CHARACTER SHEAVES, TENSOR CATEGORIES AND NON-ABELIAN FOURIER TRANSFORM

Abstract. Notes from a course given by Victor Ostrik in Luminy, 2010. Notes by Geordie Williamson.

1. Character sheaves

Character sheaves were invented by Lusztig. The proofs are very difficult. The goal of the work to be described in these lectures is to find more conceptual arguments for some of the main theorems of character sheaves.

The goal of today's lecture will be to give an introduction to character sheaves.

Problem: Let G be a reductive group. The problem is to compute the character table of $G(\mathbb{F}_q)$, e.g. $GL_n(\mathbb{F}_q)$, ..., $E_8(\mathbb{F}_q)$.

Main idea: Use geometry associated to G.

1.1. **Set-up.** We will jump between base fields $k = \overline{\mathbb{F}}_q$ and $k = \mathbb{C}$. If X is defined over $\mathbb{F}_q$, this yields the Frobenius endomorphism Fr of X.

Throughout, *sheaf* will either mean ℓ-adic constructible sheaves or, if $k = \mathbb{C}$, sheaves in the classical topology. Throughout E will denote the coefficient field for the sheaves. That is $E = \overline{\mathbb{Q}}_\ell$ or $E = \overline{\mathbb{Q}}$.

The main example we will have in mind are local systems (otherwise known as locally constant sheaves). There is an equivalence of categories between local systems and representations of $\pi_1(X)$.

Constructible sheaves are those sheaves which are not too far from being local systems. One can imagine that they are obtained by "gluing local systems" along strata.

We will denote by $D(X)$ the bounded derived category of constructible sheaves on X. If $K \in D(X)$, $H^i(K)$ will denote its cohomology sheaves, and $H^i_x(K)$ will denote the stalk of $H^i(K)$ at $x \in X$. Given a morphism $f : X \to Y$ of complex varieties we have the functors $f_*, f_!$ from $D(X) \to D(Y)$ and $f^*, f^!$ from $D(Y)$ to $D(X)$. We also have Verdier duality $\mathbb{D}$ and the tensor product $\otimes$. All functors we consider are derived.

We recall the definition of perverse sheaves. These are sheaves satisfying

$$\dim \operatorname{supp} H^i(K) \leq -i \quad \dim \operatorname{supp} H^i(\mathbb{D}K) \leq -i.$$

This a weird condition, but the important point is that they form an abelian category $P(X)$ called the category of perverse sheaves on X. Given $U \subset X$ locally closed smooth irreducible and $\mathcal{L}$ a local system on U we can form $IC(\overline{U}, \mathcal{L})$, its perverse extension. Every irreducible perverse sheaf can be obtained in this way.

1.2. **Sheaf-function correspondence.** (Due to Grothendieck.) Start with X defined over $\mathbb{F}_q$. We have the Frobenius endomorphism Fr. We consider $\mathcal{F}$ a sheaf on X. Assume we have an isomorphism (called a **Weil structure**)

$$\phi : Fr^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$

Given $x \in X(\mathbb{F}_q) = \{x \in X \mid \operatorname{Fr}(x) = x\}$, we have

$$\mathcal{F}_x \overset{\phi^{-1}}{\xrightarrow{\sim}} (\operatorname{Fr}^* \mathcal{F})_x = \mathcal{F}_{\operatorname{Fr}(x)} = \mathcal{F}_x$$

hence we obtain a linear operator $H^i_x(\mathcal{F}) \to H^i_x(\mathcal{F})$. We define

$$\chi_{\mathcal{F},\phi}(x) = \sum_i (-1)^i \operatorname{Tr}(\phi, H^i_x(\mathcal{F})) \in E$$

Hence $\mathcal{F}, \phi$ leads to $\chi_{\mathcal{F},\phi} : X(\mathbb{F}_q) \to E$. If we iterate we get

$$\chi_{\mathcal{F},\phi^n} : X(\mathbb{F}_{q^n}) \to E.$$

Forming characteristic functions is compatible with f^*, $f_!$, $\otimes$.

Slogan: Natural functions on $X(\mathbb{F}_q)$ come from sheaves.

Note that characters are very natural functions, and so we can hope that they might come from nice sheaves.

It is very important above that the ϕ involves a choice. But if $\mathcal{F}$ is irreducible then ϕ is well-defined up to a scalar.

1.3. **Character sheaves.** If $G = T$ is a torus. A local system is **Kummer** if $\mathcal{L}^{\otimes n} \cong E$ a constant sheaf for some $n > 0$ with $(n, q) = 1$.

