SIX LECTURES ON DELIGNE-LUSZTIG THEORY


1. Lecture 1

Here the example of $SL_2$ was considered in detail and the character table was discussed.

2. Lecture 2: Finite groups of Lie type

Fix $k = \overline{k}$.

Algebraic group (affine) over $k$ is an affine algebraic smooth variety $G$ endowed with a group structure.

Consider the algebra of regular functors $k[G]$ on $G$. Then $G = \text{Spec} k[G]$ and $k[G]$ is a (commutative) Hopf algebra.

Example 2.1. 

- $G_a = \text{Spec} k[T]$, $G_a(k) = k$ as an additive group.
- $G_m = \text{Spec} k[T, T^{-1}]$, $G_a(k) = k^*$ multiplicative group.

These are all the one-dimensional groups.

- $GL_n = \text{Spec} k[T_{ij}]_{1 \leq i,j \leq n}[\det(T_{ij})^{-1}]$, $GL_n(k)$ consists of invertible $n \times n$ matrices over $k$.

Fact 2.2. Every algebraic group is isomorphic to a closed subgroup of $GL_n$ for some $n$. (This is the reason for the terminology “linear algebraic group”).

The notion of a semi-simple (diagonalisable) element of $GL_n$ gives rise to the notion of a semi-simple element for any $G$. Similarly we have the notion of a unipotent element of $GL_n$ (conjugate to an upper uni-triangular matrix). This gives rise to the notion of a unipotent element in any $G$.

Fact 2.3 (Jordan Decomposition). Let $x \in G$. There exists a unique pair $(u, s)$ in $G$ such that $x = us$, $[u, s] = 1$, such that $u$ is unipotent and $s$ is semi-simple.

(This is an analogue of the $p, p'$ decomposition of elements in a finite group.)
$G^0$ is defined to be the maximal connected closed subgroup of $G$ and $G/G^0$ is a finite group.

$R_u(G)$: unipotent radical = largest closed normal connected unipotent subgroup.

A nice fact is that $R_u(G/R_u(G)) = 1$.

So an arbitrary finite group looks like:

(2.1) $G/G^0$ finite

(2.2) $G = G^0/R_u(G)$ connected, reductive

(2.3) $R_u(G)$ unipotent

A connected algebraic group $G$ is solvable if it has a chain of closed normal subgroups with 1-dimensional quotients (that is, $\mathbb{G}_a$ or $\mathbb{G}_m$).

Fact 2.4. A unipotent group is solvable.

Definition 2.5. A torus is an algebraic group isomorphic to $\mathbb{G}_m^r$ for some $r$.

Proposition 2.6.

- If $G$ is an algebraic group, there exists a maximal closed torus in $G$. All maximal tori are conjugate.
- There exists a maximal closed solvable subgroup (called a Borel subgroup). All Borel subgroups are conjugate.

Proposition 2.7. $G$ solvable, $T$ maximal torus in $G$. Then

$G = R_u(G) \rtimes T$, semi-direct product

Example 2.8. $G = GL_n \supset B \supset U$. Here $B$ upper triangular matrices, $U$ is unitriangular matrices, $T$ is diagonal matrices.

Proposition 2.9.

- If $B$ is a Borel subgroup then $N_G(B) = B$.
- If $T \subset G$ is a maximal torus, then $C_G(T) = T$. Moreover $W := N_G(T)/T$ is a finite group, the Weyl group.

Example 2.10. $G = GL_n$, $N_G(T) =$ monomial matrices, $W = S_n$ the symmetric group.

2.1. Finite fields. $\mathbb{F}_q$ a finite field, $k = \overline{\mathbb{F}_q}$, $q = p^m$. $X_0$ algebraic variety over $\mathbb{F}_q$, $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$.

The (geometric) Frobenius endomorphism $F : X \to X$ which acts on $a = a_0 \otimes \alpha \in k[X] = \mathbb{F}_q[X] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ then $F(a) = a_0^q \otimes \alpha$.

Example 2.11. $X_0 = \text{Spec} \mathbb{F}_q[X_{ij}, \det(X_{ij})^{-1}]$, $X = GL_n$. $F(X_{ij}) = X_{ij}^q$, $g \in GL_n(\overline{\mathbb{F}_q})$, $g = g_{ij}$ then $F(g) = (g_{ij}^q)$.

There is another possibility: $F'(g) = (g^q)^{-1}$ on $G$. Note that $(F')^2 = F^2$. ($F'$ is “a” Frobenius because $(F')^2$ is one.)
Given an $F$-rational structure is the same as giving the endomorphism $F$.

We can look at $(X(F_q))^F = X_0(F_q)$, a finite set.

If $G$ is an algebraic group defined over $F_q$, or equivalently endowed with a Frobenius endomorphism $F$. Then $G^F$ is a finite group.

**Example 2.12.** $GL^F_n = GL_n(F_q)$ and $GL^F_n = U_n(F_q)$.

We define the Lang map $L : G \to G : g \mapsto g^{-1}F(g)$.

Clearly $\ker L = G^F$.

**Theorem 2.13.** If $G$ is connected then $L$ is surjective.

The Lang map (and the above theorem) is absolutely fundamental to all that follows! It allows us to pass between the algebraic group $G$ and its points over finite fields.

**Proof.** (Sketch) The key point is that $(dF)_1 = 0$. Hence $(dL)_1$ is equal to multiplication by $-1$. Hence $(dL)_1$ is bijective. Hence $L(G)$ contains a dense open subset (algebraic geometry: the image contains an open subset). Fix $x \in G$. Consider the map $L_x : g \mapsto g^{-1} \cdot x \cdot F(g)$ then $(dL_x)_1$ is bijective. Hence $L_x(G)$ contains a dense open subset. Hence $L(G) \cap L_x(G) \neq \emptyset$. Hence there exists $g, h \in G$ such that $g^{-1}F(g) = h^{-1}xF(h)$ and hence $x = L(gh^{-1})$.

Let $G$ be an algebraic group and $H$ a closed normal subgroup. We obtain an algebraic group $G/H$ and one has a canonical morphism $G \to G/H$ of algebraic groups.

(This can be done neatly using the functor of points approach).

Suppose $G$ is defined over $F_q$ and $H$ is an $F$-stable. Then we have $G^F/H^F \hookrightarrow (G/H)^F$.

**Proposition 2.14.** If $H$ is connected, then $G^F/H^F \cong (G/H)^F$.

This is false if $H$ is not connected.

**Example 2.15.** Take $Z \subset SL_n$ where $Z$ denotes the centre.

[Exercise: Work out this example for $SL_2$.]

Hence $SL_n(F_q)/\pm 1 \neq (SL_2/\pm 1)^F$.

