An example of higher representation theory

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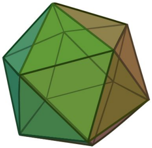

Geometric and categorical representation theory,
Mooloolaba, December 2015.

First steps in representation theory.

We owe the term *group(e)* to Galois (1832).

En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que

$$G = H + HS + HS' + \dots.$$

1. Écrite la veille de la mort de l'auteur. (Insérée en 1832 dans la *Revue encyclopédique*, numéro de septembre, page 568.) (J. LIOUVILLE.)

Et aussi il peut se diviser en groupes qui ont tous les mêmes substitutions, en sorte que

$$G = H + TH + T'H + \dots.$$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre*.

Il est aisé de voir que, quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.

Au contraire, quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations : l'une aura un groupe de M permutations, l'autre un de N permutations.

Lors donc qu'on aura épuisé sur le groupe d'une équation tout ce qu'il y a de décompositions propres possibles sur ce groupe, on arrivera à des groupes qu'on pourra transformer, mais dont les permutations seront toujours en même nombre.

Si ces groupes ont chacun un nombre premier de permutations, l'équation sera soluble par radicaux ; sinon, non.

$H \subset G$ is a subgroup

Letter to Auguste Chevalier in 1832

written on the eve of Galois' death

notion of a normal subgroup

notion of a simple group

notion of a soluble group

main theorem of Galois theory

Mathematicians were studying group theory for 60 years before they began studying *representations* of finite groups.

The first character table ever published. Here G is the alternating group on 4 letters, or equivalently the symmetries of the tetrahedron.

dnung 3 zwei inverse Classen (2) und (3) $= (2')$. Sei ρ eine prim

bische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	-1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Die Werthe von χ_0 sind zugleich die von $f = e$.

Frobenius, *Über Gruppencharaktere*, S'ber. Akad. Wiss. Berlin, **1896**.

Now $G = S_5$, the symmetric group on 5 letters of order 120:

[1013] Frobenius: Über Gruppencharaktere. 29

$h = 120$

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	$\chi^{(6)}$	h_α
χ_0	1	5	5	4	4	6	1	1
χ_1	1	1	1	0	0	−2	1	15
χ_2	1	1	−1	2	−2	0	−1	10
χ_3	1	−1	−1	1	1	0	1	20
χ_4	1	−1	1	0	0	0	−1	30
χ_5	1	0	0	−1	−1	1	1	24
χ_6	1	1	−1	−1	1	0	−1	20

Conway, Curtis, Norton, Parker, Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray.* Oxford University Press, 1985.

$\mathrm{M} = \mathrm{F}_1$

However around 1900 other mathematicians took some convincing at to the utility of representation theory...

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

– Burnside, *Theory of groups of finite order*, 1897.
(One year after Frobenius' definition of the character.)

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

– Burnside, *Theory of groups of finite order*, Second edition, 1911. (15 years after Frobenius' definition of the character table.)

Representation theory if largely useful because often ...

...out of group actions one can produce *linear* actions.

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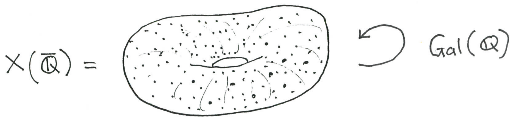

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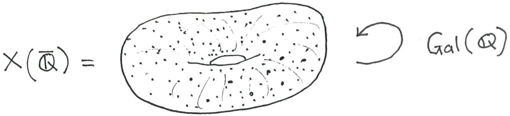

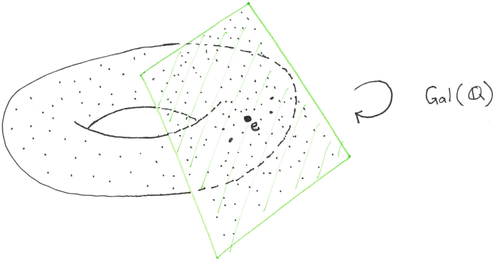

Categories can have symmetry too!

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What "linear" means is more subtle.

Usually it means to study categories in which one has operations like direct sums, limits and colimits, kernels . . .