Construction: Consider $n : T \to T : x \to x^n$ with n as above. The claim is that

$$n_* E = \text{direct sum of Kummer local systems.}$$

Classification: Kummer local systems are in bijection with

$$X(T) \otimes \mathbb{Z}_{(p)}/X(T).$$

Theorem 1.1. (1) *Frobenius invariant Kummer local systems are in bijection with $Hom(T(\mathbb{F}_q), E^*)$.*
(2) *$\mathcal{L}$ a Frobenius invariant Kummer local system. Then $\chi_{\mathcal{L},\phi}$ is a character of $T(\mathbb{F}_q)$. We choose ϕ so that $\phi_e = \mathrm{Id}$.*

Hence we have a perfect theory for the torus!

The only problem is that we are not very interested in the calculation of the character table of T. (It is too easy!)

We now move on to semi-simple and reductive groups.

1.4. **Principal series representations of $G(\mathbb{F}_q)$.** Choose $B(\mathbb{F}_q)$ and consider $B(\mathbb{F}_q) \to T(\mathbb{F}_q)$. Now choose a character λ of $T(\mathbb{F}_q)$ over E.

$\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\lambda)$. This will give a character of $G(\mathbb{F}_q)$ which is irreducible for λ "generic enough". It is not irreducible in general. These are the principal series representations of $G(\mathbb{F}_q)$. They form a large part of the representations, but not all.

We have

$$\deg \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\lambda) = [G(\mathbb{F}_q) : B(\mathbb{F}_q)] = |\mathcal{B}(\mathbb{F}_q)|$$

where $\mathcal{B}$ is the flag variety of G. That is the variety parametrising all Borel subgroups of G.

We now give a geometric construction of the principal series representations. Consider $\widetilde{G} = \{(x, B) \in G \times \mathcal{B} \mid x \in B\}$. We have obvious maps

$$\begin{tikzcd} \widetilde{G} \arrow[r, "h"] \arrow[d, "\pi"] & T = B/[B,B] \\ G & \end{tikzcd}$$

To λ we can associate $\mathcal{L}$ a Kummer local system on T.

Definition 1.2. $K_{\mathcal{L}} = \pi_! h^* \mathcal{L}$.

(Note that π is proper, so it doesn't matter if we write $\pi_!$ or π_*.)

Lemma 1.3. $\chi_{K_{\mathcal{L}},\phi} = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\lambda)$.

Proof. Compare computation of $\chi_{K_{\mathcal{L}},\phi}$ and the usual formula for the induced character. ☐

Note that this is still quite far from all representations of G.

In the above we used a particular isomorphism between $K_{\mathcal{L}}$ and its Frobenius pull-back.

Question 1.4. Are there any other Weil structures on $K_{\mathcal{L}}$?

Yes! This is apparent from the following formula.

Lemma 1.5. $\dim \operatorname{Hom}(K_{\mathcal{L}}, K_{\mathcal{L}'}) = |\{w \in W \mid w(\mathcal{L}) = \mathcal{L}'\}|$.

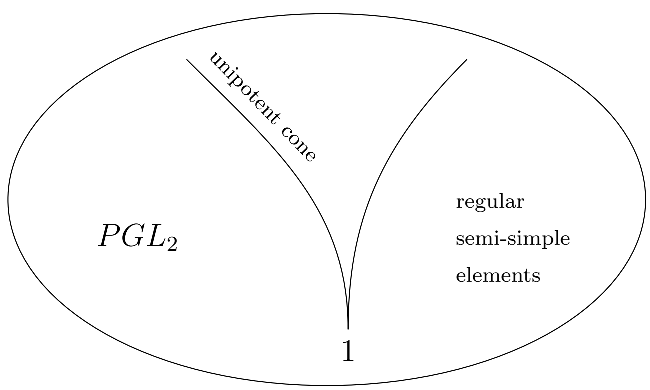
Proof. This follows from the fact that the morphism π is "small". $K_{\mathcal{L}}$ is an intersection cohomology extension of some local system on G^{reg}. Hence this reduces to a calculation on G^{reg}. □

Take $\mathcal{L}$ such that $\mathrm{Fr}^* \mathcal{L} \cong w(\mathcal{L})$. Assume that $\mathcal{L}$ is generic (i.e. $St_W(\mathcal{L}) = \{e\}$). Hence $K_{\mathcal{L}}$ is irreducible, and hence

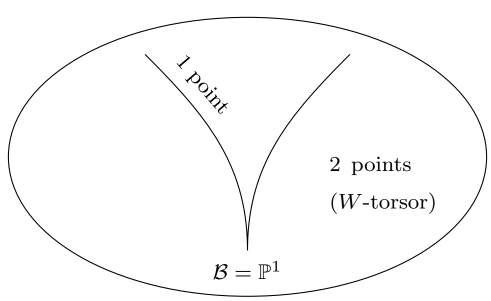
$$\phi : K_{\mathcal{L}} \xrightarrow{\sim} \mathrm{Fr}^* K_{\mathcal{L}} = K_{\mathrm{Fr}^* \mathcal{L}} = K_{w(\mathcal{L})}.$$

If we compute $\chi_{K_{\mathcal{L}},\phi}$ we get an irreducible character of $G(\mathbb{F}_q)$. This character is highly non-trivial. (For example it is difficult a priori to say what its degree is.)

