KOSZUL DUALITY AND APPLICATIONS IN REPRESENTATION THEORY

Abstract. Notes from lectures given by Wolfgang Soergel in Luminy, September 2010. Notes by Geordie Williamson.

1. Lecture 1: Koszul duality

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ a semi-simple Lie algebra, Borel and Cartan. (Certainly on may take $\mathfrak{sl}_n$ containing the upper triangular matrices, and diagonal matrices.)

We consider

$$U(\mathfrak{g}) \supset Z \supset Z^+ = Ann_Z\mathbb{C}$$

where Z denotes the center.

We consider

$$\mathcal{O} := \left\{ M \in \mathfrak{g} - mod \;\middle|\; \begin{array}{c} \text{finitely generated } /\mathfrak{g} \\ \text{locally finite } /\mathfrak{b} \\ \text{semi-simple } /\mathfrak{h} \end{array} \right\}$$

A good reason to study to study $\mathcal{O}$ is that is "almost" Harish-Chandra modules for $G(\mathbb{C})$.

$\mathcal{O}$ contains $\mathcal{O}_0$ as a direct summand.

$$\mathcal{O}_0 = \{M \in \mathcal{O} \mid (Z^+)^N M = 0 \text{ for } N \text{ big enough}\}.$$

Note that modules in $\mathcal{O}_0$ are certainly not unitary, and we study to what extent semi-simplicity fails when we leave unitary representations.

Set

$$\begin{array}{c} L = (\text{sum of all simples from } \mathcal{O}_0) \\ \downarrow \\ I = \text{injective hull of } L. \end{array}$$

(Note that I is well-defined up to non-unique isomorphism.)

Theorem 1.1 (Koszul self-duality for $\mathcal{O}_0$)**.** *There exists an isomorphism of finite dimensional $\mathbb{C}$-algebras:*

$$\mathrm{End}_{\mathcal{O}_0} I \cong \bigoplus Ext^i_{\mathcal{O}_0}(L, L).$$

(This was conjecture by Beilinson and Ginzburg.)

If $\lambda \in \mathfrak{h}^*$ this leads to $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda = \Delta(\lambda) \in \mathcal{O}$. This is a *Verma module* and corresponds to a principal series Harish-Chandra module.

One has a unique simple quotient

$$\Delta(\lambda) \to L(\lambda)$$

the unique simple quotient.

There is a bijection between $\mathfrak{h}^* \xrightarrow{\sim} \operatorname{irr} \mathcal{O}$ which associates $L(\lambda)$ to $\lambda \in \mathfrak{h}^*$.

This restricts to a bijection

$$W \xrightarrow{\sim} \operatorname{irr} O_0$$
$$x \mapsto L(x \cdot 0)$$

Where the W-action on $\mathfrak{h}^*$ is shifted by $-\rho$ where ρ denotes the half-sum of the positive roots.

Hence $L = \oplus_{x \in W} L(x \cdot 0)$ and we have the injective hull $L(x \cdot 0) \hookrightarrow I(x \cdot 0)$. Then

$$I = \bigoplus_{x \in W} I(x \cdot 0).$$

Hence the above theorem becomes

$$\operatorname{End}_{\mathfrak{g}}(\bigoplus I(xw_0 \cdot 0)) \xrightarrow{\sim} \operatorname{Ext}^\bullet_{\mathcal{O}_0}(\bigoplus L(x \cdot 0))$$

This isomorphism preserves the idempotents. The idempotent 1_x is mapped to the idempotent 1_x.

Let us recall some general theory. Let $\mathcal{A}$ be an abelian category in which every object is of finite length and assume $\mathcal{A}$ admits an injective generator I. (This means that every simple object has a non-zero morphism to I.)

Then we have an equivalence of categories:

$$\mathcal{A} \xrightarrow{\sim} \operatorname{End}_{\mathcal{A}} I - \text{Modfl}$$
$$M \mapsto \operatorname{Hom}_{\mathcal{A}}(M, I).$$

(where Modfl denotes modules of finite length.)

