SIX LECTURES ON DELIGNE-LUSZTIG THEORY

Abstract. Lectures given by Raphael Rouquier in Hilary term, Oxford, 2010. Notes of lectures 2, 3, 4 and 6 by Geordie Williamson, lecture 5 by David Craven.

1. Lecture 1

Here the example of SL_2 was considered in detail and the character table was discussed.

2. Lecture 2: Finite groups of Lie type

Fix $k = \overline{k}$.

Algebraic group (affine) over k is an affine algebraic smooth variety G endowed with a group structure.

Consider the algebra of regular functors $k[G]$ on G. Then $G = \operatorname{Spec} k[G]$ and $k[G]$ is a (commutative) Hopf algebra.

Example 2.1.
- $\mathbb{G}_a = \operatorname{Spec} k[T]$, $\mathbb{G}_a(k) = k$ as an additive group.
- $\mathbb{G}_m = \operatorname{Spec} k[T, T^{-1}]$, $\mathbb{G}_a(k) = k^*$ multiplicative group.

These are all the one-dimensional groups.

- $GL_n = \operatorname{Spec} k[T_{ij}]_{1 \leq i,j \leq n}[\det(T_{ij})^{-1}]$, $GL_n(k)$ consists of invertible $n \times n$ matrices over k.

Fact 2.2. Every algebraic group is isomorphic to a closed subgroup of GL_n for some n. (This is the reason for the terminology "linear algebraic group").

The notion of a semi-simple (diagonalisable) element of GL_n gives rise to the notion of a semi-simple element for any G. Similarly we have the notion of a unipotent element of GL_n (conjugate to an upper uni-triangular matrix). This gives rise to the notion of a unipotent element in any G.

Fact 2.3 (Jordan Decomposition). Let $x \in G$. There exists a unique pair (u, s) in G such that $x = us$, $[u, s] = 1$, such that u is unipotent and s is semi-simple.

(This is an analogue of the p, p' decomposition of elements in a finite group.)

G^0 is defined to be the maximal connected closed subgroup of G and G/G^0 is a finite group.

$R_u(G)$: unipotent radical = largest closed normal connected unipotent subgroup.

A nice fact is that $R_u(G/R_u(G)) = 1$.

So an arbitrary finite group looks like:

$$G/G^0 \text{ finite} \tag{2.1}$$

$$G = \ G^0/R_u(G) \text{ connected, reductive} \tag{2.2}$$

$$R_u(G) \text{ unipotent} \tag{2.3}$$

A connected algebraic group G is solvable if it has a chain of closed normal subgroups with 1-dimensional quotients (that is, $\mathbb{G}_a$ or $\mathbb{G}_m$).

Fact 2.4. A unipotent group is solvable.

Definition 2.5. A torus is an algebraic group isomorphic to $\mathbb{G}_m^r$ for some r.

Proposition 2.6.
- *If G is an algebraic group, there exists a maximal closed torus in G. All maximal tori are conjugate.*
- *There exists a maximal closed solvable subgroup (called a Borel subgroup). All Borel subgroups are conjugate.*

Proposition 2.7. *G solvable, T maximal torus in G. Then*

$$G = R_u(G) \rtimes T. \textit{ semi-direct product}$$

Example 2.8. $G = GL_n \supset B \supset U$. Here B upper triangular matrices, U is unitriangular matrices, T is diagonal matrices.

Proposition 2.9.
- *If B is a Borel subgroup then $N_G(B) = B$.*
- *If $T \subset G$ is a maximal torus, then $C_G(T) = T$. Moreover $W := N_G(T)/T$ is a finite group, the Weyl group.*

Example 2.10. $G = GL_n$, $N_G(T) =$ monomial matrices, $W = S_n$ the symmetric group.

2.1. **Finite fields.** $\mathbb{F}_q$ a finite field, $k = \overline{\mathbb{F}_q}$, $q = p^m$. X_0 algebraic variety over $\mathbb{F}_q$, $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$.

The (geometric) Frobenius endomorphism $F : X \to X$ which acts on $a = a_0 \otimes \alpha \in k[X] = \mathbb{F}_q[X] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ then $F(a) = a_0^1 \otimes \alpha$.

Example 2.11. $X_0 = \operatorname{Spec} \mathbb{F}_q[X_{ij}, \det(X_{ij})^{-1}]$, $X = GL_n$. $F(X_{ij}) = X_{ij}^q$, $g \in GL_n(\overline{\mathbb{F}_q})$, $g = g_{ij}$ then $F(g) = (g_{ij}^q)$.

There is another possibility: $F'(g) = {}^tF(g)^{-1}$ on G. Note that $(F')^2 = F^2$. (F' is "a" Frobenius because $(F')^2$ is one.)

Given an $\mathbb{F}_q$-rational structure is the same as giving the endomorphism F.

We can look at $(X(\overline{\mathbb{F}_q}))^F = X_0(\mathbb{F}_q)$, a finite set.

If G is an algebraic group defined over $\mathbb{F}_q$, or equivalently endowed with a Frobenius endomorphism F. Then G^F is a finite group.

Example 2.12. $GL_n^F = GL_n(\mathbb{F}_q)$ and $GL_n^{F'} = U_n(\mathbb{F}_q)$.

We define the Lang map $\mathcal{L} : G \to G : g \mapsto g^{-1}F(g)$.

Clearly $\ker \mathcal{L} = G^F$.

Theorem 2.13. *If G is connected then $\mathcal{L}$ is surjective.*

The Lang map (and the above theorem) is absolutely fundamental to all that follows! It allows us to pass between the algebraic group G and its points over finite fields.

Proof. (Sketch) The key point is that $(dF)_1 = 0$. Hence $(d\mathcal{L})_1$ is equal to multiplication by -1. Hence $(d\mathcal{L})_1$ is bijective. Hence $\mathcal{L}(G)$ contains a dense open subset (algebraic geometry: the image contains an open subset). Fix $x \in G$. Consider the map $\mathcal{L}_x : g \mapsto g^{-1} \cdot x \cdot F(g)$ then $(d\mathcal{L}_x)_1$ is bijective. Hence $\mathcal{L}_x(G)$ contains a dense open subset. Hence $\mathcal{L}(G) \cap \mathcal{L}_x(G) \neq \emptyset$. Hence there exists $g, h \in G$ such that $g^{-1}F(g) = h^{-1}xF(h)$ and hence $x = \mathcal{L}(gh^{-1})$. $\square$

Let G be an algebraic group and H a closed normal subgroup. We obtain an algebraic group G/H and one has a canonical morphism $G \to G/H$ of algebraic groups.

(This can be done neatly using the functor of points approach).