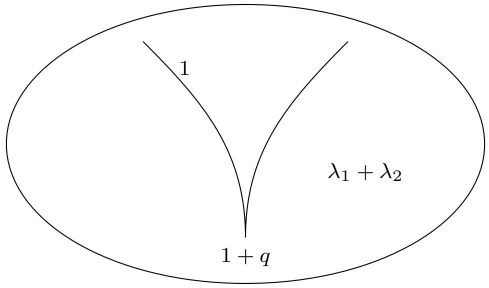
Example 1.6. We can picture $G = PGL_2$ as follows:



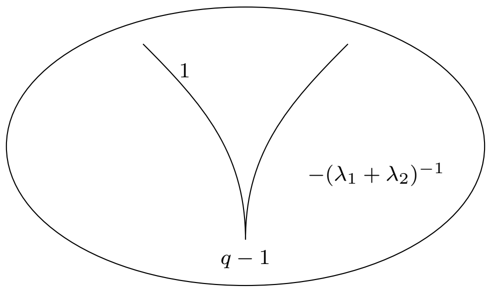
We can try to see what $K_{\mathcal{L}}$ looks like in this picture for various choices of $\mathcal{L}$. First note that the fibres of the map $\widetilde{G} \to G$ have following form:



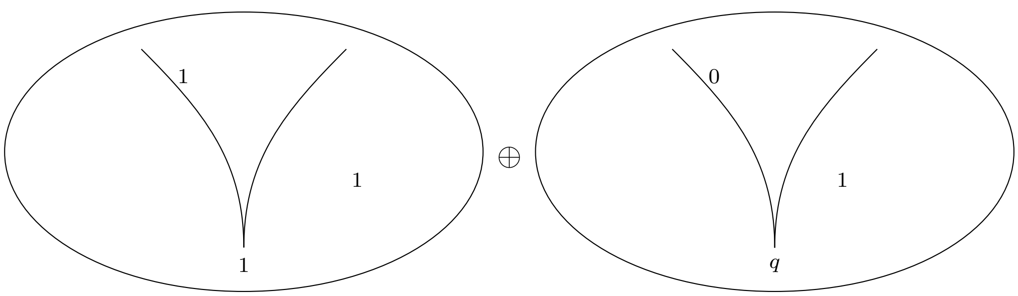
If $\mathrm{Fr}^* \mathcal{L} = \mathcal{L}$ we get the following characteristic function (λ_1 and λ_2 are roots of unity):



If we consider $\mathrm{Fr}^* \mathcal{L} = \mathcal{L}^{-1}$ (with Weil structure as explained above), we get:

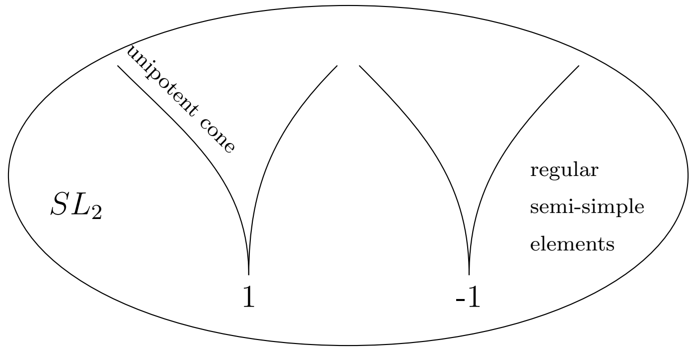


If we take $\mathcal{L}$ non generic, that is $\mathcal{L} \cong \mathcal{L}^{-1}$, then $K_{\mathcal{L}}$ is reducible and we have a direct sum decomposition:

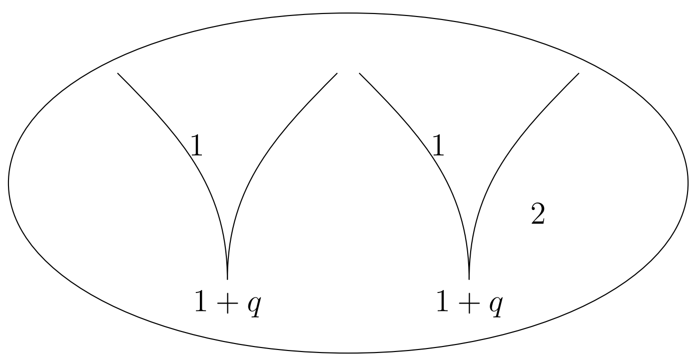


One can show that **all** irreducible characters of G are constructed in this way! The same is true for GL_n, PGL_n, U_n, etc. But it is not true for any other group, starting from SL_2!