Proof. Functor is exact and fully faithful on $\oplus I$. Given any object we resolve it

$$M \hookrightarrow \oplus_{finite} I \to \oplus_{finite} I$$

Hence the functor is fully faithful always. □

This means that if we set

$$A = \operatorname{End}_{\mathfrak{g}}(\bigoplus I(xw_0 \cdot 0))$$

then $\mathcal{O}_0 \cong A-\text{Modfl}$. (We use the fact that $\mathcal{O}_0$ has the above nice properties.)

Example 1.2. Let us consider $\mathfrak{g} = \mathfrak{sl}_2$ and $M \in \mathcal{O}_0$. There are two simple objects $L(0) = \mathbb{C}$ and $L(-2\rho) = \Delta(-2\rho)$. The only weight spaces which M can have belong to $\mathbb{N}(-2\rho)$.

If we let X, H, Y be as usual. $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We can picture M as

$$\begin{array}{c} M_0 \\ X \uparrow \downarrow Y \\ M_{-2} \\ X \uparrow \downarrow Y \\ M_{-4} \end{array}$$

and we have a functor that only remembers the M_0 and M_{-2} weight spaces. We get a functor to representations of the quiver

$$\begin{array}{c} \bullet \\ X \uparrow \downarrow Y \\ \bullet \end{array}$$

and one can check that this gives an equivalence with representations that satisfy $xy = 0$. We get that $\dim_{\mathbb{C}} A = 5$ with basis

deg		
2	yx	
1	x	y
0	1_e	1_s

We now explain why the theorem can be seen as saying that A is a self-dual Koszul ring.

Definition 1.3. A positively graded $\mathbb{Z}$-graded ring $A = \bigoplus_{i \geq 0} A^i$ is called *Koszul* if

(1) A^0 is semi-simple,
(2) As a module for A, $A^0 = A/A^{\geq 0}$ admits a graded projective resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to A$$

with P_i generated in degree i (that is $P_i = AP_i^i$).

(Note that the fact that A^0 is semi-simple in degree 0 is clear in the above example. The idempotent projectors give the decomposition.)

Example 1.4. (1) $A = A^0$ is semi-simple (this is a silly example).
(2) $A = k[X]$ then we have a graded projective resolution

$$k[X]X \to k[X] \to k.$$

(3) $A = SV$ where $\dim_k V < \infty$ and we have a resolution

$$\ldots \to SV \otimes \Lambda^2 V \to SV \otimes V \to SV \to k$$

(this is the Koszul complex).

We first give a more intrinsic alternative definition of the Koszul property. A positively $\mathbb{Z}$-graded ring A is called Koszul if $M = M^m$ and $N = N^n$ and both M and N are graded A-modules. (We say that M and N are *pure*.) Then

$$\mathrm{Ext}^i_{A-\mathrm{gMod}}(M, N) = 0$$

unless $i = n - m$. Hence extensions between M and N only live on the diagonal.

Remark 1.5. This can be seen as a graded version of semi-simple. This is the "closest that a graded ring can come to being semi-simple".

We now explain what self-dual means.

Definition 1.6. If A is a positively graded $\mathbb{Z}$-graded ring then we can define

$$E(A) = \mathrm{Ext}^\bullet_A(A^0, A^0)$$

Theorem 1.7. *If A is Koszul and A^1 is finitely generated as a left A^0-module then $E(A)$ has the same property. Moreover*

$$E(E(A)) \cong A$$

canonically.

$E(A)$ is called the *Koszul dual* of A.

Remark 1.8. In the above case, $\mathrm{Ext}^\bullet_A(A^0, A^0)$ is formal, and this is why it makes sense to consider it.