Suppose G is defined over $\mathbb{F}_q$ and H is an F-stable. Then we have $G^F/H^F \hookrightarrow (G/H)^F$.

Proposition 2.14. *If H is connected, then $G^F/H^F \cong (G/H)^F$.*

This is false if H is not connected.

Example 2.15. Take $Z \subset SL_n$ where Z denotes the centre.

[Exercise: Work out this example for SL_2.]

Hence $SL_s(\mathbb{F}_q)/\pm 1 \neq (SL_2/\pm 1)^F$.

Lemma 2.16. *If G is a connected algebraic variety acting on X. Assume that everything is defined over $\mathbb{F}_q$. If $\mathcal{O}$ is an F-stable orbit of G on X (means fixed, but not point-wise). Then $\mathcal{O}^F \neq \emptyset$.*

Proof. Choose $x \in \mathcal{O}$. Then $F(x) = g(x)$ for some $g \in G$. Hence there exists $h \in G$ such that $g = h^{-1}F(h)$ then $F(h(x)) = h(x)$. [Check]. $\square$

Proof. (Of the proposition) Set $X = G$ acted on by H. Set $\mathcal{O} = gH$ and assume that $\mathcal{O}$ is F-stable. Hence there exists a point $h \in H$ such that $F(gh) = gh$. Hence $gh = G^F$. $\square$

The above lemma can be refined as follows:

Lemma 2.17. *With the same assumptions as in the previous lemma let $x \in \mathcal{O}^F$ and $g \in G$.*

(1) *We have $g(x) \in \mathcal{O}^F$ if and only if $\mathcal{L}(g) \in \mathrm{Stab}_G(x)$.*
(2) *There is a bijection (depending on X)*

$$\{G^F - \textit{orbits on } \mathcal{O}^F\} \xrightarrow{\sim} \left\{ \begin{array}{c} F\textit{-conjugacy classes of} \\ \mathrm{Stab}_G(x)/\mathrm{Stab}_G(x)^0 \end{array} \right\}$$
$$g(x) \mapsto \mathcal{L}(g)$$

Remark 2.18. Check that (2) is well-defined. Consider $hg(x)$ where $h \in G^F$. Then $\mathcal{L}(hg) = g^{-1}h^{-1}F(h)F(g) = g^{-1}F(g) = \mathcal{L}(g)$.

Definition 2.19. If F acts on G we say that g, g' are F-conjugate if there exists an $x \in G$ such that $g' = xgF(x)^{-1}$.

Note that if G is connected then there is a unique F-conjugacy class.

Corollary 2.20. *F-conjugacy classes of G are in bijection with F-conjugacy classes of G/G^0.*

Proposition 2.21. *If G is an algebraic group defined over $\mathbb{F}_q$. There is a bijection:*

$$\left\{ \begin{array}{c} G^F\textit{-conjugacy classes contained in} \\ G\textit{-conjugacy class of } x \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} F\textit{-conjugacy classes of} \\ \textit{of } C_G(x)/C_G(X)^0 \end{array} \right\}$$

Take Θ = G-conjugacy class of x, $X = G$.

Remark 2.22. Centralisers are connected for GL_n. (Also works for unitary groups : non-dependence on choice of F).

Example 2.23. Take $G = SL_w$. Then G^F conjugacy classes of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for $a \in \mathbb{F}_q^*$ is in bijection with $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$.

This is even worse for symplectic groups!

Take G an algebraic group defined over $\mathbb{F}_q$.

Fact 2.24. (1) There exist F-stable Borel subgroups.
(2) They are all G^F-conjugate.
(3) There exist F-stable maximal tori.
(4) There is a bijection:

$$\left\{ \begin{array}{c} G^F\text{-conjugacy classes} \\ \text{of } F\text{-stable maximal tori} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} F\text{-conjugacy classes} \\ \text{of } W \end{array} \right\}$$

3. Lecture 3: Deligne-Lusztig induction

3.1. **The Bruhat decomposition.** k is an algebraically closed field and G a reductive connected algebraic group over k. (Note: reductive usually contains connected as an assumption!)

Recall from last time that all Borel subgroups in G are conjugate. Set

$$\mathcal{B} = \{\text{Borel subgroups in G}\}$$

If we fix $B \in \mathcal{B}$ there is a bijection $G/B \xrightarrow{\sim} \mathcal{B} : gB \mapsto gBg^{-1}$. (Uses the two crucial facts: all Borel subgroups are conjugate, and B is its own normaliser.) In fact G/B can be endowed with the structure of an algebraic variety.

In fact, G/B is smooth and projective. Smoothness follows from the fact that it is a homogenous space. Set $\mathfrak{g} := \operatorname{Lie} G$. Then we can identify $\mathcal{B}$ with the set of Borel subalgebras of $\mathfrak{g}$. But

$$\{\text{Borel subalgebras of } \mathfrak{g}\} \hookrightarrow \left\{ \begin{array}{c} \text{Grassmannian of dimension} \\ N = \dim B \text{ subspaces in } \mathfrak{g} \end{array} \right\}$$

the latter is the Grassmannian is projective. The condition that a subspace be a Borel subalgebra is a closed condition.

Remark 3.1. There exists an irreducible finite dimensional representation L of G and a line $\ell \subset L$ such that $\operatorname{Stab}_G(\ell) = B$, then $G/B \hookrightarrow \mathbb{P}(L) : gB \mapsto g\ell$.

Example 3.2. $G = GL_2$ and $B = \left\{ \begin{array}{cc} * & * \\ 0 & * \end{array} \right\}$. Then

$$\begin{array}{rcl} G/B & \xrightarrow{\sim} & \mathbb{P}^1 = \{\ell \in k^2 \mid \dim \ell = 1\} \\ gB & \mapsto & ge_1. \end{array}$$

Example 3.3. $G = GL_n$,

$$\mathcal{F}\ell_n = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^n \mid \dim V_i = i\}$$

Then

$$G/B \xrightarrow{\sim} \mathcal{F}\ell_n$$

by acting on the standard flag.

Let $T \subset B$ be a maximal torus, $W = N_G(T)/T$. Given $w \in W$, $\dot{w}$ denotes a lift to w.

Theorem 3.4.

$$G = \bigsqcup_{w \in W} B\dot{w}B$$

Corollary 3.5.

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B$$

In fact, $B\dot{w}B/B \cong \mathbb{A}^{\ell(w)}$. One may show that $B\dot{w}B$ is a homogenous space for U, and can be realised as an explicit quotient (by certain root subgroups).

Recall that $B = U \rtimes T$ and $U = R_u(B)$.