(Using these operations one can try to "categorify linear algebra" by taking sums, cones etc.

If we are lucky Ben Elias will have more to say about this.)

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Example: Given a finite group G its "$\mathbb{C}$-linear shadow" is the character table (essentially by semi-simplicity). However the subtle homological algebra of kG if kG is not semi-simple means that $\operatorname{Rep} kG$ or $D^b(\operatorname{Rep} kG)$ is better thought of as its k-linear shadow.

First steps in higher representation theory.

Monoids, groups and algebras are categorified by forms of tensor (=monoidal) categories.

Fix an additive tensor category $\mathcal{A}$.

This means we have a bifunctor of additive categories:

$$(M_1, M_2) \mapsto M_1 \otimes M_2$$

together with a unit $\mathbb{1}$, associator, ...

Examples: Vect_k, $\mathrm{Rep}\, G$, G-graded vector spaces, $\mathrm{Fun}(M, M)$ (endofunctors of an additive category), ...

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What exactly this means can take a little getting used to.

As in classical representation theory it is often more useful to think about an "action" of $\mathcal{A}$ on $\mathcal{M}$.

① $\begin{array}{cc} \mathcal{A} & \mathcal{M} \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \end{array}$
$(A, M) \longmapsto A \bullet M$ "objects act on objects"

(often visible on Grothendieck group)

② $\left(\begin{tikzcd} A' \\ A \arrow[u, "f"] \end{tikzcd}, M \right) \longmapsto \begin{tikzcd} A' \bullet M \\ A \bullet M \arrow[u, "f \bullet M"'] \end{tikzcd}$ "morphisms act on objects"

③ $\left(A, \begin{tikzcd} M' \\ M \arrow[u, "g"'] \end{tikzcd} \right) \longmapsto \begin{tikzcd} A \bullet M' \\ A \bullet M \arrow[u, "A \bullet g"'] \end{tikzcd}$ "objects act on morphisms"

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An $\mathcal{A}$-module is a recipe $M \mapsto \mathrm{nat} \cdot M$ and a host of maps

$$\mathsf{Hom}_{\mathcal{A}}(\mathrm{nat}^{\otimes m}, \mathrm{nat}^{\otimes n}) \to \mathsf{Hom}_{\mathcal{M}}(\mathrm{nat}^{\otimes m} \cdot M, \mathrm{nat}^{\otimes n} \cdot M)$$

satisfying an even larger host of identities which I will let you contemplate.

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Theorem

(Classification of representations of $\mathrm{Rep}\, SU_2$*.) These are all.*

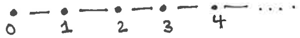
Let $\{L_i\}$ denote the simple objects in $\mathcal{M}$.

Draw an edge $L_i \longrightarrow L_j$ if $L_i \overset{\oplus}{\subset} \mathrm{nat} \cdot L_j$.

<u>Exercise:</u> nat self-dual $\Rightarrow \left(L_i \longrightarrow L_j \iff L_j \longrightarrow L_i \right)$.

$\mathrm{Vect}_{\mathbb{C}}$

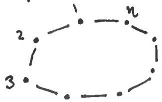
Rep SU_2



Rep BI

Rep S^1 $\left(\mathbb{C}[x,x^{-1}]^{S_2} \subset \mathbb{C}[x,x^{-1}] \right)$

Rep μ_n



Remarkably, the action of $\operatorname{Rep} SU_2$ on the Grothendieck group of $\mathcal{M}$ already determines the structure of $\mathcal{M}$ as an $\operatorname{Rep} SU_2$-module!

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This is an example of “rigidity” in higher representation theory.

An example of higher representation theory
(joint with Simon Riche).

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It is a little like contemplating homotopy groups of spheres: amazing mathematics has emerged from consideration of these problems, although the complete picture is still a long way off.

For the rest of the talk fix a field k and a connected reductive group G like GL_n (where we will state a theorem later) of Sp_4 (where we can draw pictures).