Example 1.7. In this case the picture is different. There are "two unipotent cones"; the second part is obtained by translating the unipotent cone by -1:

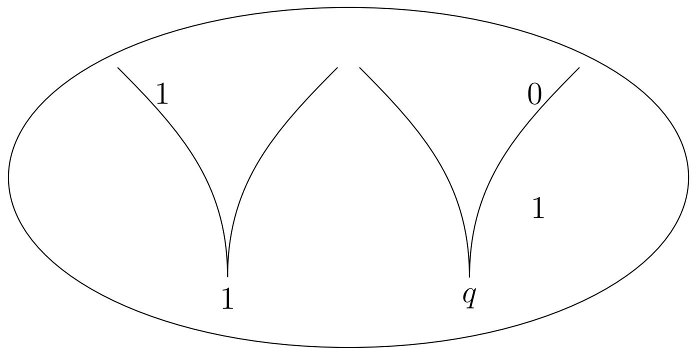


If $\mathcal{L}$ is generic (that is $\mathcal{L} \ncong \mathcal{L}^{-1}$) we get the same behavior as before:

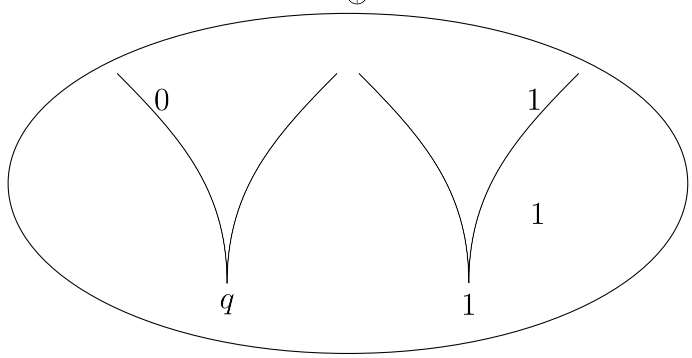


If $\mathcal{L} \cong E$ we get a similar picture to before. ($K_{\mathcal{L}}$ decomposes into two pieces: trivial character plus Steinberg character.)

Something unexpected happens when we take $\mathcal{L} \cong \mathcal{L}^{-1} \neq E$. In this case $K_{\mathcal{L}}$ decomposes as follows:

$\alpha =$ 

$\oplus$

$\beta =$ 

It turns out that neither α nor β are irreducible characters!

But there exist characters A, B, C and D such that

$$\alpha = \frac{A+B+C+D}{2}, \quad \beta = \frac{A+B-C-D}{2}.$$

Let use define

$$\gamma = \frac{A-B+C-D}{D}, \quad \delta = \frac{A-B-C+D}{2}.$$

It is easy to guess that γ and δ are characteristic functions of certain sheaves on G.

Let U denote the unipotent cone in SL_2. We have

$$\pi_1(U-e) = \mathbb{Z}/2\mathbb{Z}$$

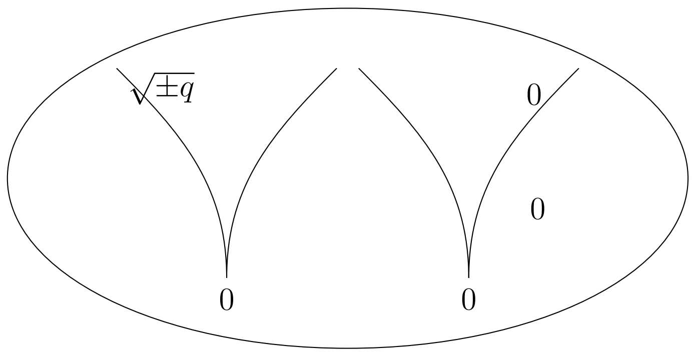
with local system on $U-e$. Hence we have a two-fold covering

$$\widetilde{U-e} \to U-e.$$

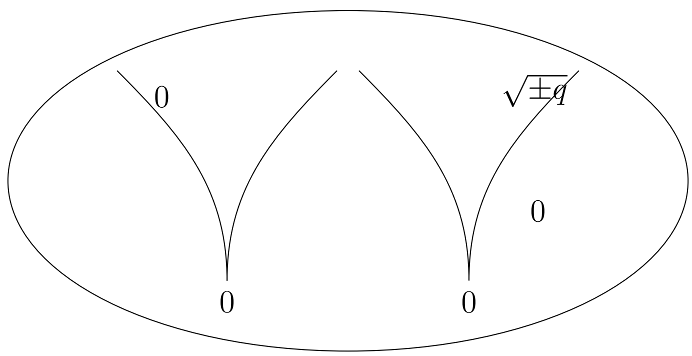
This gives the desired local systems on $U-e$ and the correct functions.

