves*. Certain compositions of these morphisms, called *double leaves*, give an R-basis for morphisms of all degrees between Bott-Samelson bimodules. Light leaves and double leaves morphisms admit a nice diagrammatic description [EWa, §1.5 and §6].

3. Intersection forms

From now on $\mathcal{D}$ denotes the diagrammatic category of Soergel bimodules as defined in [EWa] over the field $\Bbbk$. Given any ideal $I \subset W$ (i.e. $x \le y \in I \Rightarrow x \in I$)

we denote by $\mathcal{D}_I$ the ideal of $\mathcal{D}$ generated by all morphisms which factor through a Bott-Samelson bimodule $BS(\underline{y})$, where $\underline{y}$ is a reduced expression for $y \in I$.

Given $x \in W$ we denote by $\mathcal{D}^{\geq x}$ the quotient category $\mathcal{D}/\mathcal{D}_{\ngeq x}$. We write $\mathrm{Hom}_{\geq x}$ for (degree zero) morphisms in $\mathcal{D}^{\geq x}$. All Bott-Samelson bimodules $BS(\underline{x})$ corresponding to reduced expressions $\underline{x}$ for x become canonically isomorphic to B_x in $\mathcal{D}^{\geq x}$. We denote the resulting object simply by x. We have $\mathrm{End}_{\geq x}(x) = \Bbbk$.

Given any expression $\underline{w}$ in S the *intersection form* is the canonical pairing

$$I_{x,\underline{w},d} : \mathrm{Hom}_{\geq x}(x(d), BS(\underline{w})) \times \mathrm{Hom}_{\geq x}(BS(\underline{w}), x(d)) \to \mathrm{End}_{\geq x}(x(d)) = \Bbbk.$$

Lemma 3.1. *The multiplicity of $B_x(d)$ as a summand of $BS(\underline{w})$ is equal to the rank of $I_{x,\underline{w},d}$.*

Proof. Because B_x is indecomposable, $\mathrm{End}(B_x)$ is a local ring and the radical of $\mathrm{End}(B_x)$ is the kernel of the canonical surjection

$$\mathrm{can} : \mathrm{End}(B_x) \to \mathrm{End}_{\geq x}(B_x) = \mathrm{End}_{\geq x}(x) = \Bbbk.$$

Now, it is a standard fact (see e.g. [JMW09, Lemma 3.1]) that the multiplicity of $B_x(d)$ as a summand of $BS(\underline{w})$ is equal to the rank of the form

$$\mathrm{Hom}(B_x(d), BS(\underline{w})) \times \mathrm{Hom}(BS(\underline{w}), B_x(d)) \to \mathrm{End}(B_x) \stackrel{\mathrm{can}}{\to} \Bbbk.$$

However, the subspace of $\mathrm{Hom}(B_x(d), BS(\underline{w}))$ belonging to $\mathcal{D}_{\ngeq x}$ is certainly in the kernel of this form ($\mathcal{D}_{\ngeq x}$ is an ideal), and similarly for $\mathrm{Hom}(BS(\underline{w}), B_x(d))$. The form induced on the quotient is precisely the intersection form $I_{x,\underline{w},d}$. □

We now want to explain why the forms $I_{x,\underline{w},d}$ are explicit, computable and defined over $\mathbb{Z}$. Given a word $\underline{w} = s_1 s_2 \dots s_m$ in S a *subexpression* $\mathbf{e}$ of $\underline{w}$ is a sequence $\mathbf{e} = e_1 e_2 \dots e_m$ of 0's and 1's. We set $\underline{w}^{\mathbf{e}} = s_1^{e_1} \dots s_m^{e_m}$ and say that the subexpression $\mathbf{e}$ of $\underline{w}$ *expresses* $\underline{w}^{\mathbf{e}}$. Given a subexpression $\mathbf{e}$ its *Bruhat stroll* is the sequence

$$x_0 = \mathrm{id}, \quad x_1 = s_m^{e_m}, \quad x_2 = s_{m-1}^{e_{m-1}} s_m^{e_m}, \quad \dots \quad , \quad x_m = \underline{w}^{\mathbf{e}}.$$

We decorate each entry in the subsequence with a token U or D (for "up" and "down") as follows: for $1 \leq i \leq m$, e_i is decorated with U if $s_i x_{m-i} > x_{m-i}$ and is decorated with D if $s_i x_{m-i} < x_{m-i}$. The *defect* of the subsequence $\mathbf{e}$ is defined to be the number of occurences of U0 minus the number of occurences of D0.