**Lemma 2.16.** If $G$ is a connected algebraic variety acting on $X$. Assume that everything is defined over $F_q$. If $O$ is an $F$-stable orbit of $G$ on $X$ (means fixed, but not point-wise). Then $O^F \neq \emptyset$.

**Proof.** Choose $x \in O$. Then $F(x) = g(x)$ for some $g \in G$. Hence there exists $h \in G$ such that $g = h^{-1}F(h)$ then $F(h(x)) = h(x)$. [Check].
Proof. (Of the proposition) Set $X = G$ acted on by $H$. Set $O = gH$ and assume that $O$ is $F$-stable. Hence there exists a point $h \in H$ such that $F(gh) = gh$. Hence $gh = G^F$. \hfill \qed

The above lemma can be refined as follows:

**Lemma 2.17.** With the same assumptions as in the previous lemma let $x \in O^F$ and $g \in G$.

1. We have $g(x) \in O^F$ if and only if $L(g) \in \text{Stab}_G(x)$.
2. There is a bijection (depending on $X$)

$$\{G^F \text{ - orbits on } O^F\} \simeq \left\{ F\text{-conjugacy classes of } \frac{\text{Stab}_G(x)}{\text{Stab}_G(x)^0} \right\}$$

$$g(x) \mapsto L(g)$$

**Remark 2.18.** Check that (2) is well-defined. Consider $hg(x)$ where $h \in G^F$. Then $L(hg) = g^{-1}h^{-1}F(h)F(g) = g^{-1}F(g) = L(g)$.

**Definition 2.19.** If $F$ acts on $G$ we say that $g, g'$ are $F$-conjugate if there exists an $x \in G$ such that $g' = xgF(x)^{-1}$.

Note that if $G$ is connected then there is a unique $F$-conjugacy class.

**Corollary 2.20.** $F$-conjugacy classes of $G$ are in bijection with $F$-conjugacy classes of $G/G^0$.

**Proposition 2.21.** If $G$ is an algebraic group defined over $\mathbb{F}_q$. There is a bijection:

$$\left\{ G^F \text{-conjugacy classes contained in } G\text{-conjugacy class of } x \right\} \simeq \left\{ F\text{-conjugacy classes of } \frac{C_G(x)}{C_G(x)^0} \right\}$$

Take $\Theta = G$-conjugacy class of $x$, $X = G$.

**Remark 2.22.** Centralisers are connected for $GL_n$. (Also works for unitary groups : non-dependence on choice of $F$).

**Example 2.23.** Take $G = SL_w$. Then $G^F$ conjugacy classes of \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}

for $a \in \mathbb{F}_q^*$ is in bijection with $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$.

This is even worse for symplectic groups!

Take $G$ an algebraic group defined over $\mathbb{F}_q$.

**Fact 2.24.**

1. There exist $F$-stable Borel subgroups.
2. They are all $G^F$-conjugate.
3. There exist $F$-stable maximal tori.
4. There is a bijection:

$$\left\{ G^F \text{-conjugacy classes of } F\text{-stable maximal tori} \right\} \sim \left\{ F\text{-conjugacy classes of } W \right\}$$
3. LECTURE 3: DELIGNE-LUSZTIG INDUCTION

3.1. The Bruhat decomposition. \( k \) is an algebraically closed field and \( G \) a reductive connected algebraic group over \( k \). (Note: reductive usually contains connected as an assumption!)

Recall from last time that all Borel subgroups in \( G \) are conjugate. Set \( B = \{ \text{Borel subgroups in } G \} \)

If we fix \( B \in B \) there is a bijection \( G/B \cong B : gB \mapsto gBg^{-1} \). (Uses the two crucial facts: all Borel subgroups are conjugate, and \( B \) is its own normaliser.) In fact \( G/B \) can be endowed with the structure of an algebraic variety.

In fact, \( G/B \) is smooth and projective. Smoothness follows from the fact that it is a homogenous space. Set \( g := \text{Lie } G \). Then we can identify \( B \) with the set of Borel subalgebras of \( g \). But

\[
\{ \text{Borel subalgebras of } g \} \hookrightarrow \{ \text{Grassmannian of dimension } N = \dim B \text{ subspaces in } g \}
\]

the latter is the Grassmannian is projective. The condition that a subspace be a Borel subalgebra is a closed condition.

**Remark 3.1.** There exists an irreducible finite dimensional representation \( L \) of \( G \) and a line \( \ell \subseteq L \) such that \( \text{Stab}_G(\ell) = B \), then \( G/B \hookrightarrow \mathbb{P}(L) : gB \mapsto g\ell. \)

**Example 3.2.** \( G = GL_2 \) and \( B = \left\{ \begin{array}{cc} * & * \\ 0 & * \end{array} \right\}. \) Then
\[
G/B \cong \mathbb{P}^1 = \{ \ell \in \mathbb{P}^2 \mid \dim \ell = 1 \}
\[
gB \mapsto ge_1.
\]

**Example 3.3.** \( G = GL_n, \)
\[
\mathcal{F} \ell_n = \{ 0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^n \mid \dim V_i = i \}
\]
Then
\[
G/B \cong \mathcal{F} \ell_n
\]
by acting on the standard flag.

Let \( T \subseteq B \) be a maximal torus, \( W = N_G(T)/T \). Given \( w \in W, \dot{w} \) denotes a lift to \( w \).

**Theorem 3.4.**
\[
G = \bigsqcup_{w \in W} B\dot{w}B
\]
Corollary 3.5.

\[ G/B = \bigsqcup_{w \in W} BwB/B \]

In fact, \( BwB/B \cong \Delta^{\ell(w)} \). One may show that \( BwB \) is a homogenous space for \( U \), and can be realised as an explicit quotient (by certain root subgroups).

Recall that \( B = U \times T \) and \( U = R_n(B) \).

Lemma 3.6. \( N_G(T) \cap B = T \)

Proof. If \( g \in U \cap N_G(T) \) then there exists a \( t \in T \) such that \([g, t] \in T \). Hence \( T \cap U = 1 \). Hence \( g \in C_G(T) = T \). \( \square \)

There are two proofs of the Bruhat decomposition:

Proof 1. One constructs a Tits system (\( BN \)-pair). One has lots of structure with a \( BN \)-pair. (In particular, one always has a Bruhat decomposition.) \( \square \)

Proof 2. The second proof uses the fact that every \( B \)-orbit on \( G/B \) contains a unique \( T \)-fixed point. One can then study the \( B \)-orbits using the Bialynicki-Birula decomposition. \( \square \)

For \( GL_n \) the Bruhat decomposition is the Gaussian elimination. Every element can be written as a product of \( bnb' \) where \( b, b' \) are upper triangular matrices and \( n \) is a permutation matrix.