Lemma 3.6. $N_G(T) \cap B = T$

Proof. If $g \in U \cap N_G(T)$ then there exists a $t \in T$ such that $[g, t] \in T$. Hence $T \cap U = 1$. Hence $g \in C_G(T) = T$. □

There are two proofs of the Bruhat decomposition:

Proof 1. One constructs a Tits system (BN-pair). One has lots of structure with a BN-pair. (In particular, one always has a Bruhat decomposition.) □

Proof 2. The second proof uses the fact that every B-orbit on G/B contains a unique T-fixed point. One can then study the B-orbits using the Bialynicki-Birula decomposition. □

For GL_n the Bruhat decomposition is the Gaussian elimination. Every element can be written as a product of bnb' where b, b' are upper triangular matrices and n is a permutation matrix.

We have $B \setminus G/B \xrightarrow{\sim} G \setminus (G/B \times G/B)$. Now we have something intrinsic:

Corollary 3.7.

$$\begin{aligned} G\text{-orbits on } \mathcal{B} \times \mathcal{B} &\xrightarrow{\sim} W \\ \mathcal{O}(w) &\leftarrow w \end{aligned}$$

Hence one has an intrinsic description of W. $\mathcal{O}(1) = \Delta\mathcal{B} \subset \mathcal{B} \times \mathcal{B}$. Simple reflections $s \in S$ correspond to almost minimal G orbits. In fact,

$$\dim \mathcal{O}(w) = \dim(\mathcal{B}) + \ell(w).$$

We have a map $\mathcal{O}(w) \to \mathcal{B} : (B_1, B_2) \mapsto B_2$. This is a fibration and the fibres over a fixed B_2 are $BwB/B \subset G/B \xrightarrow{\sim} \mathcal{B}$.

Once we fix a Borel subgroup

$$\begin{aligned} \mathcal{O}(w) &\xrightarrow{\sim} \{(g_1B, g_2B) \mid g_1^{-1}g_n \in BwB\} \\ \mathcal{B} \times \mathcal{B} &\xrightarrow{\sim} G/B \times G/B \end{aligned}$$

We write $B_1 \xrightarrow{w} B_2$ for $(B_1, B_2) \in \mathcal{O}(w)$ and say that "B_1 is in relative position w to B_2".

How do we multiply elements in W using the intrinsic description above? If $\ell(w)+\ell(w')=\ell(ww')$ then

$$\begin{array}{rcl} \mathcal{O}(w)\times_{\mathcal{B}}\mathcal{O}(w') & \xrightarrow{\sim} & \mathcal{O}(ww') \\ (B_1\xrightarrow{w}B_2\xrightarrow{w'}B_3) & \mapsto & (B_1\xrightarrow{ww'}B_2) \end{array}$$

Hence one can recover the multiplication from G-orbits on $\mathcal{B}\times\mathcal{B}$.

3.2. **Deligne-Lusztig varieties.** Assume that k is the algebraic closure of $\mathbb{F}_q$ and that G is defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F.

Take $w\in W$, define

$$X(w):=\mathcal{O}(w)\cap\Gamma_F=\{B\in\mathcal{B}\mid B\xrightarrow{w}F(B)\}$$

where $\Gamma_F=\{(B,F(B))\mid B\in\mathcal{B}\}$.

For example, $X(1)=\mathcal{B}^F=G^F/B^F$ if B is an F-stable Borel subgroup of G. Of course,

$$\mathcal{B}=\bigsqcup_{w\in W}X(w).$$

Switch to representations: $\mathbb{C}[G^F/B^F]$ is a permutation representation and $\mathrm{End}_{\mathbb{C}G^F}(\mathbb{C}[G^F/B^F])$ is the Hecke algebra and the irreducible components of $\mathbb{C}[G^F/B^F]$ are in bijection with $\mathrm{Irr}(W)$.

For GL_2, $\mathbb{C}[G^F/B^F]=1_{G^F}\oplus(St)$. The Steinberg occuring with multiplicity 1. Note that $G^F/B^F=\mathbb{P}^1(\mathbb{F}_q)$.

Definition 3.8. $R_w:=\sum_{i\geq 0}(-1)^i[H^i(X(w))]$ in $K_0(G^F)$.

Definition 3.9. An irreducible character of G^F is called *unipotent* if there exists $w\in W$ such that $\langle\chi,R_w\rangle\neq 0$.

Those $\chi\in\mathbb{C}[G^F/B^F]$ are *principal series* unipotent characters.

Note: G acts on $\mathcal{O}(w)$ which restricts to an action of G^F on $X(w)$. G *does not* act on $X(w)$.

Fix B an F-stable Borel subgroup.

Remember that $G/B\xrightarrow{\sim}\mathcal{B}$. We claim that $X(w)\cong\{g\in G/B\mid g^{-1}F(g)\in BwB\}$. We introduce a variety $Y(w)$ above $X(w)$:

$$Y(\dot{w}):=\{gU\in G/U\mid g^{-1}F(g)\in U\dot{w}U\}.$$

We have an obvious map

$$\begin{array}{rcl} Y(\dot{w}) & \mapsto & X(w) \\ gU & \mapsto & gB \end{array}$$

On G/U we have commuting left G and right T-actions. For $t\in T$, $\mathcal{L}(gt)=t^{-1}\mathcal{L}(g)F(t)\in t^{-1}U\dot{w}UF(t)=U\dot{w}U(w^{-1}t^{-1}w)F(t)\}$. The

condition $\dot{w}^{-1}t^{-1}\dot{w}F(t) = 1$ is equivalent to $\dot{w}F(t)\dot{w}^{-1} = t$ which occurs if and only if $t \in T^{wF}$.

Hence G^F acts on the right on $Y(\dot{w})$ and T^{wF} acts on the left.

Note that there exists a d such that $(wF)^d = F^d$. Hence wF is a Frobenius endomorphism of T.

The map $G/U \to G/B$ is the quotient by T. The map $\pi : Y(\dot{w}) \to X(w)$ is a quotient by T^{wF}. Moreover, T^{wF} acts freely.

π is unramified (étale) and has Galois group T^{wF}.

Definition 3.10. Take $\theta \in \mathrm{Irr}(T^{wF})$ define

$$R_w(\theta) = \sum_{i \geq 0} (-1)^i [H^i(Y(\dot{w})) \otimes_{\overline{\mathbb{Q}_\ell}} \theta]$$

We have $R_w(1) = R_w$ because $H^i(Y(w)) \otimes_{\overline{\mathbb{Q}_\ell}} \theta = H^i(X(w), \mathcal{L}_\theta)$.