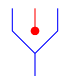
Example 3.2. If $\underline{w} = sts$ with $s \neq t$ then there are two subexpressions expressing the identity: 000 and 101. They are decorated U0 U0 U0, and D1 U0 U1 and have defects 3 and 1.

Remark 3.3. See [EWa, §2.4] for a discussion and more examples. The reader should be warned that in this paper we work from right to left (essentially due to the convention that divided difference operators act on the left), rather than from left to right as is done in [EWa].

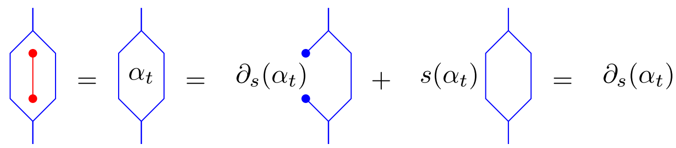
By Libedinsky's theorem [EWa, Theorem 1.1] the spaces $\mathrm{Hom}^\bullet_{\geq x}(x, BS(\underline{w}))$ and $\mathrm{Hom}^\bullet_{\geq x}(BS(\underline{w}), x)$ are free R-modules with graded basis given by light leaves corresponding to subexpressions $\mathbf{e}$ of $\underline{w}$ such that $\underline{w}^{\mathbf{e}} = x$. The degree is given by the defect. Moreover the light leaves morphisms are defined over $\mathbb{Z}$. Hence by using the generators and relations one obtains integral matrices $I_{x,\underline{w},d}$ whose reduction modulo p controls the behaviour of the category of Soergel bimodules, as made precise by Lemma 3.1.

Example 3.4. We give two examples of intersection forms (the reader is referred to [EWa] for more details):

(1) Assume that W is a dihedral group with simple reflections s and t. Let $\underline{w} = sts$, $x = s$ and $d = 0$. There is only one subexpression of defect 0: $\mathbf{e}$ =D0 U0 U1. The corresponding light leaves morphism is



Pairing it with itself we obtain:

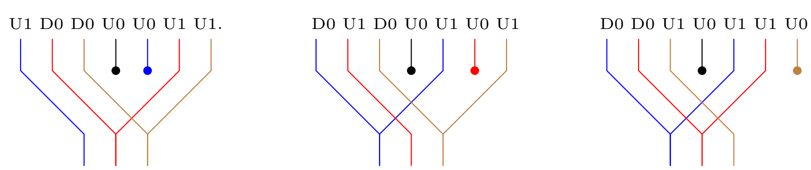


Hence in this case the intersection form is the 1×1-matrix:

$$(\partial_s(\alpha_t)) = (\langle \alpha_t, \alpha_s^\vee \rangle).$$

This example show the existence of torsion in the intersection cohomology of the Schubert variety indexed by sts if $\langle \alpha_t, \alpha_s^\vee \rangle < -1$ (as happens in B_2 and G_2).

(2) Assume that W is of type D_4 with generators s, t, u, v such that s, u and v commute. Let $\underline{w} = suvtsuv$, $x = suv$ and $d = 0$. Then there are three subexpressions of defect zero. These subexpressions, and the corresponding light leaves maps are the following:



We leave it to the reader to pair these morphisms and obtain the intersection form

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Note that the determinant of this matrix is -2. This example of 2-torsion in the D_4 flag variety was discovered by Braden [WB12, A.18].

In the following Proposition $I_{x,\underline{w},d}$ denotes the integral form of the intersection form defined above, H denotes the Hecke algebra of (W, S) and $\{C'_x\}$ its Kazhdan-Lusztig basis. We denote by

$$\mathrm{ch} : [\mathcal{D}] \xrightarrow{\sim} H$$

the isomorphism between the split Grothendieck group of Soergel bimodules and the Hecke algebra (see [Soe07, §5] or [EWa, §6.5]).

Proposition 3.5. *The following are equivalent:*

(1) *The indecomposable Soergel bimodules categorify the Kazhdan-Lusztig basis. That is,* $\mathrm{ch}(B_x) = C'_x$ *for all* $x \in W$.