Sketch. The first step is to show that if A is Koszul then A is *quadratic*. This means that A is generated in $k := A^0$ and $V = A^1$. Then V is a k-bimodule. Consider the exact sequence

$$R \to T_k V = \bigoplus_{n \geq 0} V^{\otimes n} \to A.$$

The quadratic condition says that the kernel $R \subset V \otimes_k V$ (that is, generated in degree 2). Given a quadratic ring one can form $A^!$ the quadratic dual ring. This is defined as

$$A^! = T_k V^* / \langle R^\perp \rangle.$$

(Be careful of left and right duals in the above.) The theorem is that if A is Koszul, then $E(A) \cong (A^!)^{opp}$. The theorem then follows in a straightforward way. ☐

We explained at the beginning the equivalence

$$A - \quad \mathrm{mod} \xrightarrow{\sim} \mathcal{O}_0$$
$$A^0 \mapsto L$$
$$E(A) \cong \mathrm{Ext}^\bullet_{\mathcal{O}_0}(L, L) \cong A.$$

This implies that $E(A) \cong A$. (This is the self-duality alluded to before.)

2. Second lecture

Recall from last time that we said that a graded ring $A = \bigoplus_{i \geq 0} A^i$ is *Koszul* if $M, N \in A - \mathrm{Mod}^{\mathbb{Z}}$, $M = M^m$, $N = N^n$ then

$$\mathrm{Ext}^i_{A-\mathrm{Mod}^{\mathbb{Z}}}(M, N) = 0$$

unless $n = m + i$. We also defined the Koszul dual ring

$$E(A) = \mathrm{Ext}^\bullet_A(A^0, A^0)$$

Theorem 2.1.

$$\mathrm{Der}^b(A - \mathrm{Modfl}^{\mathbb{Z}}) \xrightarrow{\sim}{}^{\kappa} \mathrm{Der}^b(\mathrm{Modfl}^{\mathbb{Z}} - E)$$

We have

$$\kappa(M\langle n \rangle) = (\kappa M)[-n]\langle -n \rangle$$

For all $p \in A^0$ we have:

$$A^0 p \mapsto pE$$

in other words, the simples are sent to projectives.

If: *A is of finite length over A^0 from left and right and E is right Noetherian.*

In this theorem we use the definition $(M\langle n \rangle)^i = M^{i-n}$.

Example 2.2. If $A = \Lambda V^*$, $E = SV$ and $\dim V < \infty$. Then

$$\mathrm{Der}^b(\Lambda V^* - \mathrm{Modfl}^{\mathbb{Z}}) \xrightarrow{\sim} \mathrm{Der}^b(SV - \mathrm{Modfl}^{\mathbb{Z}}).$$

On the right is the category of coherent sheaves on $\mathbb{P}(V)$ (after dividing out by finite dimensional representations of SV), and on the left are the problems of linear algebra. (c.f. Beilinson's theorem).

Because the above is at the heart of the Koszul duality formalism, we will give a proof.

Proof. 1) Suppose that $\mathcal{I}$ is an additive category. Consider its homotopy category $\mathrm{Hot}\,\mathcal{I}$ and chain complexes $\mathrm{Ket}\,\mathcal{I}$. Given a complex $T \in \mathrm{Ket}\,T$ then

$$E = \mathrm{Hom}_{\mathcal{I}}(T, T) = \mathrm{End}_{\mathcal{I}}(T)$$

is a complex of abelian groups, and is even a dg-ring.

We have a functor

$$\mathrm{Ket}\,\mathcal{I} \xrightarrow{\mathrm{Hom}_{\mathcal{I}}(T,-)} \mathrm{dgMod}-E$$

and it descends to a functor

$$\mathrm{Hot}\,\mathcal{I} \to \mathrm{dgHot}-E$$

We may identify both sides of

$$\mathrm{Hot}_{\mathcal{I}}(T, [n]T) \to \mathrm{dgHot}_{-E}(E, [n]E)$$

with $\mathcal{H}^n E$. Hence we get an equivalence of triangulated categories

$$\mathrm{Hot}\,\mathcal{I} \to \mathrm{dgHot}-E$$
$$\langle T\rangle_\Delta \xrightarrow{\sim} \langle E\rangle_\Delta$$

This shows the close relationship between triangulated categories and modules over differential graded algebras.