We get γ and δ as follows:

$\gamma =$



$\delta =$



2. Lecture 2

2.1. **Reminder.** G denotes a reductive algebraic group like last time. The goal is to construct a class $\hat{G}$ of irreducible perverse sheaves on G such that their characteristic functions give characters of $G(\mathbb{F}_q)$.

We have the diagram from last time

$$\begin{tikzcd} \widetilde{G} \arrow[r, "h"] \arrow[d, "\pi"] & T \\ G & \end{tikzcd}$$

It is known that $\pi_! h^* \mathcal{L} =: K_{\mathcal{L}}$ is direct sum of irreducible perverse sheaves.

Let us recall the situation that we saw for SL_2:

deg	#	algebraically	geometrically
1	1	trivial	K_E decomposes
q	1	Steinberg	
$q+1$	$\frac{q-3}{2}$	principal series	$K_{\mathcal{L}}, \quad \mathrm{Fr}^*\mathcal{L} = \mathcal{L}$
$q-1$	$\frac{q-1}{2}$	cuspidal	$K_{\mathcal{L}}, \quad Fr^*\mathcal{L} = \mathcal{L}^{-1}$
$\frac{q+1}{2}$	2	oscillator	need to add sheaves here
$\frac{q-1}{2}$	2	oscillator	

Last time we saw the importance of the Fourier transform matrix. (Note also that we have no hope to get the $\frac{1}{2}q$ as the character of sheaf.)

2.2. **General definition of $\widehat{G}$.** Recall that $B\setminus G/B$ is in bijection with W. Let $G_w := BwB$. The structure theory of reductive algebraic groups gives an isomorphism

$$G_w = N_w T N \cong T \times \text{affine space}.$$

We can consider the projection

$$pr : G_w \to T$$

and consider the pullback $pr^*\mathcal{L}$, where $\mathcal{L}$ is a Kummer local system on T. We consider then the intersecion cohomology extension of $pr^*\mathcal{L}$. We call this $A_{\mathcal{L},w}$.

Remark 2.1. Assume that $w\mathcal{L} \cong \mathcal{L}$. Then $A_{\mathcal{L},w}$ is B-equivariant with respect to B-action on G by conjugation: $b \cdot g = bgb^{-1}$.

2.3. **Equivariantisation.** If X a G-variety, and $\mathcal{F}$ a perverse sheaf on X which is B-equivariant. We can construct a new sheaf $\Gamma_B^G(\mathcal{F})$ on X which is G-equivariant.

Consider the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\pi"] \arrow[d, "pr"] & G \times_B X \arrow[r, "a"] & X \\ X & & \end{tikzcd}$$

$pr^*\mathcal{F}$ is B-equivariant on $G \times X$, hence comes from a sheaf $\widetilde{F}$ on $G \times_B X$.

Definition 2.2. $\Gamma_B^G(\mathcal{F}) = a_!\widetilde{\mathcal{F}} = a_*\widetilde{\mathcal{F}}$.

This definition yields a G-equivariant sheaf on X. If $\mathcal{F}$ is irreducible, then $\Gamma_B^G(\mathcal{F})$ is a direct sum of shifted simple perverse sheaves on X.

2.4. **General definition of character sheaves.**

Definition 2.3. $\hat{G}$ is defined to be all perverse sheaves that appear in various $\Gamma_B^G(A_{\mathcal{L},w})$ with $w(\mathcal{L}) = \mathcal{L}$.

Example 2.4.
(1) If $w = e$ then $\Gamma_B^G(A_{\mathcal{L},e}) = K_{\mathcal{L}}$.
(2) If $G = SL_2$, $w = s$ and $\mathcal{L} \cong \mathcal{L}^{-1} = s(\mathcal{L}) \neq E$ then $\Gamma_B^G(A_{\mathcal{L},s})$ gives the two "missing" perverse sheaves from last time. (This is more difficult to check geometrically, but can be done.) The interesting thing here is that things disappear when one does the "equivariantisation".

It turns out that this set $\hat{G}$ does the job (i.e. all characters can be obtained). But this relies on very difficult theorems of Lusztig.

The problem for the rest of the lectures will be to classify the objects in $\hat{G}$.

There is a geometric description of the set $\hat{G}$ due mostly to Lusztig. We will discuss a different classification, using various pieces of data attached to the Weyl group.