(2) *For all (reduced) expressions $\underline{w}$ in S, all $x \in W$ and all $m \in \mathbb{Z}$ the graded ranks of $I_{x,\underline{w},m} \otimes_{\mathbb{Z}} \mathbb{Q}$ and of $I_{x,\underline{w},m} \otimes_{\mathbb{Z}} \Bbbk$ agree.*

Proof. Soergel's theorem [Soe01, Lemma 5] implies that the indecomposable Soergel bimodules categorify the Kazhdan-Lusztig basis if $\Bbbk$ is of characteristic zero (see [EWb] for an algebraic proof of this fact). Now Lemma 3.1 says that (2) holds if and only if $BS(\underline{w})$ decomposes the same way over $\mathbb{Q}$ as it does over $\Bbbk$. Hence (1) and (2) are equivalent. □

Remark 3.6. The intersection form and the above proposition is one of the tools used by Fiebig to establish his bound [Fie12].

Remark 3.7. In fact (1) and (2) are equivalent to $I_{x,\underline{w},0} \otimes_{\mathbb{Z}} \mathbb{Q}$ and of $I_{x,\underline{w},0} \otimes_{\mathbb{Z}} \Bbbk$ having the same rank for all reduced expressions $\underline{w}$ and $x \in W$. (That is we only need to check that the forms for degree zero summands have the correct ranks. This fact plays a key role in [EWb].) We will not need this below.

Proposition 3.8. *If $p = \operatorname{char} \Bbbk \geq h$, the Coxeter number of W, and the underlying root system $(X, R, X^\vee, R^\vee)$ is that of a simply connected semi-simple algebraic group, then Lusztig's conjecture implies that both statements of Proposition 3.5 hold.*

Sketch of proof: If $p \geq h$ then $\mathcal{D}$ is equivalent to the category of Soergel bimodules $\mathcal{B}$. By abuse of notation we also denote by B_x the indecomposable Soergel bimodule parametrized by $x \in W$. We claim that $D_x = B_x \otimes_R \Bbbk$ is indecomposable as a left $C = R/(R_+^W)$-module. The fact that Lusztig's conjecture implies (1) follows (with some effort) from [Soe00].

It remains to see that D_x is indecomposable. Because the quotient of a local ring if local, it is enough to show that the natural morphism

$$\phi : \operatorname{Hom}^\bullet_{R-R}(BS(\underline{x}), BS(\underline{y})) \otimes_R \Bbbk \to \operatorname{Hom}_C(BS(\underline{x}) \otimes_R \Bbbk, BS(\underline{y}) \otimes_R \Bbbk)$$

is an isomorphism. By Soergel's hom formula [Soe07, Theorem 5.15] and [Soe00, Lemma 2.11.2] we know that both sides have the same (finite) dimension. Using the biadjointness of $B_s \otimes_R -$ one can reduce to the case when $\underline{x} = \emptyset$. Finally, in this case it is not difficult to check that ϕ is injective, and hence is an isomorphism. □

4. Proof of the theorem

Let W denote the symmetric group on $\{1, 2, \dots, a+n+b\}$ with Coxeter generators $S = \{s_1, s_2, \dots, s_{a+n+b-1}\}$ the simple transpositions. Given a subset $I \subset S$ let W_I denote the corresponding standard parabolic subgroup. Consider the sets

$$A = \{s_1, s_2, \dots, s_{a-1}\}, M = \{s_{a+1}, \dots, s_{a+n-1}\}, B = \{s_{a+n+1}, \dots, s_{a+n+b-1}\}.$$

Then W_A (resp. W_M, resp. W_B) is the subgroup of permutations of $\{1, \dots, a\}$ (resp. $\{a+1, \dots, a+n\}$, resp. $\{a+n+1, \dots, a+n+b\}$).