2) Now imagine that $\mathcal{A}$ is an abelian category. Let us say that $T \in \mathrm{Ket}\,\mathcal{A}$ is *endacyclic* if

$$\mathrm{Hot}_{\mathcal{A}}(T, [n]T) \xrightarrow{\sim} \mathrm{Der}_{\mathcal{A}}(T, [n]T)$$

for all n. For example a bounded below complex of injective objects is endacyclic.

In this case we could reinterpret the above as giving an equivalence

$$\begin{array}{ccc} \mathrm{Der}\,\mathcal{A} & & \mathrm{dgDer}-E \\ \cup & & \cup \\ \langle T\rangle_\Delta & \xrightarrow{\sim} & \langle E\rangle_\Delta \\ \cap & & \cap \\ \mathrm{Hot}\,\mathcal{A} & & \mathrm{dgHot}-E \end{array}$$

2b) If $\mathcal{A}$ is a Γ-category and T is such that $\bigoplus_{\gamma\in\Gamma} T\langle\gamma\rangle$ is endacyclic. (This can be made to make sense even if we don't have infinite sums.)

Then we set

$$E = \bigoplus_{\gamma\in\Gamma} \mathrm{Hom}(T, T\langle\gamma\rangle)$$

which is a Γ-graded differential graded ring. Then we have an equivalence

$$\begin{array}{ccc} \mathrm{Der}\,\mathcal{A} & & \mathrm{dgDer}^{\Gamma} - E \\ U & & U \\ \langle T\langle\gamma\rangle\rangle_{\gamma\in\Gamma} & \xrightarrow{\sim} & \langle E\langle\gamma\rangle_{\gamma\in\Gamma} \end{array}$$

3) Let Γ be an abelian group. Let $R = \bigoplus R^n_\gamma$ a Γ-graded differential graded ring. We can consider

$$\mathrm{dgMod}^{\Gamma}_G, \mathrm{dgHot}^{\Gamma}_R, \ldots$$

Let $\varphi : \Gamma \to \mathbb{Z}$ be a group homomorphism, we can define

$$\hat{R}^n_\gamma = R^{n+\varphi(\gamma)}_\gamma$$

and we get an equivalence

$$\mathrm{dgMod}^{\Gamma} - R \xrightarrow{\sim} \mathrm{dgMod}^{\Gamma} - \hat{R}.$$

4) We now try to put all these things together.

We set $\mathcal{A} = A - \mathrm{Mod}^{\mathbb{Z}}$ and as our endacyclic complex T we take

$$T = P^\bullet \quad \text{a projective graded resolution of } A_0.$$

We set $E = \mathrm{End}_{\mathcal{A}}(P^\bullet)$. From Step 2b) we have an equivalence

$$\begin{array}{ccc} \mathrm{Der}(A - \mathrm{Mod}^{\mathbb{Z}} & & \mathrm{dgDer}^{\mathbb{Z}} - E \\ \cup & & \cup \\ \langle A_0\langle n\rangle\rangle_{n\in\mathbb{Z}} & \xrightarrow{\sim} & \langle E\langle n\rangle_{n\in\mathbb{Z}} \end{array}$$

We now apply 3) to regrade to $\hat{E}$ such that all homology is in degree zero.

But if we have a differential graded algebra with all its cohomology concentrated in degree zero then we have quasi-isomorphisms

$$\hat{E} \leftarrow \hat{Z} \to \hat{H}$$

and hence

$$\mathrm{dgDer}^{\mathbb{Z}} - E \xrightarrow{\sim} \mathrm{dgDer}^{\mathbb{Z}} - \hat{H}.$$

But now $\hat{H} = E(A)$. This is Koszul duality! (One needs to allow direct summands if A_0 is not a field.) □

We now come back to representation theory. Recall that

$$\mathcal{O}_0 = A - \mathrm{modfg}$$

Theorem 2.3. $A \cong \mathrm{Ext}^\bullet(L)$.