2.5. **Reduction.** Set

$$\hat{G}(\mathcal{L}) = \text{all perverse constituents of } \Gamma_B^G(A_{\mathcal{L},w}).$$

Theorem 2.5.
(1) *If $\mathcal{L} = w(\mathcal{L}')$ then*
$$\hat{G}(\mathcal{L}) = \hat{G}(\mathcal{L}').$$
(2) *If $\mathcal{L} \notin W(\mathcal{L}')$ then $\hat{G}(\mathcal{L})$ is disjoint from $\hat{G}(\mathcal{L}')$.*

Main Problem: Find a classification of $\hat{G}(E) = \hat{G}_{unip.}$.

The above reduction is reminiscent of the classification of conjugacy classes in algebraic groups. The above theorem can be seen as separating the semi-simple and unipotent parts. Note also that the classification of unipotent classes is the hard part of the classification.

In our case $A_{\mathcal{L},w}$ reduces to $A_{E,w} = A_w$. Note that each A_w is $B \times B$-equivariant. Instead we can think about B-equivariant sheaves on G/B. We have to understand the constituents of $\Gamma_B^G(A_w)$.

Observation: We can convolve the various A_w. The multiplication in G factors as

$$G \times G \xrightarrow{\pi} G \times_B G \xrightarrow{m} G$$

with m proper. If we have A_w and $A_{w'}$ we can form $A_w \boxtimes A'_w$ and because of equivariance we can find $\widetilde{A_w \boxtimes A'_w}$ on $G \times_B G$ whose pullback via π is isomorphic to $A_w \boxtimes A'_w$.

Definition 2.6. The convolution of A_w and $A_{w'}$ is defined as

$$A_w * A'_w = m_!(\widetilde{A_w \boxtimes A'_w}) = m_*(\widetilde{A_w \boxtimes A'_w}).$$

Note that the decomposition theorem applies for $A_w * A'_w$. Hence this complex is a direct sum of shifts of $A_{w''}$'s.

This gives a categorical Hecke algebra $\mathcal{H}_W$ defined as the category of all semi-simple $B \times B$-equivariant complexes of sheaves on G with convolution operation above. This is a tensor category. The associativity constraint will be very important in what follows.

2.6. **The Hecke algebra.** It is well-known that

$$K(\mathcal{H}_W) = H_W$$

the classical Hecke algebra associated to the Weyl group W.

H_W is the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{T_x \mid x \in W\}$ and multiplication determined by

$$T_xT_y = T_{xy} \quad \text{if } \ell(xy) = \ell(x) + \ell(y),$$
$$(T_s - v)(T_s - v^{-1}) \quad \text{for } s \in S.$$

We recall the Kazhdan-Lusztig basis of H_W. The Kazhdan-Lusztig involution of H_W sends h to $\overline{h}$ and is determined by $v \mapsto v^{-1}$ and $\overline{(T_s + v^{-1})} = T_s + v^{-1}$.

The Kazhdan-Lusztig basis is characterised by

$$T_x \in T_x + \sum_y v^{-1}\mathbb{Z}[v^{-1}]T_v$$
$$\overline{C_x} = C_x$$

For example $C_e = T_e$ and $C_s = T_s + v^{-1}$.

Then a basic theorem is that the map sending the class of $A_w[i]$ to $v^{-i}C_w$ gives an isomorphism

$$K(\mathcal{H}_W) \to H_W.$$

We now consider the structure constants of H_W. Let us write

$$C_xC_y = \sum_y h_{xyz}C_z$$

then $h_{xyz} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$.

2.7. **Lusztig's assymptotic Hecke algebra J.** We start with Lusztig's a-function:

$$a : W \to \mathbb{Z}_{\geq 0}$$

which is defined as follows. Let us write

$$h_{xyz} = \gamma_{xyz} v^{a(z)} + \text{lower powers of } v.$$

Then $a(z)$ and γ_{xyz} is well-defined if we assume that γ_{xyz} is non-zero for some x, y.

We define

$$J_W = \text{free } \mathbb{Z}\text{-module with basis } t_x,\ x \in W$$

and multiplication

$$t_x t_y = \sum_z \gamma_{xyz} t_z.$$

Of course $\gamma_{xyz} \in \mathbb{Z}_{\geq 0}$.

Theorem 2.7. *J_W is associative, with unit $1 = \sum_{d \in \mathcal{D}} t_d$.*

Associativity is easy, the existence of a unit is very difficult!

Example 2.8. $J_W = \mathbb{Z}t_e \oplus \mathbb{Z}t_s$. Then $C_e = 1$ and $C_s^2 = (v + v^{-1})C_s$. Hence $a(e) = 0$ and $a(s) = 1$. We get $t_e^2 = t_e$, $t_s^2 = t_s$ and $t_e t_s = t_s t_e = 0$ and $1 = t_e + t_s$. Note that the decomposition $J_W = \mathbb{Z}t_e \oplus \mathbb{Z}t_s$ is a decomposition of algebras.

