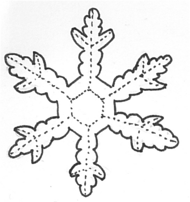
Reflection groups, Hecke algebras and Soergel bimodules

Geordie Williamson

Brandeis-Harvard-MIT-Northeastern
Joint mathematics colloquium
MIT, February 14, 2013.

Reflections usually provide our first encounter with symmetry.

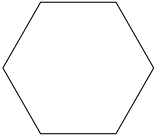
Already as children we can identify an axis of symmetry and are drawn to objects with many reflexive symmetries.



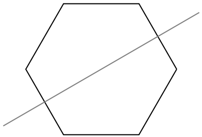
In the 1930's H. S. M. Coxeter developed a powerful tool for understanding symmetry groups generated by reflections, or *reflection groups*.

In the 1930's H. S. M. Coxeter developed a powerful tool for understanding symmetry groups generated by reflections, or *reflection groups*.

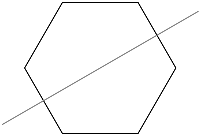
Let X be a geometric object whose symmetries are discrete and generated by reflections. For example X might be a polytope, a triangulation of a sphere, or a tesselation of hyperbolic space. Suppose that we want to understand the symmetry group of X.

$X =$ 

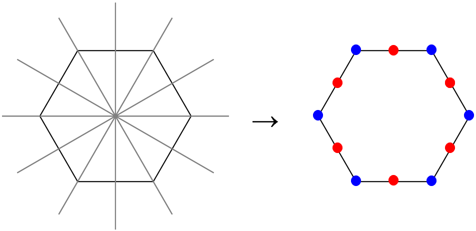
Assumption: Each axis of reflection divides X into two pieces.



Assumption: Each axis of reflection divides X into two pieces.



If we consider all axes at once then X is divided into pieces, called *alcoves*:



Second assumption: Each alcove is a simplex.

Second assumption: Each alcove is a simplex.

Coxeter noticed that each alcove provides a fundamental domain for the action of W on X. Moreover, after fixing a "fundamental" alcove $A \subset X$:

Second assumption: Each alcove is a simplex.

Coxeter noticed that each alcove provides a fundamental domain for the action of W on X. Moreover, after fixing a "fundamental" alcove $A \subset X$:

i) W, the symmetry group of X, is generated by the set

$$S = \{s \mid s \text{ is a reflection in a wall of } A\}$$

Second assumption: Each alcove is a simplex.

Coxeter noticed that each alcove provides a fundamental domain for the action of W on X. Moreover, after fixing a "fundamental" alcove $A \subset X$:

i) W, the symmetry group of X, is generated by the set

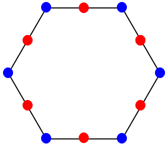
$$S = \{s \mid s \text{ is a reflection in a wall of } A\}$$

ii) W has a presentation

$$W = \langle S \mid s^2 = (st)^{m_{st}} = id \rangle$$

where m_{st} denotes the order of st (possibly infinite).

In our example



we get

$$W = \langle s, t \mid s^2 = t^2 = (st)^6 = id \rangle.$$

Groups admitting a presentation of the form

$$W = \langle S \mid s^2 = (st)^{m_{st}} = id \rangle$$

with $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ are *Coxeter groups*. The pair (W, S) is a *Coxeter system*.

Coxeter's assumptions are always satisfied for discrete groups of orthogonal or Euclidean affine transformations. Hence all such groups are Coxeter groups. Coxeter used this fact to classify all such groups.

(Actually Coxeter was only concerned with these examples. The study of general Coxeter groups began with Tits.)

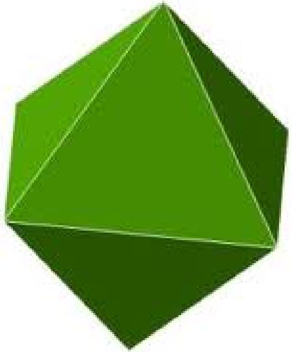
Usually one uses a Coxeter graph to encode the orders m_{st}.