We want to deduce that A is Koszul!

Proposition 2.4. *If A is a finite dimensional algebra over F a field. If there exists and isomorphism*

$$A \cong E(A) = \mathrm{Ext}^\bullet(A_0, A_0)$$

then A is Koszul!

Proof. We $k = A_0$. We have an exact sequence

$$A_{>0} = I \hookrightarrow A \to k$$

We get (by the long exact sequence)

$$\begin{array}{ccc} \mathrm{Ext}^1_A(k,k) = \mathrm{Hom}_A(I,k) & \cong & \mathrm{Hom}_A(I/I^2,k) \\ \cup & & \cup \\ \mathrm{Ext}^1_{A-\mathrm{Mod}^{\mathbb{Z}}}(k,k\langle 1\rangle) & \cong & \mathrm{Hom}_A(A_1,k) \end{array}$$

Now $\mathrm{Ext}^1_A(k,k)$ and $\mathrm{Hom}_A(A_1,k)$ are of the same dimension. Hence both inclusions above are equalities. Hence A is generated by A_1 and A_0. But we have seen that all the $\mathrm{Ext}^1_A(k,k)$ are pure. But since these generated the ring, all extensions are pure, and Koszulity follows. □

Definition 2.5. $\mathcal{O}_0^{\mathbb{Z}} = A - \mathrm{modfg}^{\mathbb{Z}}$. Then $A \cong E(A)$ has a Koszul grading. Let us call this a $\mathbb{Z}$-graded version of $\mathcal{O}_0$.

Corollary 2.6. $\mathrm{Der}^b(\mathcal{O}_0^{\mathbb{Z}}) \xrightarrow{\sim} \mathrm{Der}^b(\mathcal{O}_0^{\mathbb{Z}})$. *There exist lifts of the simple and projective modules such that*

$$\widetilde{L(x\cdot 0)} \mapsto P(\widetilde{w_0 x \cdot 0})$$
$$\kappa(M\langle n\rangle) = (\kappa M)\langle -n\rangle[-n].$$

Also:

$$\widetilde{\nabla(y\cdot 0)} \mapsto \Delta(\widetilde{w_0 y \cdot 0}).$$

(Note that $A = A^{\mathrm{op}}$.)

When people began investigating category $\mathcal{O}$ they discovered certain strange formulas of the form:

$$[\Delta(x\cdot 0), L(y\cdot 0)] = \sum \dim \mathrm{Ext}^i(\Delta(w_0 x\cdot 0), L(w_0 y\cdot 0)$$

In fact, both sides are given by the values at 1 of certain Kazhdan-Lusztig polynomials. Beilinson and Ginzburg were investigating a framework where the above strange formulas could be given an explanation. This gave rise to Koszul duality!

The above formulae have the following interpretation using Koszul duality

$$\begin{aligned}\operatorname{Ext}^i(L(y\cdot 0),\nabla(x\cdot 0)) &= \bigoplus_j \operatorname{Der}_{\mathcal{O}^{\mathbb{Z}}}(\widetilde{L(y\cdot 0)},\widetilde{\nabla(x\cdot 0)}[i]\langle j\rangle)\\ &= \operatorname{Der}_{\mathcal{O}^{\mathbb{Z}}}(P(\widetilde{w_0y\cdot 0}),\Delta(\widetilde{w_0x\cdot 0})[i-j]\langle -j\rangle)\\ &= [\Delta(w_0x\cdot 0) : L(w_0y\cdot 0)\langle -i\rangle].\end{aligned}$$

3. Lecture 3

Recall that last time we saw that we have

$$\begin{array}{ccccc}\mathcal{O} & \supset & \mathcal{O}_0 & \ni & \bigoplus_{x\in W} L(x\cdot 0)\\ & & \in & & \\ & & \bigoplus I(x\cdot 0) & & \end{array}$$

We saw an isomorphism

$$\operatorname{End}_{\mathfrak{g}}(\bigoplus I(x\cdot 0)) \xrightarrow{\sim} \operatorname{End}^{\bullet}_{\mathcal{O}_0}(\bigoplus_{x\in W} L(x\cdot 0)).$$

We saw that such an isomorphism can be used to explain certain inversion formulas for Kazhdan-Lusztig polynomials.