We use the notation of §1.1 except we shift all indices by a. That is we regard S_n as embedded in S_{a+n+b} as the standard parabolic subgroup W_M. We rename $R = \mathbb{Z}[x_1, \dots, x_{a+n+b}]$ and write $\alpha_i = x_i - x_{i+1}$ for the simple root corresponding to s_i. Fix

$$C = \partial_{w_m}(x_{a+1}^{a_m} x_{a+n}^{b_m} \partial_{w_{m-1}}(x_{1+a}^{a_{m-1}} x_{a+n}^{b_{m-1}} \dots \partial_{w_1}(x_{a+1}^{a_1} x_{a+n}^{b_1}) \dots))$$

which we assume is a non-zero integer. (Now $w_1, \dots, w_m \in W_M$ and the fact that C is a non-zero integer implies that $\sum \ell(w_i) = a + b$.)

Let $M' = M \setminus \{s_{a+1}, s_{a+n-1}\}$. Then for any $w \in W_{M'}$, ∂_w commutes with the operators of multiplication by x_{a+1} and x_{a+n}. Hence we may (and do) assume that each w_i is of minimal length in its coset $w_i W_{M'}$.

Fix a reduced expression $\underline{w}_M$ for w_M and reduced expressions $\underline{w}_i$ for each w_i. Let $\underline{w}$ be the sequence

$$\underline{w} = \underline{w}_m \underline{u}_m \underline{v}_m \dots \underline{w}_2 \underline{u}_2 \underline{v}_2 \underline{w}_1 \underline{u}_1 \underline{v}_1 \underline{w}_M.$$

where

$$\underline{u}_1 = (s_a \dots s_{a-a_1+1}) \dots (s_a s_{a-1})(s_a)$$
$$\underline{u}_2 = (s_a \dots s_{a-a_1-a_2+1}) \dots (s_a \dots s_{a-a_1-1})(s_a \dots s_{a-a_1})$$
$$\vdots$$
$$\underline{u}_m = (s_a \dots s_1) \dots (s_a \dots s_{a-a_1-\dots-a_{m-1}-1})(s_a \dots s_{a-a_1-\dots-a_{m-1}})$$

(subscripts fall by 1 within each parenthesis, and s_a occurs a_i times in $\underline{u}_i$) and

$$\underline{v}_1 = (s_{a+n} \dots s_{a+n+b_1-1}) \dots (s_{a+n} s_{a+n+1})(s_{a+n})$$
$$\underline{v}_2 = (s_{a+n} \dots s_{a+n+b_1+b_2-1}) \dots (s_{a+n} \dots s_{a+n+b_1+1})(s_{a+n} \dots s_{a+n+b_1})$$
$$\vdots$$
$$\underline{v}_m = (s_{a+n} \dots s_{a+n+b-1}) \dots (s_{a+n} \dots s_{a+n+b_1+\dots+b_{m-1}}).$$

(subscripts rise by 1 within each parenthesis, and s_{a+n} occurs b_i times in $\underline{v}_i$).

Lemma 4.1. *$\underline{w}$ is reduced.*

Proof. It is routine but tedious to draw a picture of $\underline{w}$ and convince oneself that it is reduced, using that each w_i is minimal in $w_i W_{M'}$. □

Remark 4.2. In fact Lemma 4.1 is not necessary for the main theorem, as it is enough to demonstrate an intersection form equal to $(\pm C)$ associated to any expression $\underline{w}$, reduced or not.

Set $I = A \cup M \cup B$ and let w_I be its longest element. Write $\underline{w} = s_{i_1} \dots s_{i_l}$.

Lemma 4.3. *Any subexpression* $\mathbf{e}$ *of $\underline{w}$ with $\underline{w}^{\mathbf{e}} = w_I$ has $e_j = 0$ if $s_{i_j} \in \{s_a, s_{a+n}\}$ and $e_j = 1$ if $s_{i_j} \in A \cup B$.*

Proof. Let $\mathbf{e}$ denote a subexpression of $\underline{w}$ with $\underline{w}^{\mathbf{e}} = w_I$.