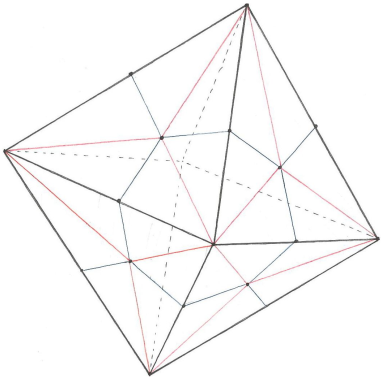
For example

encodes the group

$$W = \left\langle s, t, u \;\middle|\; \begin{array}{c} (st)^3 = (tu)^4 = (su)^2 = id \\ s^2 = t^2 = u^2 = id \end{array} \right\rangle.$$

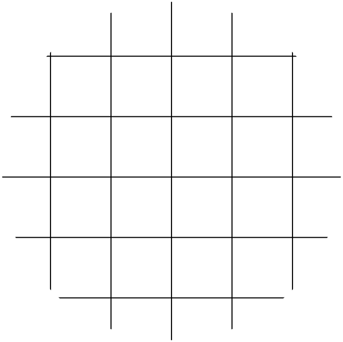
(No edge means $m_{st} = 2$ and an unlabelled edge indicates $m_{st} = 3$.)



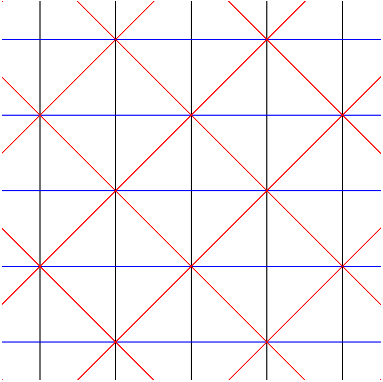


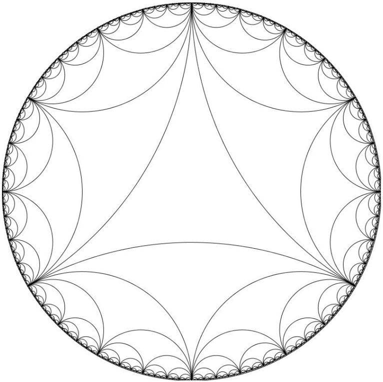

4

For an infinite example we can take X to be the square lattice:

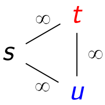


$$s \overset{4}{\text{——}} {\color{red}t} \overset{4}{\text{——}} {\color{blue}u}$$

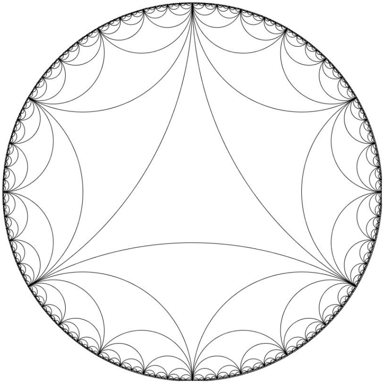




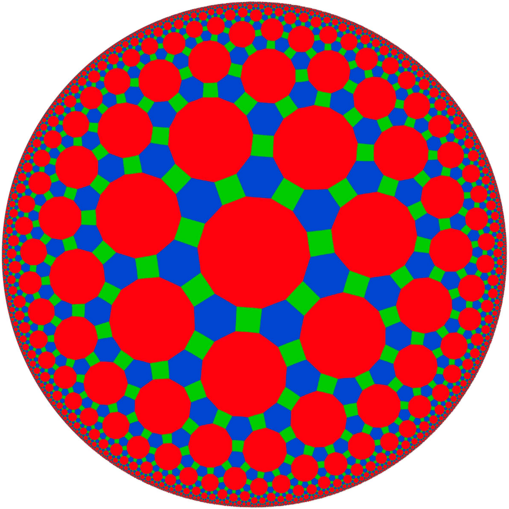
;