In fact

$$\pi_* \overline{\mathbb{Q}_\ell} = \bigoplus \mathcal{L}_\theta.$$

Example 3.11. If $G = SL_2$ then $\mathcal{B} = \mathbb{P}^1$ and $W = S_2 = \{1, s\}$. We have $X(1) = \mathbb{P}^1(\mathbb{F}_q)$ and $X(s) = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ which has a map from the Drinfeld curve $Y(s) = \{xy^q - x^q y = 1\}$.

Theorem 3.12 (Deligne-Lusztig)**.** *If θ is in general position (this means* $\mathrm{Stab}\, N_{G^F}(T^{wF})(\theta) = 1$*). Then $\pm R_w(\theta)$ is irreducible.*

This was MacDonald's conjecture. The proof will be given later.

[Generalities: suppose that $Y \to X$ is an unramified connected covering with group $\mathbb{Z}/n$ are in bijection with $H^1(X, \mathbb{Z}/n)$. In the example of SL_2, $H^1(X(s), \mathbb{Z}/n)$ subspace of $(\mathbb{Z}/n[\mathbb{P}^1(\mathbb{F}_q)])$ such that$(\sum e_i)$ which is $(\mathbb{Z}/n) \otimes (St)$. Hence $H^1(X(s), \mathbb{Z}/n)$ is a rep of $SL_n(\mathbb{F}_q)$ over $\mathbb{Z}/(n)$. it has a one-dimensional fixed subspace if and only if $n | q + 1$.

[Exericise: Check that "in general" there are no fixed points.]

$\theta \in \mu_{q+1}^\vee : \alpha \mapsto \alpha^d$ is in general position if and only if $d \neq \pm 1$.

$X(w) = \mathcal{O}(w) \cap \Gamma_F$. Note that $\mathcal{O}(w)$ is smooth and $\Gamma_F \cong \mathcal{B}$. Claim: the intersection is transverse. Hence $X(w)$ is smooth and its dimension is $\ell(w)$.

A more complicated fact is:

Theorem 3.13. *$X(w)$ is affine if $q > h$, where h is the Coxeter number.*

Proof. Later! □

Bonnafé-Rouquier: conjecture this always holds.

Rappaport: conjectures that this doesn't always hold.

In fact, Deligne and Lusztig show:

$$R_w(\theta) = H^{\ell(w)}(Y(w)) \otimes_{\overline{\mathbb{Q}}_\ell T^{wF}} \theta$$

4. Lecture 3

G algebraic group, reductive over $k = \overline{\mathbb{F}}_q$. We assume that G is defined over $\mathbb{F}_q$ or equivalently, endowed with $F : G \to G$ a Frobenius. G^F is a finite group.

If we fix $T_0 \subset B_0$ an F-stable torus contained in an F-stable torus contained in an F-stable Borel subgroup of G.

We define $W = N_G(T_0)/T_0$.

Given $w \in W$ we define $X(w) := \{gB_0 \in G/B_0 \mid \mathcal{L}(g) \in B_0wB_0\}$. Remember that the Lang map being $\mathcal{L}(g) = g^{-1}F(g)$.

One can see that

$$X(w) \xrightarrow{\sim} \{g \in G \mid \mathcal{L}(g) \in B_0wB_0\}/B_0 \xrightarrow{\sim} \{g \in G \mid \mathcal{L}(g) \in wB_0\}/(B_0 \cap (wB_0w^{-1}))$$

the last isomorphism is not obvious. If we put $B = gB_0g^{-1}$ then $\mathcal{L}(g) \in B_0wB_0$ if and only if $B \xrightarrow{w} F(B)$ if and only if there exists $h \in G$ such that $(B, F(B)) = h(B_0, wB_0w^{-1})h^{-1}$. Now $\mathcal{L}(h^{-1}g) \in wB_0$. [Check!]

$X(w)$ has a left G^F-action (because $\mathcal{L}(hg) = \mathcal{L}(g)$ if $h \in G^F$.

$$Y(w) = \{g \in G \mid \mathcal{L}(g) \in \dot{w}U_0\}/U_0 \cap (\dot{w}U_0\dot{w}^{-1})$$

Note that

$$B_0 \cap wB_0w^{-1} = T_0 \ltimes (U_0 \cap \dot{w}U_0\dot{w}^{-1})$$

The natural map

$$Y(\dot{w}) \to X(w)$$

is the quotient by T_0^{wF}.

$$\begin{aligned} \{(T \subset B), T \text{ is } F\text{-stable}\}/G^F - \text{conjugacy} &\xrightarrow{\sim} \qquad W \\ (T \subset B) &\mapsto \text{ unique } w \text{ such that } B \xrightarrow{w} F(B). \end{aligned}$$

$$\{T\ F\text{-stable maximal torus}\}/G^F - \text{conjugacy} \xrightarrow{\sim} \ F\text{-conjugacy classes in } W.$$

One can consider

$$X_{T\subset B} = \{g \in G \mid \mathcal{L}(g) \in BF(B)\}/B \xrightarrow{\sim} \{g \in G \mid \mathcal{L}(g) \in F(B)/B_0F(B)$$

and

$$Y_{T\subset B} = \{g \in G \mid \mathcal{L}(g) \in F(U)\}/U \cap F(U)$$

and $X_{T\subset B}$ is the quotient of $Y_{T\subset B}$ by T^F. In fact

$$X_{T\subset B} \to \{B' \mid B' \xrightarrow{w} FB'\} \xleftarrow{\sim} \{gB_0 \mid \mathcal{L}(g) \in B_0wB_0\}$$

the latter map being $gB_0 \mapsto gB_0b^{-1}$.

One always has a canonical isomorphism $N_G(T_0)/T_0 \xrightarrow{\sim} N_G(T)/T$ (this was explained last time).

One can identify $Y_{T\subset B} \xrightarrow{\sim} Y(\dot{w})$.

Theorem 4.1. *$\sum(-1)^i\chi_{H^i(Y_{T\subset B})}$ is independent of B. Similarly, $\sum(-1)^i\chi_{H^i(Y(\dot{w}))}$ depends only on the F-conjugacy class of w.*

If we consider $Y = \{g \in G \mid \mathcal{L}(g) \in F(U)\}/U \cap F(U)$ is a quotient by $U \cap F(U)$ of $\mathcal{L}^{-1}(F(U))$. Clearly

$$G \supset \mathcal{L}^{-1}(F(U)) \xrightarrow{\mathcal{L}} F(U) \subset G$$

Locally, the quotient map is trivial with fibres $U \cap F(U)$ which is an affine space. hence $H^*(\mathcal{L}^{-1}(F(U))) \xrightarrow{\sim} H^*(Y)$.

Note that the Lang map is a quotient of G by the finite group G^F.