We have \( B \setminus G/B \sim G \setminus (G/B \times G/B) \). Now we have something intrinsic:

Corollary 3.7.

\[ G\text{-orbits on } B \times B \sim W \]

\[ O(w) \leftarrow w \]

Hence one has an intrinsic description of \( W \). \( O(1) = \Delta B \subset B \times B \). Simple reflections \( s \in S \) correspond to almost minimal \( G \) orbits. In fact,

\[ \dim O(w) = \dim(B) + \ell(w). \]

We have a map \( O(w) \to B : (B_1, B_2) \mapsto B_2 \). This is a fibration and the fibres over a fixed \( B_2 \) are \( BwB/B \subset G/B \sim B \).

Once we fix a Borel subgroup

\[ O(w) \sim \{ (g_1B, g_2B) \mid g_2^{-1}g_n \in BwB \} \]

\[ B \times B \sim G/B \times G/B \]

We write \( B_1 \twoheadrightarrow B_2 \) for \( (B_1, B_2) \in O(w) \) and say that “\( B_1 \) is in relative position \( w \) to \( B_2 \)”.
How do we multiply elements in $W$ using the intrinsic description above? If $\ell(w) + \ell(w') = \ell(ww')$ then

$$O(w) \times_B O(w') \sim O(ww')$$

$$(B_1 \rightarrow B_2 \rightarrow B_3) \mapsto (B_1 \rightarrow B_2)$$

Hence one can recover the multiplication from $G$-orbits on $B \times B$.

3.2. Deligne-Lusztig varieties. Assume that $k$ is the algebraic closure of $\mathbb{F}_q$ and that $G$ is defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F$.

Take $w \in W$, define

$$X(w) := O(w) \cap \Gamma_F = \{B \in \mathcal{B} \mid B \rightarrow F(B)\}$$

where $\Gamma_F = \{(B, F(B)) \mid B \in \mathcal{B}\}$.

For example, $X(1) = \mathcal{B}^F = G^F/B^F$ if $B$ is an $F$-stable Borel subgroup of $G$. Of course,

$$\mathcal{B} = \bigsqcup_{w \in W} X(w).$$

Switch to representations: $\mathbb{C}[G^F/B^F]$ is a permutation representation and $\text{End}_{\mathbb{C}[G^F]}(\mathbb{C}[G^F/B^F])$ is the Hecke algebra and the irreducible components of $\mathbb{C}[G^F/B^F]$ are in bijection with $\text{Irr}(W)$.

For $GL_2$, $\mathbb{C}[G^F/B^F] = 1_{G^F} \oplus (St)$. The Steinberg occurring with multiplicity 1. Note that $G^F/B^F = \mathbb{P}^1(\mathbb{F}_q)$.

**Definition 3.8.** $R_w := \sum_{i \geq 0} (-1)^i[H^i(X(w))]$ in $K_0(G^F)$.

**Definition 3.9.** An irreducible character of $G^F$ is called unipotent if there exists $w \in W$ such that $\langle \chi, R_w \rangle \neq 0$.

Those $\chi \in \mathbb{C}[G^F/B^F]$ are principal series unipotent characters.

Note: $G$ acts on $O(w)$ which restricts to an action of $G^F$ on $X(w)$. $G$ does not act on $X(w)$.

Fix $B$ an $F$-stable Borel subgroup.

Remember that $G/B \sim \mathcal{B}$. We claim that $X(w) \cong \{g \in G/B \mid g^{-1}F(g) \in BwB\}$. We introduce a variety $Y(w)$ above $X(w)$:

$$Y(\hat{w}) := \{gU \in G/U \mid g^{-1}F(g) \in U\hat{w}U\}.$$  

We have an obvious map

$$Y(\hat{w}) \mapsto X(w)$$

$$gU \mapsto gB$$

On $G/U$ we have commuting left $G$ and right $T$-actions. For $t \in T$,

$$\mathcal{L}(gt) = t^{-1}\mathcal{L}(g)F(t) \in t^{-1}U\hat{w}UF(t) = U\hat{w}U(w^{-1}t^{-1}w)F(t).$$  

The
condition $\dot{w}^{-1}t^{-1}\dot{w}F(t) = 1$ is equivalent to $\dot{w}F(t)\dot{w}^{-1} = t$ which occurs if and only if $t \in T^wF$.

Hence $G^F$ acts on the right on $Y(\dot{w})$ and $T^wF$ acts on the left.

Note that there exists a $d$ such that $(wF)^d = F^d$. Hence $wF$ is a Frobenius endomorphism of $T$.

The map $G/U \to G/B$ is the quotient by $T$. The map $\pi : Y(\dot{w}) \to X(w)$ is a quotient by $T^wF$. Moreover, $T^wF$ acts freely.

$\pi$ is unramified (étale) and has Galois group $T^wF$.

**Definition 3.10.** Take $\theta \in \text{Irr}(T^wF)$ define

$$R_w(\theta) = \sum_{i \geq 0} (-1)^i [H^i(Y(\dot{w})) \otimes_{\mathbb{Q}_F} \theta]$$

We have $R_w(1) = R_w$ because $H^i(Y(w)) \otimes_{\mathbb{Q}_F} \theta = H^i(X(w), L_{\theta})$.

In fact

$$\pi_* \mathbb{Q}_F = \bigoplus L_{\theta}.$$  

**Example 3.11.** If $G = SL_2$ then $B = \mathbb{P}^1$ and $W = S_2 = \{1, s\}$. We have $X(1) = \mathbb{P}^1(\mathbb{F}_q)$ and $X(s) = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ which has a map from the Drinfeld curve $Y(s) = \{xy - x^qy = 1\}$.

**Theorem 3.12** (Deligne-Lusztig). If $\theta$ is in general position (this means $\text{Stab} N_{G^F}(T^wF)(\theta) = 1$). Then $\pm R_w(\theta)$ is irreducible.

This was MacDonald’s conjecture. The proof will be given later.

[Generalities: suppose that $Y \to X$ is an unramified connected covering with group $\mathbb{Z}/n$ are in bijection with $H^1(X, \mathbb{Z}/n)$. In the example of $SL_2$, $H^1(X(s), \mathbb{Z}/n)$ subspace of $(\mathbb{Z}/n[\mathbb{P}^1(\mathbb{F}_q)])$ such that $(\sum e_i)$ which is $(\mathbb{Z}/n) \otimes (S_t)$. Hence $H^1(X(s), \mathbb{Z}/n)$ is a rep of $SL_n(\mathbb{F}_q)$ over $\mathbb{Z}/(n)$. It has a one-dimensional fixed subspace if and only if $n|q + 1$.