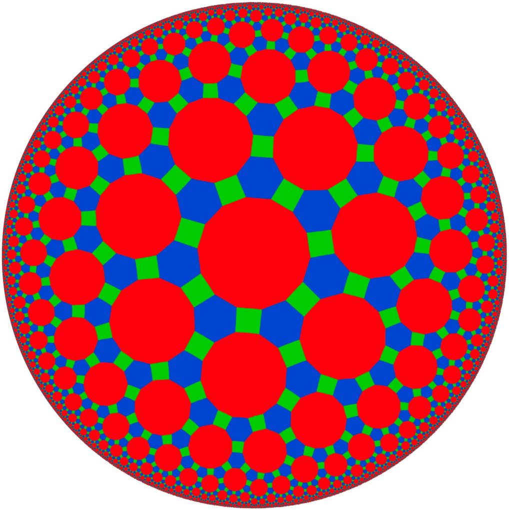


t
s
u
∞
∞
∞



;






7

Many examples of finite reflection groups are provided by the Weyl groups of compact Lie groups.

For example the symmetries of the butterfly is the Weyl group of $SU(2)$, the symmetries of the hexagon or snowflake is the Weyl group of G_2, and the symmetries of the octahedron is the Weyl group of $SO(7)$.

Many examples of finite reflection groups are provided by the Weyl groups of compact Lie groups.

For example the symmetries of the butterfly is the Weyl group of $SU(2)$, the symmetries of the hexagon or snowflake is the Weyl group of G_2, and the symmetries of the octahedron is the Weyl group of $SO(7)$.

However it is important for the story we will tell that not all Coxeter groups belong to Lie groups.

For example, amongst the Platonic solids, the symmetries of the tetrahedron and octahedron occur as Weyl groups, whereas the symmetries of the icosahedron does not. Similarly, most hyperbolic examples are not Weyl groups.

The above discussion has a converse procedure.

Given any Coxeter group (W, S) we can produce a coloured simplicial complex whose automorphisms are precisely W. This complex is called the *Coxeter complex* and will be denoted $|(W, S)|$.

Let $n = |S|$ denote the rank of W. Its construction is as follows:

- colour the n faces of the $n - 1$-simplex Δ by the set S,
- take one such simplex Δ_w for each element $w \in W$,
- glue Δ_w to Δ_{ws} along the wall coloured by s.

For example, consider the symmetric group on three letters:

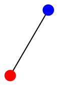
$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$

For example, consider the symmetric group on three letters:

$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$

For example, consider the symmetric group on three letters:

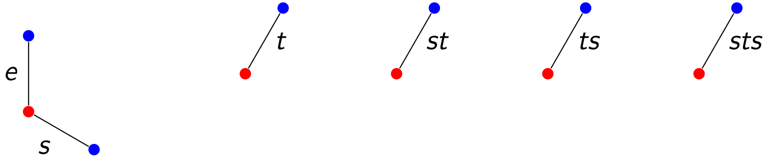
$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$



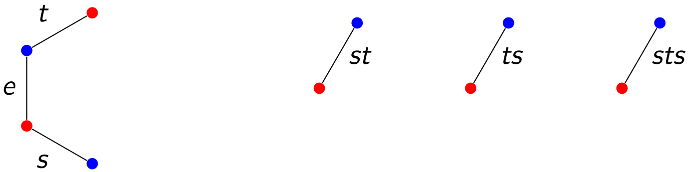
For example, consider the symmetric group on three letters:

$$W = \langle {\color{red}s}, {\color{blue}t} \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$