This is an amazing topological situation!!

$$\mathcal{L}^{-1}(F(U)) \xrightarrow{\mathcal{L}} \text{affine space}$$

the map is unramified with group G^F. (Thinking topologically, we get a map $\pi_1(F(U)) \twoheadrightarrow G^F$) (and this map is onto for a semi-simple group).

(Hence the fundamental group of a vector space is very complicated!!)

We can even make $\pi(\mathbb{A}^1) \twoheadrightarrow G^F$ (this is a Lefschetz type principle).

Theorem 4.2 (Raynaud, formerly Abyankar's conjecture)**.** *If G is a finite group, then there exists a $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}}_q}) \to G$ if and only if G is generated by its p-Sylow subgroups. That is $O^{p'}G = G$.*

(uses p-adic geometry in an essential way).

$G = GL(V)$ where $V = k^n$.

$T_0 = \left\{\begin{matrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & * \end{matrix}\right\}$. $F((x_{ij})) = (x_{ij}^q)$ and W is the symmetric group.

$\mathcal{B} = \{\text{Borel subgroups}\} \xrightarrow{\sim} \{0 = D_0 \subset D_1 \subset \cdots \subset D_n = V \mid \dim D_i = i\}$

The bijection is given by sending B to the unique complete flag fixed by B. These are "complete flags".

Also $G/B_0 \cong \mathcal{B}$.

Let U_0 be the subgroup of upper triangular matrices. Then G/U_0 is a T_0-torsor over G/B and we may identify

$$G/U_0 \xrightarrow{\sim} \{0 = D_0 \subset \cdots \subset D_n, (e_1, \dots, d_n) \mid e_i \in D_i/D_{i-1} \setminus \{0\}\}$$

T_0 acts on the $(e_1, \dots, e_n)$ in the natural way.

Let D and D' be two flags. Then $D \xrightarrow{w} D'$ if we have

$$\mathrm{Gr}_D^{w(i)} \mathrm{Gr}_{D'}^i(V) \neq 0$$

for all i.

We will be discussing $w = (1, \dots, n)$.

$D \xrightarrow{w} D'$ if and only if $D'_i + D_i = D_{i+1}$ for $i < n-1$ and $D'_{n-1} + D_1 = V$.

Hence $D \xrightarrow{w} F(D)$ if and only if $D_2 = D_1 + F(D_1), \dots, D_i = D_1 + F(D_1) + \cdots + F^{i-1}D_1$. Hence

$$\begin{aligned} X(w) &\hookrightarrow \mathbb{P}(V) \\ D &\mapsto D_1 \end{aligned}$$

Hence the image is $\ell \in \mathbb{P}(V) \mid \ell + F(\ell) + \cdots + F^{n-1}(\ell) = V\} = \mathbb{P}(V) \setminus \{\text{hyperplanes defined over } \mathbb{F}_q\}$.

In $Y(w)$ a point is determined by $e_i's$ and hence by e_1. The condition is that $e_2 = F(e_1) \mod e_1$, $e_3 = F^2(e_1) \mod (e_1, F(e_1))$ and $e_n = F^{n-1}(e_1) \mod (e_1, \dots, F^{n-1}(e_1))$ and lastly

$$e_1 = F^n(e_1) \mod (Fe_1, \dots, \dots, F^{n-1}(e_1)).$$

These conditions force

$$e_1 \wedge F(e_1) \wedge \cdots \wedge F^{n-1}(e_1) = F^n(e_1) \wedge F(e_1) \wedge \cdots \wedge F^{n-1}(e_1)$$

hence

$$e_1 \wedge \cdots \wedge F^{n-1}e_1 = (-1)^{n-1} F(e_1 \wedge \cdots \wedge F^{n-1}e_1)$$

if we write $e_1 = (x_1, \dots, x_n)$ then we get the matrix

$$\det((x_i)_{1 \le i,j \le n}^{q^{j-1}})^{q-1} = (-1)^{n-1}.$$

Proposition 4.3.

$$Y(w) \xrightarrow{\sim} \{x_1, \dots, x_n) \in \mathbb{A}^n \mid \det((x_i)_{1 \le i,j \le n}^{q^{j-1}})^{q-1} = (-1)^{n-1}\}$$

$GL_n(\mathbb{F}_q)$ acts naturally on V. $T_0^{wF} = \mathbb{F}_{q^n}^*$ acts diagonally.

The van de Monde determinant is up to scalar the product of all linear forms over $\mathbb{F}_q$. There are $\frac{q^n-1}{q-1}$ linear forms (i.e. points in $\mathbb{P}(V^*)(\mathbb{F}_q)$).

For $n = 2$ $X(w) = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ and $Y(w) = \{(x, y) \mid (xy^q - x^q y)^{q-1} = -1\} = \bigsqcup_{\zeta^{q-1}=-1} \{(x, y) \mid xy^q - x^q y = \zeta\}$. The connected components are permuted by $GL_2(\mathbb{F}_q)$ and the components are isomorphic to $\{(x, y) \mid xy^q - x^q y = 1\}$. $SL_2(\mathbb{F}_q)$ fixes the components.

5. Lecture 5: Cohomology

Let $k = \bar{\mathbb{F}}_q$ and ℓ be a prime with $\ell \nmid q$, let $\Lambda = \bar{\mathbb{Q}}_\ell$ (which is isomorphic with $\mathbb{C}$ for all intents and purposes), and consider varieties over k. There is a functor H^*, ℓ-adic cohomology, between varieties over k and graded Λ-vector spaces. Note that there is no cohomology theory with values in $\mathbb{Q}$. Serre proved that there is a suitable elliptic curve X with automorphism group G, and the representation of G on $H^1(X)$ is not defined over $\mathbb{Q}$.

There is another cohomology theory from varieties over k to graded Λ-vector spaces, H_c^*, the cohomology with compact support, in some sense dual to H^*. This embeds X as an open set in $\bar{X}$, called proper (the algebraic geometry version of compact). If X is proper (e.g., projective) then $H_c^*(X) = H^*(X)$. It is often easier to work with H_c^*, but sometimes things look nicer with H^*.

Trace map: let X be smooth and connected, of dimension d. There is a map $tr : H_c^{2d}(X) \xrightarrow{\sim} \Lambda$. This leads to Poincaré duality: there exists a perfect pairing $H^i(X) \times H^{2d-i}(X)_c \to \Lambda$. A nice summary of the results on etale cohomology in Digne–Michel, and in Carter. In particular, $H^i(X) = H_c^{2d-i}(X)^*$.

For example, $H^*(\mathbb{A}^n) = \Lambda$, concentrated in degree 0. Therefore $H_c^*(\mathbb{A}^n) = \Lambda$, concentrated in degree $2n$. Finally, $H^0(X)$ is a free Λ-module of rank the number of connected components, and $H^i(X) = 0$ for $i \notin [0, 2d]$.