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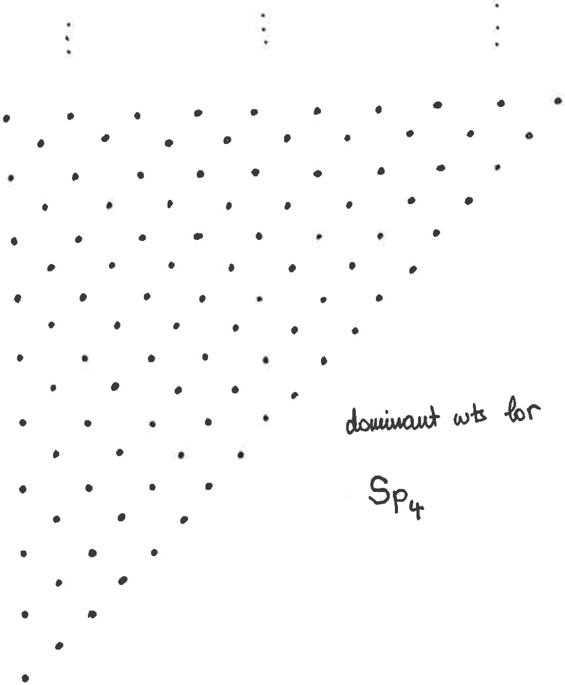
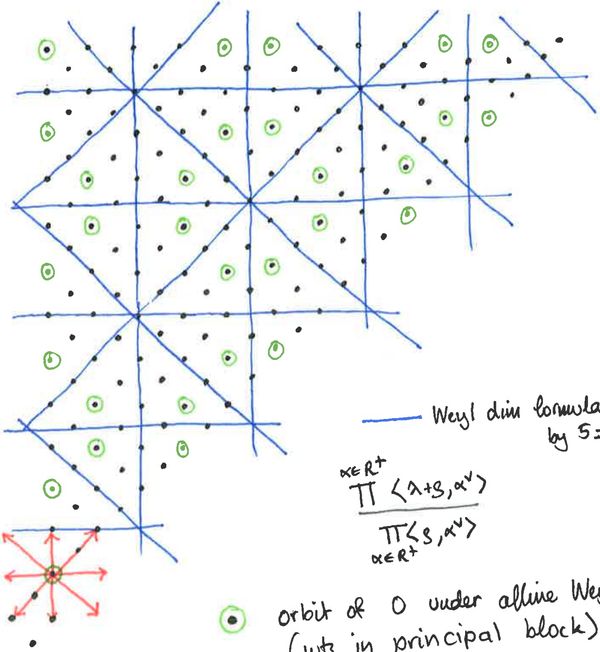
If k is of characteristic 0 then $\mathrm{Rep}\, G$ looks "just like representations of a compact Lie group". In positive characteristic one still has a classification of simple modules via highest weight, character theory etc. However the simple modules are usually much smaller than in characteristic zero.



dominant wts for

Sp_4

$p=5$



Weyl dim formula divisible by $5=p$.

$$\frac{\prod_{\alpha\in R^+} \langle \lambda+\rho, \alpha^\vee\rangle}{\prod_{\alpha\in R^+} \langle \rho, \alpha^\vee\rangle}$$

orbit of 0 under affine Weyl group (wts in principal block).

$$\mathrm{Rep}_0 \overset{\oplus}{\subset} \mathrm{Rep}\, G \text{ the principal block.}$$

$$\mathrm{Rep}_0 \subset \mathrm{Rep}\, G \text{ depends on } p!$$

$$\mathrm{Rep}_0 \overset{\oplus}{\subset} \mathrm{Rep}\, G \text{ the principal block.}$$

On Rep_0 one has the action of *wall-crossing functors*:

"matrix coefficients of tensoring with objects in $\mathrm{Rep}\, G$"

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Let W denote the affine Weyl group and $S = \{s_0, \dots, s_n\}$ its simple reflections. For each $s \in S$ one has a wall-crossing functor Ξ_s. These generate the category of translation functors.

$$\langle \Xi_{s_0}, \Xi_{s_1}, \dots, \Xi_{s_n} \rangle \circlearrowright \mathrm{Rep}_0 .$$

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Easy: On Grothendieck groups one has canonically:

$$(\langle \Xi_{s_0}, \Xi_{s_1}, \dots, \Xi_{x_n} \rangle \circlearrowright [\mathrm{Rep}_0]) \cong (\mathbb{Z}W \circlearrowright \mathbb{Z}W \otimes_{\mathbb{Z}W_f} \mathrm{sgn})$$

"Rep_0 categorifies the anti-spherical module."