3. Lecture 3

3.1. **Reminder.** We begin with a quick reminder of what happened last time. H_W is the Hecke algebra. Recall that H_W is the Grothendieck ring of $D_B(G/B)$. (We ignore technicalities N vs B etc.)

The simple perverse sheaves on G/B correspond to C_x. We have the structure constants h_{xyz} of the Kazhdan-Lusztig basis. We defined

$$a : W \to \mathbb{Z}_{\geq 0}$$

by the condition

$$h_{xyz} = \gamma_{xyz} v^{a(z)} + \text{lower terms}.$$

We have J_W with basis t_x and multiplication

$$t_x t_y = \sum \gamma_{xyz} t_z.$$

Given any subset $A \subset W$ we denote by $J_A \subset J_W$ the abelian group

$$J_A := \sum_{x \in A} \mathbb{Z} t_x$$

Definition 3.1. $W = \sum C$ is a decomposition into **two-sided cells** if

$$J_W = \oplus_C J_C$$

is a decomposition into algebras, and it is the finest decomposition with this property.

One can see in examples that this partition is highly non-trivial. (Note that this is equivalent to the standard definition of two-sided cells.)

Question: Compute J_C for any C.

This really means to compute the structure constants. (That is, we need to understand J_C as a based ring, not just as a ring.)

The classification of cells is well-known.

Example 3.2.
(1) A_1, $W = \{e\} \sqcup \{s\}$.
(2) A_2, $W = \{e\} \sqcup \{w_0\} \sqcup$ rest .

We will see that

$$J_C = K(\mathcal{I}_C)$$

where $\mathcal{I}_C$ is a tensor category. The objects of $\mathcal{I}_S$ are direct sums of A_w (intersection cohomology extensions) where $w \in C$. The tensor product is given by

$$X \otimes Y := {}^pH^{a(C)}(X * Y) \text{ modulo lower terms.}$$

Note the important fact that a is constant on each two-sided cell.

One can check that $K(\mathcal{I}_S)$ gives exactly the assymptotic ring J_C.

This is a tensor category with very nice properties.

(1) It has an associativity constraint. This is inherited from convolution.
(2) More subtle: $\mathcal{I}_S$ has a unit object. (This follows directly from the fact that J_C has a unit object.) It is important to remember that the unit object is not indecomposable.
(3) It is rigid. It has a duality with very nice properties.
(4) This category is semi-simple.

3.2. Another tensor category.

Let Γ be a finite group and let Y be a finite Γ-set. In this case we can construct a category

$$\mathrm{Coh}_\Gamma(Y \times Y)$$

This category has a tensor structure via convolution. Namely, given F_1 and F_2 their convolution is defined by

$$F_1 * F_2 := p_{13*}(p_{12}^* F_1 \otimes p_{13}^* F_2)$$

where $p_{ij} : Y \times Y \times Y \to Y \times Y$ denotes the projection to the i and j components.

Conjecture 3.3. *(Lusztig) For any C there exists $\Gamma(C)$ and $Y(C)$ together with a monoidal equivalence of categories*

$$\mathcal{I}_C \xrightarrow{\sim} \mathrm{Coh}_{\Gamma(C)}(Y(C) \times Y(C))$$

Theorem 3.4. *(Bezrukavnikov-Finkelberg-Ostrik) Assume that C is not exceptional, then the conjecture above is true.*

Remark 3.5. There are just 3 exceptional cells, and they occur in types E_7 and E_8. Lusztig's conjecture fails for these cells. But everything is under control. In particular, one can say precisely how the category on the left looks like. (The K_0 statement still holds.)

Remark 3.6. The combinatorics is still unsatisfactory. The map from the Weyl group elements to simple objects on the right hand side is not explicit.

Example 3.7.
(1) If $Y = pt$ in this case $\mathrm{Coh}_\Gamma(\mathrm{pt}) = \mathrm{Rep}(\Gamma)$.
(2) If $Y = \Gamma/\{e\}$ then $\mathrm{Coh}_\Gamma(Y \times Y) = \mathrm{Vect}_\Gamma$, vector spaces with Γ-action.
(3) If $Y = \Gamma/H$ then $\mathrm{Coh}_\Gamma(Y \times Y) = \mathrm{Coh}_H(\Gamma/H)$.
(4) If Y decomposes into orbits, then the unit object decomposes.