If X is a variety and $U \subset X$ is open, write $Z = X \setminus U$, then there is a long exact sequence

$$0 \to H_c^0(U) \to H_c^0(X) \to H_c^0(Z) \to H_c^1(U) \to H_c^1(X) \to H_c^1(Z) \to \cdots .$$

Let $X = \mathbb{P}^n$, $U = \mathbb{A}^n$, $Z = \mathbb{P}^{n-1}$, so by induction on n,

$$H^*(\mathbb{P}^n) = H_c^*(\mathbb{P}^n) = \begin{cases} \Lambda & * \in \{0, 2, \ldots, 2n\} \\ 0 & \text{otherwise} \end{cases} .$$

- If X is quasi-projective, and acted on by a group G, then $H^*(X/G) = H^*(X)^G$.
- If X is acted on by a connected algebraic group G, then G acts trivially on $H^*(X)$. If H is a finite subgroup of G, then H still acts trivially on $H^*(X)$. This is a useful method to prove that some groups act trivially on $H^*(X)$. (In particular, in extending actions of abelian groups to tori.)

A variety over k has only finitely many coefficients involved in its definition, so we think of it as a variety over some finite field, which we might as well take to be $\mathbb{F}_q$. We can endow a variety X with a Frobenius

endomorphism F. Via H^*, we get a graded Λ-vector space with an action of F. The computational power of the theory comes from the action of F. The same is true for H_c^*. At first blush, the endomorphism on graded vector spaces could be anything, but with a bit of work one may show that it is invertible. Hence F gives an automorphism on the graded Λ-vector spaces $H^*(X)$ and $H_c^*(X)$. (This has something to do with sheaves and sites.)

If X is connected, then F acts trivially on $H^0(X)$. However, the trace map doesn't work so well, so we get $H_c^{2n}(X) \xrightarrow{\sim} \Lambda(n)$, where this means that the map F acts by q^n. For Poincaré duality as well, we get $H^i(X) \times H^{2d-i}(X)_c \to \Lambda(n)$. By being extra-careful in the computation above, we get $H_c^*(\mathbb{A}^n) = \Lambda(n)$ concentrated in degree $2n$, and

$$H^i(\mathbb{P}^n) = H_c^i(\mathbb{P}^n) = \begin{cases} \Lambda(i/2) & i \in \{0, 2, \dots, 2n\} \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 5.1. *The eigenvalues of F on on $H_c^*(X)$ have modulus q^a, where a is a semi-integer and are algebraic integers.*

Notice that the eigenvalues might well not be q^a themselves.

5.1. **Lefschetz Trace Formula.** We want to know $|X^F| = |X(\mathbb{F}_q)|$. We have

$$|X^F| = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F \mid H_c^i(X)).$$

Example 5.2. First let $X = \mathbb{A}^n$: $|X^F| = q^n$. Then F acts by q^n on $H_c^{2n}(X)$.

If $X = \mathbb{P}^n$, then $|X^F| = 1 + q + \cdots + q^n$. To see this, if $U \subset X$ is F-stable and open, and $Z = X \setminus U$, then $X^F = U^F \amalg Z^F$, and we can basically add them.

Corollary 5.3. *If g is finite order automorphism of X defined over $\mathbb{F}_q$ (i.e., commuting with F), then*

$$\operatorname{Tr}(g \mid H_c^*(X)) = \sum (-1)^i \operatorname{Tr}(g \mid H_c^i(X)) = \lim_{t \to \infty} \sum_{n=1}^{\infty} |X^{gF^n}| t^n.$$

Proof. gF^n is a Frobenius endomorphism on X relative to $\mathbb{F}_{q^n}$. Therefore $|X^{gF^n}| = \operatorname{Tr}(gF^n \mid H_c^*(X))$. We may choose F to be upper triangular, with diagonal $\lambda_1, \dots, \lambda_r$ and g to be diagonal with entries $\alpha_1, \dots, \alpha_r$. Set $\varepsilon_i = 1$ if in even H^* and -1 if in odd degree. Then

$$\operatorname{Tr}(gF^n \mid H_c^*(X)) = \sum_{l=1}^{r} \varepsilon_l \alpha_l \lambda_l^n,$$

and so

$$\sum_{n=1}^{\infty} |X^{gF^n}| t^n = \sum_l \varepsilon_l \alpha_l \sum_n (\lambda_l t)^n = \sum_l \varepsilon_l \alpha_l \left(\frac{1}{1-\lambda_l t} - 1 \right).$$

As $t \to \infty$ the last term in the brackets tends to -1, and so this becomes $\sum_l \varepsilon_l \alpha_l$. □

Example 5.4. Let $X = \{xy^q - x^q y = 1\} \subset \mathbb{A}^2$, a closed subset. $F(x,y) = (x^q, y^q)$. Then $X^F = \emptyset$. This means that $\mathrm{Tr}(F \mid H_c^*(X)) = 0$. If X is an affine variety, then $H^i(X) = 0$ if i is greater then $\dim X$, so $H_c^0(X) = H^2(X)^* = 0$. We also know that $H_c^2(X) = \Lambda$ where F acts by q. The only missing case is $H_C^1(X)$. the trace being 0 means that $q = \mathrm{Tr}(F \mid H_c^1(X))$.

If $g \in SL_2(\mathbb{F}_q) \times \mu_{q+1}$, then knowing $|X^{gF^n}|$ for $n \geq 1$ determines $\mathrm{Tr}(g \mid H_c^1(X))$.

Let G be a reductive algebraic group over $k = \bar{\mathbb{F}}_q$, with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. If T is an F-stable maximal torus contained in a (not necessarily F-stable) Borel B, and θ an irreducible character of T^F, we get

$$R_{T\subset B}^G(\theta) = \sum_i (-1)^i H^i(Y_{T\subset B}) \otimes_{T^F} \theta.$$

If you vary θ and T you get lots of virtual representations, and the first result is the following.

Theorem 5.5. *We have*

$$\langle R_{T\subset B}^G(\theta), R_{T'\subset B'}^G(\theta')\rangle_{G^F} = |T^F| \cdot \left\{ g \in G^F \mid gTg^{-1} = T',\ g\theta g^{-1} = \theta' \right\}.$$

In particular, if (T', θ') is not G^F-conjugate to (T, θ), then this scalar product is 0.