But first we should an idea how the above isomorphism comes about.

There exists a functor:

$$\mathbb{V} : \mathcal{O}_0 \to \mathbb{C} - \mathrm{Mod}$$

this functor is exact, $L(x\cdot 0)$ if $x \neq w_0$ and $L(w_0\cdot 0) = \Delta(w_0\cdot 0) \mapsto \mathbb{C}$.

In fact, this functor is given by $\operatorname{Hom}(?, I(w_0\cdot 0))$. Hence this functor descends to

$$\mathbb{V} : \mathcal{O}_0 \to Z - \mathrm{Mod}$$

where Z-denotes the center of the enveloping algebra. One can show that this functor is fully-faithful on injective modules.

Hence we may identify

$$\operatorname{End}_{\mathfrak{g}}(\bigoplus I(x\cdot 0)) = \operatorname{End}_Z(\bigoplus \mathbb{V}I(x\cdot 0)).$$

To handle the other side we need localisation:

$$\mathcal{D}_{G/B} - \mathrm{Mod} \overset{\sim}{\to}^{\Gamma} U/Z^+U - \mathrm{Mod}$$

This leads to an equivalence of categories (cheating a bit):

$$\mathcal{P}_N(G/B) \xrightarrow{\sim} \mathcal{O}_0$$

The left-hand category is perverse sheaves constructible on Bruhat cells. Under this equivalence $L(w_0x\cdot 0)$ corresonds to $\mathcal{L}_x := \mathcal{IC}(\overline{BxB/B})$.

Hence the right hand side above becomes

$$\mathrm{Ext}^\bullet_{\mathcal{O}_0}(\bigoplus L_x) \xrightarrow{\sim} \mathrm{Der}^\bullet_{G/B}(\bigoplus \mathcal{L}_x)$$

On $\mathrm{Der}_{G/B} = \mathrm{Der}(\mathbb{C} - \mathrm{Mod}_{G/B})$ we have the functor

$$\mathbb{H}^\bullet(?) : \mathrm{Der}_{G/B} \to H^\bullet(G/B) - \mathrm{Mod}.$$

It turns out this functor is fully-faithful on morphisms between intersection cohomology complexes and so we have an isomorphism

$$\mathrm{Der}^\bullet_{G/B}(\bigoplus \mathcal{L}_x) \xrightarrow{\sim} \mathrm{End}_{H^\bullet(G/B)-\mathrm{Mod}}(\bigoplus \mathbb{H}^\bullet \mathcal{L}_x).$$

Now take $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ we have the Harish-Chandra isomorphism

$$Z \xrightarrow{\sim} (S\mathfrak{h})^{W\cdot}$$

when we complete at Z^+ we obtain an isomorphism

$$Z^\vee_{Z^+} \xrightarrow{\sim} (S\mathbf{h})^\vee_0.$$

We also have the Borel isomorphism

$$(S\mathbf{h}^\bullet)^+_0 \twoheadrightarrow H^\bullet(G/B).$$

One can show isomorphisms

$$\mathrm{End}_Z(\bigoplus \mathbb{V}I(x \cdot 0)) = \mathrm{End}_{(S\mathfrak{h})^\vee_0-\mathrm{Mod}}(\bigoplus \mathbb{V}I(x \cdot 0))$$

Hence the above reduces to showing an isomorphism

$$(S\mathfrak{h})^\vee_0 - \mathrm{Mod} \ni \mathbb{V}I(x \cdot 0) \overset{?}{\cong} \mathbb{H}^\bullet(\mathcal{L}_x) \in (S\mathfrak{h}^*)^\vee_0 - \mathrm{Mod}.$$

Note that the left hand module is a module over $(S\mathfrak{h})^\vee_0$ and the right hand module is a module over $(S\mathfrak{h}^*)^\vee_0$. Hence we need to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. This can be done using a Killing form, but it is much better to regard the above using Langlands dual groups.