[Exercise: Check that “in general” there are no fixed points.]

$\theta \in \mu _{q+1}^\vee : \alpha \mapsto \alpha^d$ is in general position if and only if $d \neq \pm 1$.

$X(w) = \mathcal{O}(w) \cap \Gamma_F$. Note that $\mathcal{O}(w)$ is smooth and $\Gamma_F \cong B$. Claim: the intersection is transverse. Hence $X(w)$ is smooth and its dimension is $\ell(w)$.

A more complicated fact is:

**Theorem 3.13.** $X(w)$ is affine if $q > h$, where $h$ is the Coxeter number.

**Proof.** Later! 

Bonnafé-Rouquier: conjecture this always holds.

Rappaport: conjectures that this doesn’t always hold.
In fact, Deligne and Lusztig show:

\[ R_w(\theta) = H^{l(w)}(Y(w)) \otimes_{\mathbb{Q}^F} T_w \theta \]

4. Lecture 3

Let \( G \) be an algebraic group, reductive over \( k = \mathbb{F}_q \). We assume that \( G \) is defined over \( \mathbb{F}_q \) or equivalently, endowed with \( F : G \to G \) a Frobenius. \( G^F \) is a finite group.

If we fix \( T_0 \subset B_0 \) an \( F \)-stable torus contained in an \( F \)-stable torus contained in an \( F \)-stable Borel subgroup of \( G \).

We define \( W = N_G(T_0)/T_0 \).

Given \( w \in W \) we define \( X(w) = \{ gB_0 \in G/B_0 \mid \mathcal{L}(g) \in B_0wB_0 \} \).

Remember that the Lang map being \( L(g) = g - 1 \).

One can see that \( X(w) \sim \{ g \in G \mid \mathcal{L}(g) \in wB_0 \}/B_0 \sim \{ g \in G \mid \mathcal{L}(g) \in wB_0 \}/(B_0 \cap (wB_0w^{-1})) \)

the last isomorphism is not obvious. If we put \( B = gB_0g^{-1} \) then \( \mathcal{L}(g) \in B_0wB_0 \) if and only if \( B \overset{w}{\to} F(B) \) if and only if there exists \( h \in G \) such that \( (B, F(B)) = h(B_0, wB_0w^{-1})h^{-1} \). Now \( \mathcal{L}(h^{-1}g) \in wB_0 \). [Check!]

\( X(w) \) has a left \( G^F \)-action (because \( L(hg) = L(g) \) if \( h \in G^F \).

\( Y(w) = \{ g \in G \mid \mathcal{L}(g) \in wU_0 \}/U_0 \cap (wU_0w^{-1}) \)

Note that \( B_0 \cap wB_0w^{-1} = T_0 \ltimes (U_0 \cap wU_0w^{-1}) \)

The natural map

\( Y(w) \to X(w) \)

is the quotient by \( T_0^wF \).

\( \{(T \subset B), T \text{ is } F\text{-stable}\}/G^F \text{ – conjugacy} \sim \)

\( (T \subset B) \mapsto \text{unique } w \text{ such that } B \overset{w}{\to} F(B) \).

\( \{T \text{ } F\text{-stable maximal torus}\}/G^F \text{ – conjugacy} \sim \) \( F\text{-conjugacy classes in } W \).

One can consider

\( X_{T \subset B} = \{ g \in G \mid \mathcal{L}(g) \in BF(B) \}/B \sim \{ g \in G \mid \mathcal{L}(g) \in F(B)/B_0F(B) \} \)

and

\( Y_{T \subset B} = \{ g \in G \mid \mathcal{L}(g) \in F(U) \}/U \cap F(U) \)

and \( X_{T \subset B} \) is the quotient of \( Y_{T \subset B} \) by \( T^F \). In fact

\( X_{T \subset B} \to \{ B' \mid B' \overset{w}{\to} F'B' \} \sim \{ gB_0 \mid \mathcal{L}(g) \in B_0wB_0 \} \)

the latter map being \( gB_0 \mapsto gB_0b^{-1} \).
One always has a canonical isomorphism $N_G(T_0)/T_0 \sim N_G(T)/T$ (this was explained last time).

One can identify $Y_{T \subset B} \sim Y(\dot{w})$.

**Theorem 4.1.** $\sum (-1)^i \chi_{H^i(Y_{T \subset B})}$ is independent of $B$. Similarly, $\sum (-1)^i \chi_{H^i(Y(\dot{w}))}$ depends only on the $F$-conjugacy class of $w$.

If we consider $Y = \{ g \in G \mid \mathcal{L}(g) \in F(U) \}/U \cap F(U)$ is a quotient by $U \cap F(U)$ of $\mathcal{L}^{-1}(F(U))$. Clearly

$$G \supset \mathcal{L}^{-1}(F(U)) \xrightarrow{\mathcal{L}} F(U) \subset G$$

Locally, the quotient map is trivial with fibres $U \cap F(U)$ which is an affine space. Hence $H^*(\mathcal{L}^{-1}(F(U))) \sim H^*(Y)$. Note that the Lang map is a quotient of $G$ by the finite group $G^F$.

This is an amazing topological situation!!

$$\mathcal{L}^{-1}(F(U)) \xrightarrow{\mathcal{L}} \text{affine space}$$

the map is unramified with group $G^F$. (Thinking topologically, we get a map $\pi_1(F(U)) \to G^F$) (and this map is onto for a semi-simple group).

(Hence the fundamental group of a vector space is very complicated!!)

We can even make $\pi(A^1) \to G^F$ (this is a Lefschetz type principle).

**Theorem 4.2** (Raynaud, formerly Abyankar’s conjecture). If $G$ is a finite group, then there exists a $\pi_1(A^1_{F}) \to G$ if and only if $G$ is generated by its $p$-Sylow subgroups. That is $O^pG = G$.

(uses $p$-adic geometry in an essential way).

$G = GL(V)$ where $V = k^n$.

$$T_0 = \begin{cases} * & 0 & \ldots & 0 \\ 0 & * & \ldots & 0 \\ \vdots & \ddots \\ 0 & 0 & \ldots & * \end{cases}$$

$F((x_{ij})) = (x_{ij}^q)$ and $W$ is the symmetric group.

$\mathcal{B} = \{ \text{Borel subgroups} \} \sim \{ 0 = D_0 \subset D_1 \subset \cdots \subset D_n = V \mid \dim D_i = i \}$

The bijection is given by sending $B$ to the unique complete flag fixed by $B$. These are “complete flags”.

Also $G/B_0 \cong \mathcal{B}$.