Main conjecture: This action of wall-crossing functors can be upgraded to an action of the Hecke category.

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The *Hecke category* is a fundamental monoidal category in representation theory. It categorifies the Hecke algebra and has several incarnations:

$D^b(B\backslash G/B)$, parity sheaves, Soergel bimodules, moment graph sheaves (Fiebig), mixed modular category (Achar-Riche), ...

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Following earlier work of Soergel and insistence from Rouquier, it has recently been presented by generators and relations by Libedinsky, Elias-Khovanov, Elias, Elias-W.

Hecke category for $W = \tilde{A}_2$

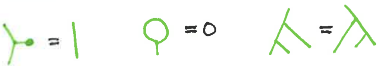
Generators: (objects) words in generators, thought of as coloured sequences:

Generators: (morphisms) isotopy classes of (linear combinations of) finite planar graphs; generated by

(for all colours)

(for all pairs of colours)

Relations:



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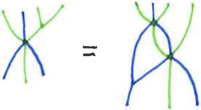

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Consequences of the conjecture...

Recall that

$$\langle \Xi_{s_0}, \Xi_{s_1}, \dots, \Xi_{x_n} \rangle \circlearrowright \mathrm{Rep}_0$$

categorifies the "anti-spherical module"

$$\mathbb{Z}W \circlearrowright \mathbb{Z}W \otimes_{\mathbb{Z}W_f} \mathrm{sgn} = \mathbb{Z}W / \mathbb{Z}W \langle (1+s) | s \text{ finite simple reflection} \rangle$$

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Using the Hecke category $\mathcal{H}$ one can also categorify the anti-spherical module in an "obvious" way. This yields an $\mathcal{H}$-module

$$\mathcal{H} \circlearrowright \mathcal{AS} := \mathcal{H} / \langle B_x \mid x \in W^f \rangle$$

(where $W^f := \{w \in W \mid ws > w \text{ for finite simple reflections } s\}$).

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This may be seen as an instance of higher representation theory. The mere existence of an action forces an equivalence. In the proof an important role is played by the "easy" isomorphism on Grothendieck groups considered above.

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This gives a strong form of the "independence of p" of Andersen-Jantzen-Soergel, and answers a question of Wolfgang Soergel from the 1990s.

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$$\mathsf{Rep}_0 \cong \mathcal{AS}.$$

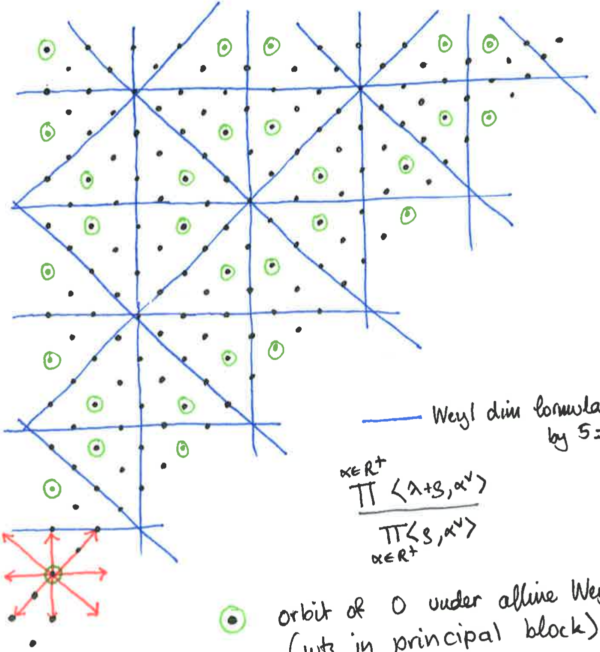
In particular:

- Rep_0 admits a grading (because $\mathcal{AS}$ does).
- Rep_0 admits a graded integral form over $\mathbb{Z}' := \mathbb{Z}[1/h!]$.
- Rep_0 can be described by generators and relations.