We now take another approach. Suppose that H acts on X, an algebraic variety over $\mathbb{C}$. Suppose additionally that H acts with finitely many orbits. We can consider

$$\bigoplus_\pi \mathcal{L}^\pi \in \mathrm{Der}_H(X)$$

the sum of all simple perverse sheaves (there are finitely many based on our assumptions).

We can set

$$\mathrm{Ext}^\bullet_H(X) := \mathrm{Ext}^\bullet_H(\bigoplus_\pi \mathcal{L}^\pi).$$

What we learnt before is that

$$\mathcal{O}_0 \cong \mathrm{Ext}^\bullet_{N^\vee}(G^\vee/B^\vee) - \mathrm{modfg}\,.$$

3.1. **Koszul duality for real groups.** We now try to learn what the analogy of this statement is for real groups.

We take $G/\mathbb{C}$ a reductive connected algebraic group. γ is an anti-holomorphic quasi-split involution. (This means that there is a Borel subgroup fixed by γ.)

Here I am relying on work of Adams, Barbarsch and Vogan. (In fact, most of the work is due to them.) We consider the first Galois cohomology

$$H^1(\Gamma; G(\mathbb{C}))$$

Γ denotes the Galois group of $\mathbb{R}$ over $\mathbb{C}$ and γ denotes its non-trivial element. Given $\delta \in H^1(\Gamma; G(\mathbb{C}))$ this leads to $G(\mathbb{R}; \delta)$. (Note that this does not classify real forms; these are classified by $H^1(\Gamma; \operatorname{Aut} G(\mathbb{C}))$.)

We consider

$$\bigoplus_{\delta \in H^1(G; G(\mathbb{C}))} \mathcal{M}(G(\mathbb{R}, \delta))$$

where $\mathcal{M}$ denotes smooth admissible representations (or equivalently, Harish-Chandra bimodules).

We take $\chi \in \operatorname{Max} Z$ and $Z \subset U(\mathfrak{g})$ denotes the centre as before. We consider

$$\bigoplus_{\delta \in H^1(G; G(\mathbb{C}))} \mathcal{M}(G(\mathbb{R}, \delta))_\chi$$

where the subscript χ denotes the modules killed by some power of χ.

Given (G, γ, χ) then Adams, Barbarsch and Vogan associate a variety $X(\chi)$ together with an action of $G^\vee$. ($X(\chi)$ is a variant of the Langlands parameter space.)

If χ integral then $X(\chi)$ is simply the cycles $G^\vee \times_{P^\vee(\chi)} Z^1(\Gamma, G^\vee)$. We could write this as the set

$$X(\chi) = G^\vee \times_{P^\vee(\chi)} \{g \in G^\vee \, | g(g^\gamma) = 1\}.$$

where $P^\vee(\chi)$ is the parabolic corresponding to χ. ($P^\vee(\chi) = G^\vee$ if χ is maximally singular, and $P^\vee(\chi) = B^\vee$ if χ is regular.)

As before we can form

$$\operatorname{Ext}^\bullet_{G^\vee} X(\chi) - \operatorname{Nil}$$

where Nil denotes modules of finite dimension killed by high degrees.

The conjecture is

$$\bigoplus_{\delta \in H^1(G; G(\mathbb{C}))} \mathcal{M}(G(\mathbb{R}, \delta))_\chi \cong \operatorname{Ext}^\bullet_{G^\vee} X(\chi) - \operatorname{Nil}$$

this looks a bit crazy (hopefully!).

First we should check that both sides have the same number of simple objects. On the right hand the simple objects are parametrised by π.