Let $U_0$ be the subgroup of upper triangular matrices. Then $G/U_0$ is a $T_0$-torsor over $G/B$ and we may identify

$$G/U_0 \sim \{ 0 = D_0 \subset \cdots \subset D_n, (e_1, \ldots, e_n) \mid e_i \in D_i/D_{i-1} \setminus \{0\} \}$$

$T_0$ acts on the $(e_1, \ldots, e_n)$ in the natural way.
Let $D$ and $D'$ be two flags. Then $D \xrightarrow{w} D'$ if we have

$$\text{Gr}_D^{w(i)} \cap \text{Gr}_{D'}(V) \neq 0$$

for all $i$.

We will be discussing $w = (1, \ldots, n)$.

$D \xrightarrow{w} D'$ if and only if $D'_i + D_i = D_{i+1}$ for $i < n - 1$ and $D'_{n-1} + D_1 = V$.

Hence $D \xrightarrow{w} F(D)$ if and only if $D_2 = D_1 + F(D_1), \ldots, D_i = D_1 + F(D_1) + \cdots + F^{i-1}D_1$. Hence

$$X(w) \mapsto \mathbb{P}(V)$$
$$D \mapsto D_1$$

Hence the image is $\ell \in \mathbb{P}(V) \setminus \{\ell + F(\ell) + \cdots + F^{n-1}(\ell) = V\} = \mathbb{P}(V) \setminus \{\text{hyperplanes defined over } \mathbb{F}_q\}$.

In $Y(w)$ a point is determined by $e'_i$s and hence by $e_1$. The condition is that $e_2 = F(e_1) \pmod{e_1}, e_3 = F^2(e_1) \pmod{e_1, F(e_1)}$ and $e_n = F^{n-1}(e_1) \pmod{e_1, \ldots, F^{n-1}(e_1)}$ and lastly

$$e_1 = F^n(e_1) \pmod{Fe_1, \ldots, F^{n-1}(e_1)}.$$

These conditions force

$$e_1 \wedge F(e_1) \wedge \cdots \wedge F^{n-1}(e_1) = F^n(e_1) \wedge F(e_1) \wedge \cdots \wedge F^{n-1}(e_1)$$

hence

$$e_1 \wedge \cdots \wedge F^{n-1}e_1 = (-1)^{n-1}F(e_1 \wedge \cdots \wedge F^{n-1}e_1)$$

if we write $e_1 = (x_1, \ldots, x_n)$ then we get the matrix

$$\det((x_{i,j})^{q^{j-1}}_{1 \leq i, j \leq n})^{q-1} = (-1)^{n-1}.$$

**Proposition 4.3.**

$$Y(w) \sim \{x_1, \ldots, x_n\} \in \mathbb{A}^n \mid \det((x_{i,j})^{q^{j-1}}_{1 \leq i, j \leq n})^{q-1} = (-1)^{n-1}\}$$

$GL_n(\mathbb{F}_q)$ acts naturally on $V$. $T_0^{\nu F} = \mathbb{F}_q^*$ acts diagonally.

The van de Monde determinant is up to scalar the product of all linear forms over $\mathbb{F}_q$. There are $q^n - 1$ linear forms (i.e. points in $\mathbb{P}(V^*) (\mathbb{F}_q)$).

For $n = 2$ $X(w) = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ and $Y(w) = \{(x, y) \mid (xy^q - x^qy)^{q-1} = -1\} = \bigcup_{\zeta^{q-1} = -1}\{(x, y) \mid xy^q - x^qy = \zeta\}$. The connected components are permuted by $GL_2(\mathbb{F}_q)$ and the components are isomorphic to $\{(x, y) \mid xy^q - x^qy = 1\}$. $SL_2(\mathbb{F}_q)$ fixes the components.
5. Lecture 5: Cohomology

Let $k = \bar{\mathbb{F}}_q$ and $\ell$ be a prime with $\ell \nmid q$, let $\Lambda = \bar{\mathbb{Q}}_\ell$ (which is isomorphic with $\mathbb{C}$ for all intents and purposes), and consider varieties over $k$. There is a functor $H^*$, $\ell$-adic cohomology, between varieties over $k$ and graded $\Lambda$-vector spaces. Note that there is no cohomology theory with values in $\mathbb{Q}$. Serre proved that there is a suitable elliptic curve $X$ with automorphism group $G$, and the representation of $G$ on $H^1(X)$ is not defined over $\mathbb{Q}$.

There is another cohomology theory from varieties over $k$ to graded $\Lambda$-vector spaces, $H^*_c$, the cohomology with compact support, in some sense dual to $H^*$. This embeds $X$ as an open set in $\bar{X}$, called proper (the algebraic geometry version of compact). If $X$ is proper (e.g., projective) then $H^*_c(X) = H^*(X)$. It is often easier to work with $H^*_c$, but sometimes things look nicer with $H^*$.

Trace map: let $X$ be smooth and connected, of dimension $d$. There is a map $tr : H^{2d}_c(X) \xrightarrow{\sim} \Lambda$. This leads to Poincaré duality: there exists a perfect pairing $H^i(X) \times H^{2d-i}_c(X) \rightarrow \Lambda$. A nice summary of the results on etale cohomology in Digne–Michel, and in Carter. In particular, $H^i(X) = H^{2d-i}_c(X)^{*}$.

For example, $H^*(\mathbb{A}^n) = \Lambda$, concentrated in degree 0. Therefore $H^*_c(\mathbb{A}^n) = \Lambda$, concentrated in degree $2n$. Finally, $H^0(X)$ is a free $\Lambda$-module of rank the number of connected components, and $H^i(X) = 0$ for $i \notin [0, 2d]$.

If $X$ is a variety and $U \subset X$ is open, write $Z = X \setminus U$, then there is a long exact sequence

$$0 \rightarrow H^0_c(U) \rightarrow H^0_c(X) \rightarrow H^0_c(Z) \rightarrow H^1_c(U) \rightarrow H^1_c(X) \rightarrow H^1_c(Z) \rightarrow \cdots.$$  

Let $X = \mathbb{P}^n$, $U = \mathbb{A}^n$, $Z = \mathbb{P}^{n-1}$, so by induction on $n$,

$$H^*(\mathbb{P}^n) = H^*_c(\mathbb{P}^n) = \begin{cases} \Lambda & * \in \{0, 2, \ldots, 2n\} \\ 0 & \text{otherwise} \end{cases}.$$  

• If $X$ is quasi-projective, and acted on by a group $G$, then $H^*(X/G) = H^*(X)^G$.

• If $X$ is acted on by a connected algebraic group $G$, then $G$ acts trivially on $H^*(X)$. If $H$ is a finite subgroup of $G$, then $H$ still acts trivially on $H^*(X)$. This is a useful method to prove that some groups act trivially on $H^*(X)$. (In particular, in extending actions of abelian groups to tori.)