This gives a strong form of the "independence of p" of Andersen-Jantzen-Soergel, and answers a question of Wolfgang Soergel from the 1990s.

(The statement should be true of $\mathbb{Z}$. Achar-Riche have very related results. There is probably a $\mathbb{Z}^{\infty}$-grading coming from $V \mapsto V^{\mathrm{Fr}} \otimes \mathrm{St}$.)

$p=5$



Weyl dim formula divisible by $5=p$.

$$\frac{\prod_{\alpha\in R^+}\langle\lambda+\rho,\alpha^\vee\rangle}{\prod_{\alpha\in R^+}\langle\rho,\alpha^\vee\rangle}$$

orbit of 0 under affine Weyl group (wts in principal block).

A major motivation for this work was trying to get character formulas in terms of the Hecke category.

When taken over a field of characteristic zero the Hecke category is the home of the *Kazhdan-Lusztig basis*, and *Kazhdan-Lusztig polynomials*.

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Unfortunately, the p-canonical basis is far from simple. However these results and conjectures tell us precisely where the difficulty lies. Achar-Riche and Rider are close to showing our tilting conjectures for any G and $p > h$.

An example of this philosophy: Let ${}^{\mathrm{f}}W$ denote minimal representatives for $W_f \backslash W$.

Then there exist finite subsets $X_J, X_L, X_{A,p} \subset {}^{\mathrm{f}}W$ such that:

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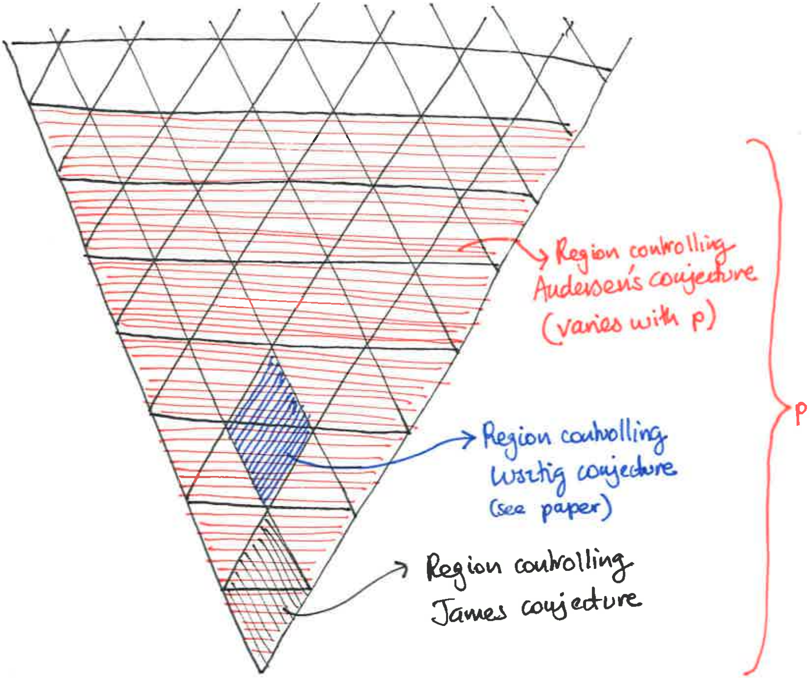
Actually, point (2) is still conjectural. Need tilting character formulas for GL_n for $p \leqslant n$. Should follow from work in progress of Elias-Losev.

Point (1) may be compared to a result of Fiebig giving necessary conditions for Lusztig's conjecture in terms of the *spherical* module.



Region controlling
Andersen's conjecture
(varies with p)
Region controlling
Wittig conjecture
(see paper)
Region controlling
James conjecture
p

Thanks!

Slides:

people.mpim-bonn.mpg.de/geordie/Mooloolaba.pdf

Paper (all 135 pages!):

people.mpim-bonn.mpg.de/geordie/tilting-total.pdf