Any expression $\underline{y}$ for w_A contains a subexpression of the form $s_{a-1}s_{a-2}\dots s_1$ (think about what happens to $1 \in \{1, \dots, a\}$). In $\underline{w}$, s_1 only occurs once. Left of s_1 there is only one occurence of s_{a-1}, s_{a-2}, etc. We conclude that the restriction of $\mathbf{e}$ to $(s_a s_{a-1} \dots s_2 s_1)$ in $\underline{u}_m$ is equal to $(01 \dots 11)$. Now any expression for w_A starting in $s_{a-1} \dots s_1$ has to contain a subexpression to the right of the form $s_{a-1} \dots s_2$ (think about what happens to $2 \in \{1, \dots, a\}$). Continuing in this way we see that the restriction of $\mathbf{e}$ to each $\underline{u}_j$ has the form

$$(01 \dots 1) \dots (01 \dots 1)(01 \dots 1).$$

Similar arguments apply to each $\underline{v}_i$ and the result follows. □

Lemma 4.4. *There is a unique subexpression* $\mathbf{e}$ *of $\underline{w}$ such that $\underline{w}^{\mathbf{e}} = w_I$ and* $\mathbf{e}$ *has defect zero.*

Proof. By the previous lemma we must have $e_j = 0$ (resp. 1) if $s_{i_j} \in \{s_a, s_{a+n}\}$ (resp. $s_{i_j} \in W_{A\cup B}$). Because each e_j with $s_{i_j} \in \{s_a, s_{a+n}\}$ is U0 and because $W_{A\cup B}$ and W_M commute we only have to understand subexpressions of

$$\underline{w}' = \underline{w}_m \underline{w}_{m-1} \dots \underline{w}_1 \underline{w}_M$$

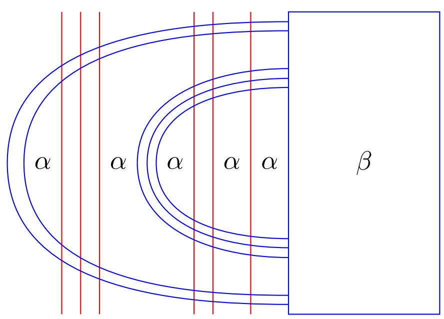
of defect $-(a+b) = -\sum_{i=1}^{m} \ell(w_i)$. Now it is easy to see that the only subexpression of $\underline{w}'$ fulfilling these requirements is

$$(0\dots0)(0\dots0)\dots(0\dots0)(1\dots1). \qquad \square$$

Recall that the intersection form associated to $\underline{w}$ at w_I in degree 0 is given by pairing light leaves maps of defect zero. By the above lemma there is only one subsequence of defect zero, and hence the intersection form is a 1×1 matrix. The following lemma calculates this matrix:

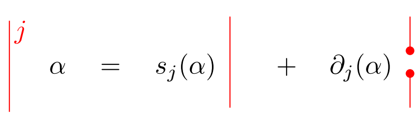
Lemma 4.5. *The intersection form of $\underline{w}$ at w_I is $(\pm C)$.*

Proof. Note that the light leaves map corresponding to the unique defect 0 subexpression $\mathbf{e}$ of $\underline{w}$ is simple: the last terms corresponding to $\underline{w}_M$ are U1 and all e_j with $s_j \in \{s_{a+1}, \dots, a_{a+n-1}\}$ before that are D0; all e_j with $s_j \in \{s_a, s_{a+n}\}$ are U0; and all other e_j (corresponding to simple reflections in $A \cup B$) are U1. We may picture the pairing of the light leaves with itself schematically as follows:



Here β denotes a morphism composed entirely of braid relations ($2m_{st}$-valent vertices in the terminology of [EWa]), blue lines indicate simple reflections belonging to the set M, α stands for either α_a or α_{n+a} and red lines stand for simple reflections belonging to $A \cup B$.

Every time we move an α through a red line we apply the nil Hecke relation:



However the expression in the red lines is reduced. Hence the second term (with the broken line) always contributes zero to the intersection form (it belongs to the ideal $\mathcal{D}_{\not\geq w_I}$). Hence we can ignore the red lines if we replace each occurence of α by $w(\alpha)$ for some $w \in W_{A\cup B}$. Any such $w(\alpha)$ is of the form

$$w(\alpha) = \alpha + \sum_{j\in A\cup B} \lambda_j \alpha_j.$$

Each α_j with $j \in A \cup B$ is invariant under W_M and hence if we apply the nil Hecke relation again we can slide all such terms outside all blue lines. Any diagram with a positive degree polynomial on the left or right is necessarily zero (because

$\mathrm{End}^\bullet_{\geq w_I}(w_I) = R$ is concentrated in non-negative degrees). Hence we can replace each $w(\alpha)$ by α. We conclude that we can simply ignore the red lines in the above picture.