A variety over $k$ has only finitely many coefficients involved in its definition, so we think of it as a variety over some finite field, which we might as well take to be $\mathbb{F}_q$. We can endow a variety $X$ with a Frobenius
endomorphism $F$. Via $H^*$, we get a graded $\Lambda$-vector space with an
action of $F$. The computational power of the theory comes from the
action of $F$. The same is true for $H^*_c$. At first blush, the endomorphism
on graded vector spaces could be anything, but with a bit of work one
may show that it is invertible. Hence $F$ gives an automorphism on the
graded $\Lambda$-vector spaces $H^*(X)$ and $H^*_c(X)$. (This has something to do
with sheaves and sites.)

If $X$ is connected, then $F$ acts trivially on $H^0(X)$. However, the
trace map doesn’t work so well, so we get $H^*_c(X) \sim \Lambda(n)$, where this
means that the map $F$ acts by $q^n$. For Poincaré duality as well, we get
$H^i(X) \times H^{2d-i}(X)_c \rightarrow \Lambda(n)$. By being extra-careful in the computation
above, we get $H^*_c(\mathbb{A}^n) = \Lambda(n)$ concentrated in degree $2n$, and

$$ H^i(\mathbb{P}^n) = H^i_c(\mathbb{P}^n) = \begin{cases} \Lambda(i/2) & i \in \{0, 2, \ldots, 2n\} \\ 0 & \text{otherwise} \end{cases}. $$

**Theorem 5.1.** The eigenvalues of $F$ on $H^*_c(X)$ have modulus $q^a$,
where $a$ is a semi-integer and are algebraic integers.

Notice that the eigenvalues might well not be $q^a$ themselves.

5.1. **Lefschetz Trace Formula.** We want to know $|X^F| = |X(\mathbb{F}_q)|$. We have

$$ |X^F| = \sum_{i \geq 0} (-1)^i \text{Tr}(F | H^i_c(X)). $$

**Example 5.2.** First let $X = \mathbb{A}^n$: $|X^F| = q^n$. Then $F$ acts by $q^n$ on
$H^*_c(\mathbb{A}^n, X)$.

If $X = \mathbb{P}^n$, then $|X^F| = 1 + q + \cdots + q^n$. To see this, if $U \subset X$ is
$F$-stable and open, and $Z = X \setminus U$, then $X^F = U^F \amalg Z^F$, and we can
basically add them.

**Corollary 5.3.** If $g$ is finite order automorphism of $X$ defined over $\mathbb{F}_q$
(i.e., commuting with $F$), then

$$ \text{Tr}(g | H^*_c(X)) = \sum_{l=1}^{\infty} (-1)^l \text{Tr}(g | H^*_c(X)) = \lim_{t \to \infty} \sum_{n=1}^{\infty} |X^{gF^n}| t^n. $$

**Proof.** $gF^n$ is a Frobenius endomorphism on $X$ relative to $\mathbb{F}_q$. Therefore
$|X^{gF^n}| = \text{Tr}(gF^n | H^*_c(X))$. We may choose $F$ to be upper triangular,
with diagonal $\lambda_1, \ldots, \lambda_r$ and $g$ to be diagonal with entries
$\alpha_1, \ldots, \alpha_r$. Set $\varepsilon_i = 1$ if in even $H^*$ and $-1$ if in odd degree. Then

$$ \text{Tr}(gF^n | H^*_c(X)) = \sum_{l=1}^{r} \varepsilon_l \alpha_l \lambda_l^n, $$
and so
\[ \sum_{n=1}^{\infty} |X^{gF^n}| t^n = \sum_l \varepsilon_l \alpha_l \sum_n (\lambda_l t)^n = \sum_l \varepsilon_l \alpha_l \left( \frac{1}{1 - \lambda_l t} - 1 \right). \]
As \( t \to \infty \) the last term in the brackets tends to \(-1\), and so this becomes \( \sum_l \varepsilon_l \alpha_l \).

**Example 5.4.** Let \( X = \{ xy^q - x^q y = 1 \} \subset \mathbb{A}^2 \), a closed subset. \( F(x, y) = (x^q, y^q) \). Then \( X^F = \emptyset \). This means that \( \text{Tr}(F \mid H^*_c(X)) = 0 \).

If \( X \) is an affine variety, then \( H^i(X) = 0 \) if \( i \) is greater than \( \dim X \), so \( H^0_c(X) = H^2(X)^* = 0 \). We also know that \( H^2_c(X) = \Lambda \) where \( F \) acts by \( q \). The only missing case is \( H^1_c(X) \). The trace being \( 0 \) means that \( q = \text{Tr}(F \mid H^1_c(X)) \).

If \( g \in SL_2(\mathbb{F}_q) \times \mu_{q+1} \), then knowing \( |X^{gF^n}| \) for \( n \geq 1 \) determines \( \text{Tr}(g \mid H^1_c(X)) \).

Let \( G \) be a reductive algebraic group over \( k = \mathbb{F}_q \), with Frobenius endomorphism \( F \) defining an \( \mathbb{F}_q \)-structure. If \( T \) is an \( F \)-stable maximal torus contained in a (not necessarily \( F \)-stable) Borel \( B \), and \( \theta \) an irreducible character of \( T^F \), we get
\[ R^G_{T \subset B}(\theta) = \sum_i (-1)^i H^i(Y_{T \subset B}) \otimes_{T^F} \theta. \]
If you vary \( \theta \) and \( T \) you get lots of virtual representations, and the first result is the following.

**Theorem 5.5.** We have
\[ \langle R^G_{T \subset B}(\theta), R^G_{T' \subset B'}(\theta') \rangle_{GF} = |T^F| \cdot \{ g \in G^F \mid gTg^{-1} = T', g\theta g^{-1} = \theta' \}. \]
In particular, if \( (T', \theta') \) is not \( GF \)-conjugate to \( (T, \theta) \), then this scalar product is 0.

**Proof.** We have
\[ H^i_c(Y_{T \subset B} \times_{GF} Y_{T' \subset B'}) \cong \bigoplus_{i_1+i_2=i} H^i_{T \subset B} \otimes_{AG^F} H^{i_2}_{T' \subset B'} \]
(Kuenneth formula). We will also need to understand the tensor product of \( \theta \) and \( \theta' \), but concentrate on the \( Y \)-s. This is
\[ Z = L^{-1}(F(U)) \times_{GF} L^{-1}(F(U')) = \{(x, x', y) \in F(U) \times F(U') \times G \mid xF(y) = yx' \}. \]
Decompose according to the Bruhat decomposition of \( G \), i.e., \( G = \Pi_{w \in W} BwB \). The variety \( Z \) may be decomposed as the disjoint union of the \( Z_w \), which have an action of \( T^F \times T'^F \) acting on it.