If $\Gamma = \{e\}$ then

$$\mathrm{Coh}_\Gamma(Y \times Y) = \mathrm{Vect}_{Y\times Y} = \text{matrices over vector spaces.}$$

One can also think about this as

$$\mathrm{Fun}(\mathrm{Vect}_Y, \mathrm{Vect}_Y) = \mathrm{Fun}(Y) - \text{bimodules.}$$

Lusztig specifies these groups very explicitly. We have

$$\Gamma(C) = \begin{cases} e & \text{in type } A \\ (\mathbb{Z}/2)^\alpha & \text{in type } BCD \\ S_k & \text{for } 1 \le k \le 5 \text{ in type } EFG \end{cases}$$

3.3. **A brief aside.** I can't resist mentioning the following connection. If $k = \mathbb{C}$ then two-sided cells correspond to special nilpotent elements in G. Given a special nilpotent element e let $C(e)$ denote the corresponding two-sided cell. To e one may associate W_e a W-algebra. The center of W_e is the same as the center of the enveloping algebra.

Conjecture 3.8. *(Bezrukavnikov-Ostrik) We have a bijection*

$$\mathrm{irr}^{fd}_{\chi} W_e \xrightarrow{\sim} Y(C(e))$$

where χ denotes a regular central character.

We are very close to being able to prove this conjecture, and can do so in exceptional type.

3.4. **Back to character sheaves.**

Theorem 3.9. *(Lusztig) We have a bijection*

$$\hat{G}_{unip} \xrightarrow{\sim} \bigcup_C \mathcal{M}(\Gamma(C))$$

where

$$\mathcal{M}(\Gamma(C)) = \{(x \in \Gamma(C), \rho \in \mathrm{irr}(C_\Gamma(x))\}/\Gamma(C).$$

One can identify the above set with the irreducible objects in a category

$$\begin{aligned}\mathcal{M}(\Gamma(C)) &= \{(x \in \Gamma(C); \rho \in \mathrm{irr}(C_\Gamma(x))\}/\Gamma(C) = \\ &= \mathrm{irr}\,\mathrm{Coh}_{\Gamma(C)}(\Gamma(C)) = \mathrm{irr}(Z(\mathrm{Rep}(\Gamma)))\end{aligned}$$

where Z denotes the Drinfeld centre.

We now define the Drinfeld centre. Let $\mathcal{C}$ denote a tensor category. The category $Z(\mathcal{C})$ is the category with objects

$$(X, \phi) \text{ where} X \in \mathbb{C}, \phi : X \otimes ? \xrightarrow{\sim} ? \otimes X$$

where ϕ subject to a natural condition (two diagrams should commute).

Note that this construction is quite subtle. It is very hard to predict what will come out, even if one starts with a category that one knows very well!

Theorem 3.10. *(Special case of a Müger's Morita invariance of Drinfeld centre.) The category*

$$Z(\mathrm{Coh}_\Gamma(Y \times Y))$$

does not depend on Y (up to canonical equivalence).

Example 3.11. The theorem means that $Z(\mathrm{Rep}(\Gamma)) \cong Z(\mathrm{Vect}_\Gamma)$. This is not immediately obvious!

Corollary 3.12. *We have a bijection*

$$\hat{G}_{unip} \xrightarrow{\sim} \bigsqcup \mathrm{Irr}(Z(\mathcal{I}_C)).$$

This suggests the following:

Question 3.13. Is this coming from a tensor equivalence of some categories?

The answer to this question is positive over $\mathbb{C}$. It is very desirable to have a similar result of $\mathbb{F}_q$.

Let us think about the above theorem in the case when we have $H \subset \Gamma$. In this case

$$Z(Sh_H(\Gamma/H)).$$

(Here we want to think about constructible sheaves which doesn't make a shred of difference because we work on finite sets!)

The above result gives us an equivalence

$$Z(Sh_H(\Gamma/H)) \cong Sh_\Gamma(\Gamma).$$

Let us try to replace Γ and H by algebraic groups!

We might hope that something similar holds for $\Gamma = G$ and $H = B$. This situation has suddenly become extremely complicated! It is even difficult to state what one might hope to hold.

This is because if one takes the Drinfeld center of a triangulated tensor category there is no reason to expect that one again obtains a triangulated category. (This comes down to the non-canonicity of cones.)

This problem can be dealt with in two different ways.

(1) Ben-Zvi and Nadler. Use the theory of $(\infty, 1)$-categories. These solve the problem of non-canonicity of cones.
(2) The Bezrukavnikov-Finkelberg-Ostrik approach: use usual abelian categories. If we think about

$$Sh_B(G/B) = Sh_G(G/B \times G/B) = \text{Harish-Chandra bimodules.}$$

It is important that Harish-Chandra bimodules form a tensor category. One obtains the abelian category of character sheaves as the centre.