Proof. We have

$$H_c^i(Y_{T\subset B} \times_{G^F} Y_{T'\subset B'}) \cong \bigoplus_{i_1+i_2=i} H_c^{i_1}(Y_{T\subset B}) \otimes_{\Lambda G^F} H_c^{i_2}(Y_{T'\subset B'})$$

(Kü nneth formula). We will also need to understand the tensor product of θ and θ', but concentrate on the Ys. This is

$$Z = \mathcal{L}^{-1}(F(U)) \times_{G^F} \mathcal{L}^{-1}(F(U')) = \{(x, x', y) \in F(U) \times F(U') \times G \mid xF(y) = yx'\}.$$

Decompose according to the Bruhat decomposition of G, i.e., $G = \amalg_{w\in W} BwB$. The variety Z may be decomposed as the disjoint union of the Z_w, which have an action of $T^F \times T'^F$ acting on it.

The action of $T^F \times T'^F$ on Z_w extends to an action of

$$H_w = \{(t, t') \in T \times T' \mid \mathcal{L}(t') = F(w)^{-1}\mathcal{L}(t)F(w)\}.$$

Now we may use the fact that H_w° acts trivially on $H^*(Z_w)$. (We extend the action of $T^F \times {}'^F$ to one of an algebraic group, as we said would be useful before.) If $\theta \otimes_{\Lambda T^F} H_c^*(Z_w) \otimes_{\Lambda T'^F} \theta' \neq 0$ then $(\theta \otimes \theta')|_{H_w^\circ} = 1$. One readily checks that there exists w such that $(\theta \otimes \theta')|_{H_w^\circ} = 1$ if and only if $(T', \theta') \sim_{G^F} (T, \theta)$. $\square$

Geometrric conjugacy is $(T, \theta) \sim (T', \theta')$. Let λ be a geometric conjugacy class. Define

$$\xi(G^F, \lambda) = \{\chi \in \mathrm{Irr}(G^F) \mid \langle R_{T \subset B}^G(\theta), \chi \neq 0 \text{ for some } (T, \theta) \in \lambda\}.$$

Theorem 5.6. $\mathrm{Irr}(G^F) = \coprod_\lambda \xi(G^F, \lambda)$. *Also*

$$\xi(G^F, \lambda) = \{\chi \in \mathrm{Irr}(G^F) \mid \langle H_c^i(Y_{T \subset B}) \otimes_{T^F} \theta, \chi \rangle \neq 0 \textit{ for some } i, (T\theta) \in \lambda\}.$$

Lusztig series: unipotent characters are $\xi(G^F, 1)$, where 1 is the geometric conjugacy class of $(T, 1)$. Also,

$$\xi(G^F, \lambda) \xrightarrow{\sim} \xi(H^F, 1)$$

for some H (Jordan decomposition).

Remark:

$$\mathrm{Cl}(G^F) \xrightarrow{\sim} \coprod_{s \text{ semisimple}/G^F} \mathrm{UnipCl}(C_{G^F}(s),$$

with (x) on the right being sent to (xs) on the left.

G and G^* are Langlands dual: take $T \subset G$ and $T^* \subset G^*$ be F-stable and F^*-stable maximal tori. Then $X(T) \xrightarrow{\sim} Y(T^*)$ compatible with F and F^*, going between roots and coroots. For example, $GL_n^* = GL_n$ and $SL_n^* = PGL_n$.

If T is a torus, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, and $Y(T) \otimes_{\mathbb{Z}} k^* \xrightarrow{\sim} T$ given by $\zeta \otimes x \mapsto \zeta(x)$. As $T^F = \ker_T(F - 1)$, we get a short exact sequence

$$0 \to T^F \to Y(T) \otimes_Z M(\mathbb{Q}/ZM)_{p'} \to^{F-1} Y(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/ZM)_{p'} \to 0.$$

Choose d such that F^d acts trivially on $Y(T)$. We have another sequence

$$0 \to Y(T) \to^{F-1} Y(T) \to T^F \to 0$$

with the last map being $N : \zeta \mapsto N_{F^d/F}(\zeta)(\alpha)$, the norm map. (Here α is a generator of $\mathbb{F}_{q^d}$. Then (T, θ) and (T', θ') are geometrically conjugate if $(T, \theta \circ N) \sim_G (T'\theta' \circ N')$.

6. Lecture 6

G a reductive algebraic group of $k = \overline{\mathbb{F}_q}$ endowed with F a Frobenius. $\Lambda = K = \overline{\mathbb{Q}_\ell}$ where ℓ does not divide q.

Given $T \subset B$ a maximal F-stable torus (B need not be stable). We have

$$R^G_{T\subset B} : K_0(KT^F) \to K_0(KG^F)$$

recall that $Y_{T\subset B}$ has commuting left G^F and right T^F actions and so $\sum_i (-1)^i [H^i(U, K) \otimes_{KT^F} -]$. A crucial property was the orthoganality relations.

$$\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle_{G^F} = 0$$

Corollary 6.1. $R^G_{T\subset B}(\theta) = R^G_{T\subset B}(\theta)$

Proof. $\langle f, f\rangle = \langle f', f'\rangle = \langle f, f'\rangle = \langle f', f\rangle$ and hence $\langle f - f', f - f'\rangle = 0$ and so $f = f'$. □

Remark 6.2. $H^*(T_{T\subset B})$ depends on B, only the alternating sum is independent.

The choice of a Borel $T \subset B$ yields an element $w \in W$ and $\dim Y_{T\subset B} = \ell(w)$.

Definition 6.3. We say that θ is in general position if $(g(T, \theta)g^{-1} = (T, \Theta)$ implies that $g \in T^F$.

Corollary 6.4. *If θ is in general position then $R^G_T(\theta)$ is irreducible.*

Problem: define $\phi \in K_0(KG^F)$ to be *uniform* if it is a linear combination of $R^G_T(\theta)$'s.

(Bad) fact: In general (i.e. type $\neq A$), not all class functions are uniform. (This means the lattice generated by $R^G_T(\theta)$'s is of strictly smaller rank.)

(Good) fact: $[KG^F]$ is uniform.

This means that for all $\chi \in \mathrm{Irr}(G^F)$ there exists a (T, θ) with $\langle \chi, R^G_T(\theta)\rangle \neq 0$.

$\mathbb{F}_q = k^F$ and $\mathbb{F}_{q^r} = k^{(F^r)}$. We have

$$\begin{tikzcd} T^{F'} \arrow[r, "N_{\mathbb{F}_{q^r}/\mathbb{F}_q}"] \arrow[dr, "\theta_r"'] & T^F \arrow[d, "\theta"] \\ & K' \end{tikzcd}$$

($N_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ is the norm of $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$.)

Given (T, θ) relative to F we have (T, θ_r) relative to F'.

Definition 6.5. (T,θ) and (T',θ') are geometrically conjugate if there exists an r such that (T,θ_r) and $(T',\theta_{r'})$ are conjugate over G^F.