The action of \( T^F \times T'^F \) on \( Z_w \) extends to an action of
\[ H_w = \{(t, t') \in T \times T' \mid \mathcal{L}(t') = F(w)^{-1} \mathcal{L}(t) F(w) \}. \]
Now we may use the fact that $H_w^* \varphi$ acts trivially on $H^*(Z_w)$. (We extend the action of $T^F \times T^F$ to one of an algebraic group, as we said would be useful before.) If $\theta \otimes_{AT^F} H_c(Z_w) \otimes_{AT^F} \theta' \neq 0$ then $(\theta \otimes \theta')|_{H_w^*} = 1$. One readily checks that there exists $w$ such that $(\theta \otimes \theta')|_{H_w^*} = 1$ if and only if $(T', \theta') \sim_{G^F} (T, \theta)$. □

Geometric conjugacy is $(T, \theta) \sim (T', \theta')$. Let $\lambda$ be a geometric conjugacy class. Define

$$\xi(G^F, \lambda) = \{ \chi \in \text{Irr}(G^F) \mid \langle R_{T \subset B}^G(\theta), \chi \rangle \neq 0 \text{ for some } (T, \theta) \in \lambda \}.$$ 

**Theorem 5.6.** $\text{Irr}(G^F) = \coprod \xi(G^F, \lambda)$. Also

$$\xi(G^F, \lambda) = \{ \chi \in \text{Irr}(G^F) \mid \langle H^t(Y_{T \subset B}) \otimes_{T^F} \theta, \chi \rangle \neq 0 \text{ for some } i, (T \theta) \in \lambda \}.$$ 

Lusztig series: unipotent characters are $\xi(G^F, 1)$, where 1 is the geometric conjugacy class of $(T, 1)$. Also,

$$\xi(G^F, \lambda) \sim_{G^F} \xi(H^F, 1)$$

for some $H$ (Jordan decomposition).

Remark:

$$\text{Cl}(G^F) \sim \coprod_{s \text{ semisimple}/G^F} \text{UnipCl}(C_{G^F}(s)),$$

with $(x)$ on the right being sent to $(xs)$ on the left.

$G$ and $G^*$ are Langlands dual: take $T \subset G$ and $T^* \subset G^*$ be $F$-stable and $F^*$-stable maximal tori. Then $X(T) \sim_{G^F} Y(T^*)$ compatible with $F$ and $F^*$, going between roots and coroots. For example, $GL_n^* = GL_n$ and $SL_n^* = PGL_n$.

If $T$ is a torus, $Y(T) = \text{Hom}(\mathbb{G}_m, T)$, and $Y(T) \otimes \mathbb{C} \rightarrow \mathbb{C}$ given by $\zeta \otimes x \mapsto \zeta(x)$. As $T^F = \ker(T(F - 1))$, we get a short exact sequence

$$0 \rightarrow T^F \rightarrow Y(T) \otimes \mathbb{C} \rightarrow 0.$$

Choose $d$ such that $F^d$ acts trivially on $Y(T)$. We have another sequence

$$0 \rightarrow Y(T) \rightarrow F^{-1} Y(T) \rightarrow T^F \rightarrow 0$$

with the last map being $N : \zeta \mapsto N_{F^d/F}(\zeta)(\alpha)$, the norm map. (Here $\alpha$ is a generator of $\mathbb{F}_{q^d}$. Then $(T, \theta)$ and $(T', \theta')$ are geometrically conjugate if $(T, \theta \circ N) \sim_G (T' \theta' \circ N')$. 

6. Lecture 6

$G$ a reductive algebraic group of $k = \mathbb{F}_q$ endowed with $F$ a Frobenius. Let $\Lambda = K = \mathbb{Q}_\ell$ where $\ell$ does not divide $q$.

Given $T \subset B$ a maximal $F$-stable torus ($B$ need not be stable). We have

$$R_{T \subset B}^G : K_0(KT^F) \to K_0(KG^F)$$

recall that $Y_{T \subset B}$ has commuting left $G^F$ and right $T^F$ actions and so $\sum_i (-1)^i[H^i(U, K) \otimes_{K^F} \cdots]$. A crucial property was the orthogonality relations.

$$\langle R_{T \subset B}^G(\theta), R_{T \subset B}^G(\theta') \rangle_{G^F} = 0$$

**Corollary 6.1.** $R_{T \subset B}^G(\theta) = R_{T \subset B}^G(\theta')$

**Proof.** $\langle f, f \rangle = \langle f', f' \rangle = \langle f, f' \rangle = \langle f', f \rangle = 0$ and so $f = f'$.

**Remark 6.2.** $H^*(T_{T \subset B})$ depends on $B$, only the alternating sum is independent.

The choice of a Borel $T \subset B$ yields an element $w \in W$ and $\dim Y_{T \subset B} = \ell(w)$.

**Definition 6.3.** We say that $\theta$ is in general position if $(g(T, \theta)g^{-1} = (T, \Theta)$ implies that $g \in T^F$.

**Corollary 6.4.** If $\theta$ is in general position then $R_T^G(\theta)$ is irreducible.

Problem: define $\phi \in K_0(KG^F)$ to be uniform if it is a linear combination of $R_T^G(\theta)$’s.

(Bad) fact: In general (i.e. type $\neq A$), not all class functions are uniform. (This means the lattice generated by $R_T^G(\theta)$’s is of strictly smaller rank.)

(Good) fact: $[KG^F]$ is uniform. This means that for all $\chi \in \text{Irr}(G^F)$ there exists a $(T, \theta)$ with $\langle \chi, R_T^G(\theta) \rangle \neq 0$.

$\mathbb{F}_q = k^F$ and $\mathbb{F}_{q^r} = k^{(F^r)}$. We have

$\xymatrix{ T^F_{q^r/F_q} \ar[r]^{N_{q^r/F_q}} \ar[dr]_{\theta_r} & T^F \ar[d]^\theta \\
K^r &}$

($N_{q^r/F_q}$ is the norm of $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$.)

Given $(T, \theta)$ relative to $F$ we have $(T, \theta_r)$ relative to $F^r$.
Definition 6.5. \((T, \theta)\) and \((T', \theta')\) are geometrically conjugate if there exists an \(r\) such that \((T, \theta^r)\) and \((T', \theta'^r)\) are conjugate over \(G^F\).

Theorem 6.6. If \((T, \theta)\) is not geometrically conjugate to \((T', \theta')\) then \(R_T^G(\theta)\) and \(R_{T'}^G(\theta')\) have no irreducible characters in common.