Now, when moving the left most power of α through a blue line it must break. We conclude that the value of the intersection form is

$$C' = \partial_{w_m}(\alpha_a^{a_m}\alpha_{a+n}^{b_m}\partial_{w_{m-1}}(\alpha_a^{a_{m-1}}\alpha_{a+n}^{b_{m-1}}\dots\partial_{w_1}(\alpha_a^{a_1}\alpha_{a+n}^{b_1})\dots)).$$

Finally, $\alpha_a = x_a - x_{a+1}$ and $\alpha_{a+n} = x_{a+n} - x_{a+n+1}$. Because x_a and x_{a+n+1} commute with W_M we can apply the argument of the previous paragraph to get

$$\begin{aligned} C' &= \partial_{w_m}((-x_{a+1})^{a_m}x_{a+n}^{b_m}\partial_{w_{m-1}}((-x_{a+1})^{a_{m-1}}x_{a+n}^{b_{m-1}}\dots\partial_{w_1}(((-x_{a+1})^{a_1}x_{a+n}^{b_1})\dots)) \\ &= (-1)^a C. \end{aligned}$$ □

5. (Counter)-examples

We use the notation of §1.2.

5.1. $n < 4$**:** One checks easily using (1.1) and (1.2) that for $n = 2, 3$ one can only obtain $C = \pm 1$.

5.2. $n = 4$**:** Using (1.1) and (1.2) we see that

$$Y = x_1(\partial_1(x_1^2 X_{\mathrm{id}}) = X_{12} + X_{21}.$$

Now consider the operator $F : H \to H$ given by

$$F(h) = \partial_{13}(\omega_{13}\partial_{12}(\omega_{13}h)).$$

where $\omega_{13} = x_1(-x_4)$. Using (1.1) and (1.2) (Figure 1 might be helpful) one checks easily that X_{32}, X_{13} and X_{12} are in the kernel of F, that F preserves the submodule $\mathbb{Z}X_{21} \oplus \mathbb{Z}X_{23}$ and that in this basis F is given by the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix determines the Fibonacci recursion! Hence for $i \geq 1$ we have

$$F^i(x_1(\partial_1(x_1^2 X_{\mathrm{id}})) = F^i Y = F_{i+1}X_{21} + F_i X_{23}$$

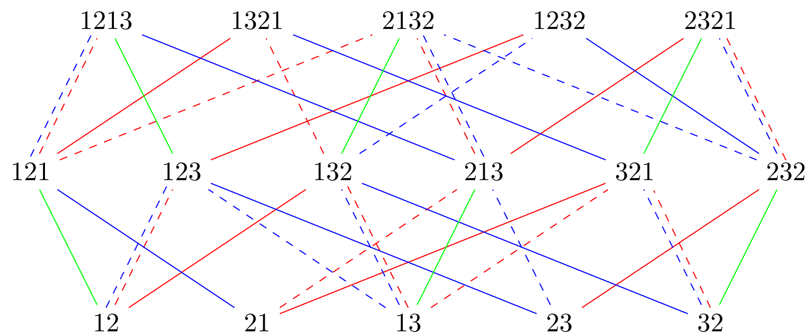
where $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ etc. denote the Fibonacci numbers.

We conclude from the main theorem that the Fibonacci numbers F_{i+1} and F_i occur as torsion in SL_{4i+7}. (For example $\partial_{12}(F^i(x_1(\partial_1(x_1^2)))) = F_{i+1}$.) By Carmichael's theorem [Car14] the first $n \gg 0$ Fibonacci numbers have at least n distinct prime factors. Now by the prime number theorem we conclude that the torsion in SL_n grows at least as fast as some constant times $n \log n$. Hence no linear bound is sufficient for Lusztig's conjecture.

$n = 5$: In the following table we list some examples of torsion in SL_N found using the SL_5 flag variety. The entries in the list were found by random computer searches, and are probably not optimal.

N	10	14	16	18	20	22	26	30	34	38	...	100	200
p	3	7	13	23	43	61	139	421	839	1867	...	5674199	1225837367

FIGURE 1. Part of the Bruhat graph for $S_4 : 1 - 2 - 3$.



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