Theorem 6.6. *If (T,θ) is not geometrically conjugate to (T',θ') then $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ have no irreducible characters in common.*

This yields a partition

$$\mathrm{Irr}(G^F) = \bigsqcup_{\substack{\lambda=(T,\theta) \\ \text{up to geom. conjugacy}}} \mathcal{E}(G^F,\lambda)$$

where

$$\mathcal{E}(G^F,\lambda) = \{\chi \mid \langle \chi, R_T^G(\theta)\rangle \neq 0 \text{ for some } (T,\theta)\in\lambda\ \}$$

$(T,1)$ and $(T',1)$ are always geometrically conjugate! (This follows from the fact that all maximal tori are conjugate over k.

Example 6.7. $G = SL_2$, $T\subset B$ F-stable and $T'\subset B'$ but B' are not F-stable.

$$R_T^G(1) = \mathrm{Id} + St \tag{6.1}$$

$$R_{T'}^G(1) = \mathrm{Id} - St \tag{6.2}$$

This decomposition was described by Lusztig. He also defined a "Jordan decomposition". We will only note that

$$\mathcal{E}(G^F,\lambda) \xrightarrow{\sim} \mathcal{E}(H^F,1)$$

for some reductive group H, depending on λ. (H^F is endoscopic??)

Hence the main difficulty is in describing unipotent characters $\mathcal{E}(G^F,\lambda)$.

Remark 6.8. Green function = values of $R_T^G(\theta)$ at a unipotent element.

One can express $\chi_{R_T^G(\theta)(g)}$ in terms of Green functions.

The explanation is of a geometrical nature:

If X is an algebraic variety of k with a Frobenius F defining an $\mathbb{F}_q$-structure and if g is an endomorphism of X of finite order (commuting with F) then we would like to compute the trace of g on $H_c^*(X,K)$.

Theorem 6.9.

$$\mathrm{Tr}(g, H_c^*(X,K)) = \mathrm{Tr}(u, H_c^*(X^s,K)$$

where $g = su$, $[s,u]=1$, s is semi-simple (p') and u is unipotent p.

Proof. Step 1: $\mathrm{Tr}(g, H_c^*(X,K))$ is independent of ℓ ($\ell\neq p$). This is clear because

$$\mathrm{Tr}(g, H_c^*(X,K)) = -\lim_{t\to\infty}\sum_{n=1}^{\infty} |X^{gF^n}|t^n$$

(see last lecture!)

Step 2: Write $X = \bigsqcup X_i$ where all points of X_i have the same stabilizer in $\langle g \rangle$. (Stabilizer decomposition.)

The X_i has a free action of $\overline{g}$. Because

$$\mathrm{Tr}(g, H_c^*(X, K) = \sum_i \mathrm{Tr}(g, H_c^*(X_i, K)$$

we can assume that g acts freely.

Assume that $s \neq 1$ and pick $\ell \neq p$ and $\ell | o(s)$. $H = \langle g \rangle$ acts freely on X and that the order of H is divisible by ℓ.

Lemma 6.10. *X an algebraic variety, and G a finite group acting freely on X and ℓ is a prime. Then*

$$\sum (-1)^i [H_c^i(X)]$$

is a virtual projective character (for $\ell \neq p$).

Proof. In fact $R\Gamma_c(X, \mathbb{Z}_\ell)$ complex of $\mathbb{Z}_\ell G$-modules is quasi-isomorphic to a bounded complex of finitely generated projective modules.

$$R\Gamma_c(X, \mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell G} V = R\Gamma_c(X/G, (\pi_* \mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell G} V)$$

$\pi : X \to X/G$ but

$$(\pi_* \mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell G} V)_x = (\pi_* \mathbb{Z}_\ell)_x \otimes^L_{\mathbb{Z}_\ell G} V \cong V$$

Then use the fact that if M is a complex of $\mathbb{Z}_\ell G$-modules and assume that there exists a finite interval I such that

$$H^i(M \otimes^L_{\mathbb{Z}_\ell G} V) = 0 \text{ for } i \in I$$

Then M is quasi-isomorphic to a bounded complex of projective $\mathbb{Z}_\ell G$-modules. □

We are back at $g = su$ acting freely on X and $\ell | o(s)$. We know that

$$\sum (-1)^i [H_c^i(X, K)]$$

is virtually projective with respect to ℓ.

Hence the character of g is zero.

(If we take $Q \times R$ where Q is an ℓ-group, and R is an ℓ'-group then a projective character is $KQ \otimes \phi$ and so the character value vanishes (x, y) and $y \in R$ if $x \neq 1$, $x \in Q$.

□

Example 6.11. Take $X = \mathbb{A}^1$, $g : x \mapsto x + 1$, $g^p = 1$ and $\mathbb{Z}/p\mathbb{Z}$ acts freely. Now

$$H_c^*(X) = \begin{cases} K & * = 2 \\ 0 & * \neq 2 \end{cases}$$

Note that $\mathrm{Tr}(g, H_c^*(X)) = 1$ and $X^g = \emptyset$.

Note the big difference with characteristic 0 coefficients! This should be zero!!

6.1. **Green functions for GL_n.** We now consider $G = GL_n$ and $F : (a_{ij}) \mapsto (a_{ij}^q)$.

We now consider the unipotent characters $R_{T_w}^G(1)$ and T_w of type (w) with $w \in \Sigma_n$ (in bijection with a partition of n).

Define almost characters $\chi \in \mathrm{Irr}(W)$

$$R_\chi := \frac{1}{|W|} \sum_{w \in W} \chi(w) R_{T_w}^G(1)$$

these are the *almost characters*.

Theorem 6.12. *R_χ is an irreducible character (for G of type A!) and the map*

$$\begin{array}{rcl} \mathrm{Irr}(W) & \xrightarrow{\sim} & \mathcal{E}(GL_n(q), 1) \\ \chi & \mapsto & R_\chi \end{array}$$

is a bijection. In general

$$\mathrm{Irr}(GL_n(q)) = \bigsqcup_{(s) \text{ semi-simple / conjugacy}} \mathcal{E}(GL_n(q), (s))$$

and

$$\mathcal{E}(GL_n(q), (s)) \xrightarrow{\sim} \mathcal{E}(C_{GL_n(q)}(s), 1)$$

(note that $C_{GL_n(q)}$ is a product of general linear groups).

If G is not of type A: decomposing R_χ as irreducible character in $\mathcal{E}(G^F, 1)$ yields a matrix: this is part of a "non-commutative Fourier transform" matrix. (The groups involved are small groups: i.e. $(\mathbb{Z}/2)^n$, Σ_3, Σ_4 and Σ_5.)