This yields a partition
\[
\operatorname{Irr}(G^F) = \bigsqcup_{\lambda=\{(T, \theta)\} \text{ up to geom. conjugacy}} \mathcal{E}(G^F, \lambda)
\]
where
\[
\mathcal{E}(G^F, \lambda) = \{ \chi \mid \langle \chi, R_T^G(\theta) \rangle \neq 0 \text{ for some } (T, \theta) \in \lambda \}
\]

\((T, 1)\) and \((T', 1)\) are always geometrically conjugate! (This follows from the fact that all maximal tori are conjugate over \(k\).)

Example 6.7. \(G = SL_2, T \subset B\) \(F\)-stable and \(T' \subset B'\) but \(B'\) are not \(F\)-stable.

(6.1) \(R_T^G(1) = \operatorname{Id} + \operatorname{St}^G\)
(6.2) \(R_{T'}^G(1) = \operatorname{Id} - \operatorname{St}^G\)

This decomposition was described by Lusztig. He also defined a “Jordan decomposition”. We will only note that
\[
\mathcal{E}(G^F, \lambda) \xrightarrow{\sim} \mathcal{E}(H^F, 1)
\]
for some reductive group \(H\), depending on \(\lambda\). (\(H^F\) is endoscopic?)

Hence the main difficulty is in describing unipotent characters \(\mathcal{E}(G^F, \lambda)\).

Remark 6.8. Green function = values of \(R_T^G(\theta)\) at a unipotent element.

One can express \(\chi_{R_T^G(\theta)(g)}\) in terms of Green functions.

The explanation is of a geometrical nature:
If \(X\) is an algebraic variety of \(k\) with a Frobenius \(F\) defining an \(F_q\)-structure and if \(g\) is an endomorphism of \(X\) of finite order (commuting with \(F\)) then we would like to compute the trace of \(g\) on \(H^s_c(X, K)\).

Theorem 6.9.
\[
\operatorname{Tr}(g, H^s_c(X, K)) = \operatorname{Tr}(u, H^s_c(X^s, K))
\]
where \(g = su, [s, u] = 1\), \(s\) is semi-simple (\(p'\)) and \(u\) is unipotent (\(p\)).

Proof. Step 1: \(\operatorname{Tr}(g, H^s_c(X, K))\) is independent of \(\ell\) (\(\ell \neq p\)). This is clear because
\[
\operatorname{Tr}(g, H^s_c(X, K)) = -\lim_{t \to \infty} \sum_{n=1}^{\infty} |X^gF^n| t^n
\]
Step 2: Write $X = \bigsqcup X_i$ where all points of $X_i$ have the same stabilizer in $\langle g \rangle$. (Stabilizer decomposition.) The $X_i$ has a free action of $\overline{g}$. Because

$$\text{Tr}(g, H^*_c(X, K)) = \sum_i \text{Tr}(g, H^*_c(X_i, K))$$

we can assume that $g$ acts freely.

Assume that $s \neq 1$ and pick $\ell \neq p$ and $\ell | o(s)$. $H = \langle g \rangle$ acts freely on $X$ and that the order of $H$ is divisible by $\ell$.

Lemma 6.10. $X$ an algebraic variety, and $G$ a finite group acting freely on $X$ and $\ell$ is a prime. Then

$$\sum (-1)^i[H^i_c(X)]$$

is a virtual projective character (for $\ell \neq p$).

Proof. In fact $R\Gamma_c(X, \mathbb{Z}_\ell)$ complex of $\mathbb{Z}_\ell G$-modules is quasi-isomorphic to a bounded complex of finitely generated projective modules.

$$R\Gamma_c(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell G}^L V = R\Gamma_c(X/G, (\pi_*\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell G}^L V)$$

$\pi : X \to X/G$ but

$$(\pi_*\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell G}^L V_x = (\pi_*\mathbb{Z}_\ell)_x \otimes_{\mathbb{Z}_\ell G}^L V \cong V$$

Then use the fact that if $M$ is a complex of $\mathbb{Z}_\ell G$-modules and assume that there exists a finite interval $I$ such that

$$H^i(M \otimes_{\mathbb{Z}_\ell G}^L V) = 0 \text{ for } i \in I$$

Then $M$ is quasi-isomorphic to a bounded complex of projective $\mathbb{Z}_\ell G$-modules.

We are back at $g = su$ acting freely on $X$ and $\ell | o(s)$. We know that

$$\sum (-1)^i[H^i_c(X, K)]$$

is virtually projective with respect to $\ell$.

Hence the character of $g$ is zero.

(If we take $Q \times R$ where $Q$ is an $\ell$-group, and $R$ is an $\ell'$-group then a projective character is $KQ \otimes \phi$ and so the character value vanishes $(x,y)$ and $y \in R$ if $x \neq 1$, $x \in Q$.)

Example 6.11. Take $X = \mathbb{A}^1$, $g : x \mapsto x + 1$, $g^p = 1$ and $\mathbb{Z}/p\mathbb{Z}$ acts freely. Now

$$H^*_c(X) = \begin{cases} K & \ast = 2 \\ 0 & \ast \neq 2 \end{cases}$$
Note that $\text{Tr}(g, H^*_c(X)) = 1$ and $X^g = \emptyset$.

Note the big difference with characteristic 0 coefficients! This should be zero!!

6.1. **Green functions for $GL_n$.** We now consider $G = GL_n$ and $F : (a_{ij}) \mapsto (a_{ij}^q)$.

We now consider the unipotent characters $R^G_{T_w}(1)$ and $T_w$ of type $(w)$ with $w \in \Sigma_n$ (in bijection with a partition of $n$).

Define almost characters $\chi \in \text{Irr}(W)$

$$R_\chi := \frac{1}{|W|} \sum_{w \in W} \chi(w) R^G_{T_w}(1)$$

these are the **almost characters**.

**Theorem 6.12.** $R_\chi$ is an irreducible character (for $G$ of type $A$!) and the map

$$\text{Irr}(W) \sim \text{E}(GL_n(q), 1)$$

$$\chi \mapsto R_\chi$$

is a bijection. In general

$$\text{Irr}(GL_n(q)) = \bigsqcup_{(s) \text{ semi-simple / conjugacy}} \text{E}(GL_n(q), (s))$$

and

$$\text{E}(GL_n(q), (s)) \sim \text{E}(C_{GL_n(q)}(s), 1)$$

(note that $C_{GL_n(q)}$ is a product of general linear groups).

If $G$ is not of type $A$: decomposing $R_\chi$ as irreducible character in $\text{E}(G^F, 1)$ yields a matrix: this is part of a “non-commutative Fourier transform” matrix. (The groups involved are small groups: i.e. $(\mathbb{Z}/2)^n$, $\Sigma_3$, $\Sigma_4$ and $\Sigma_5$.)