Let me introduce a nice parameter set. As before we have H acting on X and we set

$$P(H,X) = \left\{ (Y,\tau) \;\middle|\; \begin{matrix} Y \subset X \text{ an orbit} \\ \tau \in (G_y/G_y^0)^\vee \text{ for } y \in Y \end{matrix} \right\}.$$

We have a bijection

$$\begin{aligned} \mathrm{Par}(H,X) &\xrightarrow{\sim} \operatorname{irr} \mathcal{P}_H(X) \\ \pi &\mapsto \mathcal{L}^\pi = \mathcal{IC}(\overline{Y},\tau). \end{aligned}$$

It is easy to see that $\mathrm{Par}(H,X)$ parametrises simple modules on the right hand side above. What ABV were able to show is a bijection

$$\mathrm{Par}(G^\vee, X(\chi)) \xrightarrow{\sim} \operatorname{irr} \bigoplus_{\delta \in H^1(G;G(\mathbb{C}))} \mathcal{M}(G(\mathbb{R},\delta))_\chi.$$

For complex groups one can use the above techniques to check that the conjecture is true for complex groups.

One can also check $SL(2;\mathbb{R})$.

One can also check that the conjecture is true for generic χ. Here the parameter space is basically a flag variety, and the proof goes as above.

Let us see what it says for $\mathbb{R}^\times$, the multiplicative group of the real numbers.

There is only one form (Hilbert's theorem 90?). We claim an equivalence

$$\mathcal{M}(\mathbb{R}^\times)_\chi \cong \mathbb{C}[t] - \mathrm{Nil} \oplus \mathbb{C}[t] - \mathrm{Nil}\,.$$

The two copies come from the action of ± 1. Then we identify $\mathbb{C}[t]$ with the enveloping algebra.

We now consider the other side. We have $\mathbb{C}^\times$ with $\gamma(z) = \overline{z}$. On the dual side we have $\mathbb{C}^\times = G^\times$ and $\gamma = \mathrm{Id}$. Hence

$$Z^1 = \{ z \in \mathbb{C}^\times \mid z^2 = 1 \}.$$

Hence we have to compute

$$\mathrm{Ext}^\bullet_{\mathbb{C}^\times}(\text{two points}).$$

We have

$$\mathrm{Ext}^\bullet_{\mathbb{C}^\times}(\mathrm{pt}) = H^\bullet_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[t]$$

and the conjecture follows.

In a similar way one can check the conjecture for tori.

One can also check the conjecture for p-adic groups in the unramified case. (The same statement holds by work of Lusztig and Ginzburg. It was checked by Suy Kato.)

So this gives some evidence.

There is another range of evidence, given by identities of Kazhdan-Lusztig polynomials. There exists a Jordan-Hölder matrix (measuring multiplicities of simple modules in standard modules)

$$\mathcal{JH}_{\pi,\Psi}$$

On the other side we have a matrix (given by stalks of intersection cohomology complexes):

$$\mathcal{IC}_{\pi,\Psi}$$

These two matrices are inverse transpose up to sign. (This is Vogan duality explained in a paper called ... IV. It was understood in ABV that Vogan duality could be interpreted in this way.

Let us set $A = \mathrm{Ext}^{\bullet}_{G^\vee}(X(\chi)) - \mathrm{Nil}$. Consider

$$\mathrm{Der}(A - \mathrm{Mod}^{\mathbb{Z}})$$

There are two ways to forget the grading on this category. There are two ways to forget the grading

$$\begin{array}{ccc} & \mathrm{Der}(A - \mathrm{Mod}^{\mathbb{Z}}) & \\ \swarrow & & \searrow \\ \mathrm{Der}(A - \mathrm{Mod}) & & \mathrm{dgDer}_A \\ \cup & & \cup \\ \mathrm{Der}(\oplus\mathcal{M}) & & \mathrm{Der}_{G^\vee}(X(\chi)) \end{array}$$

On the right, $\mathrm{Der}_{G^\vee}(X(\chi))$ can be described in terms of modules over a differential graded algebra. (This was recently completed by Olaf Schnürer.)