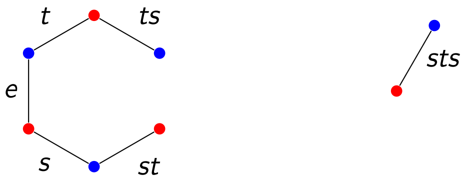
$$W = \langle {\color{red}s}, {\color{blue}t} \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$



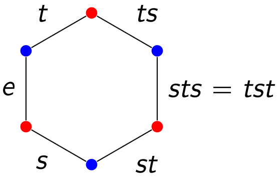
$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$



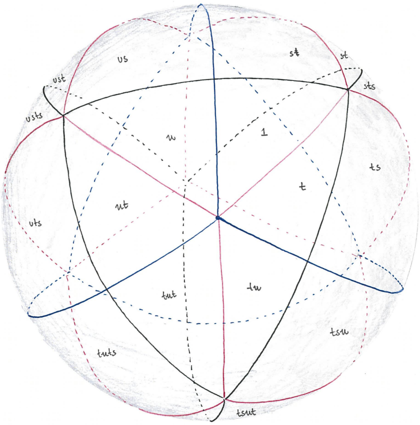
$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$



$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$



The Coxeter complex of S_4 = ● — ● — ● :



(barycentric subdivision of the tetrahedron).

By construction $|(W,S)|$ has a left action of W.

W also acts on the alcoves of $|(W,S)|$ on the right by

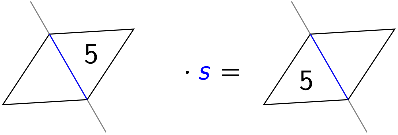
$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* simplicial, but is "local": cross the wall coloured by s.

The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

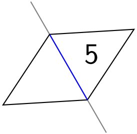
The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:

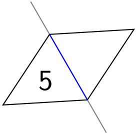


The Coxeter complex provides a convenient way of visualising the group algebra $\mathbb{Z}W$ of W. Recall that the group algebra $\mathbb{Z}W$ consists of finite formal linear combinations $\sum \lambda_w w$ of elements of W. The product in W induces a multipliction in $\mathbb{Z}W$.

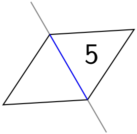
Hence we can picture an element of $\mathbb{Z}W$ as the assignment of integers to each alcove, such that only finitely many are non-zero. If we view $\mathbb{Z}W$ as a right module over itself it is easy to picture the action of the elements of S:



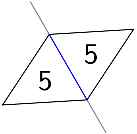
$\cdot\, s =$



Similarly ("s averaging operator")



$\cdot\, (1 + s) =$

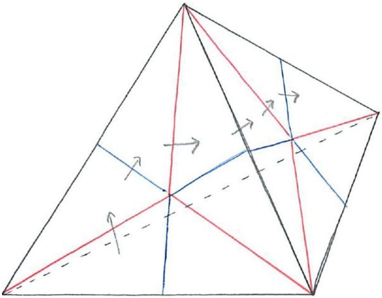


Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\begin{aligned} \ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|. \end{aligned}$$

Let $\ell : W \to \mathbb{N}$ denote the length function on W:

$$\begin{aligned} \ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|. \end{aligned}$$



The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

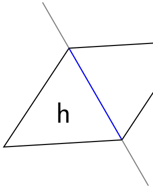
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

The Hecke algebra H is a quantization of $\mathbb{Z}W$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{H_x \mid x \in W\}$ parametrised by W. If we write $\underline{H}_s := H_s + vH_{id}$ then the multiplication in H is determined by

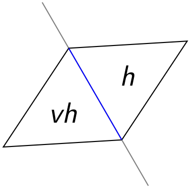
$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1}H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

We can visualise this as follows: ("quantized averaging operator")

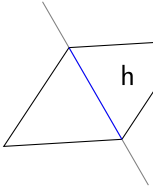
id



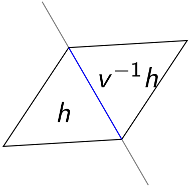
$\cdot \, \underline{H}_s =$



id



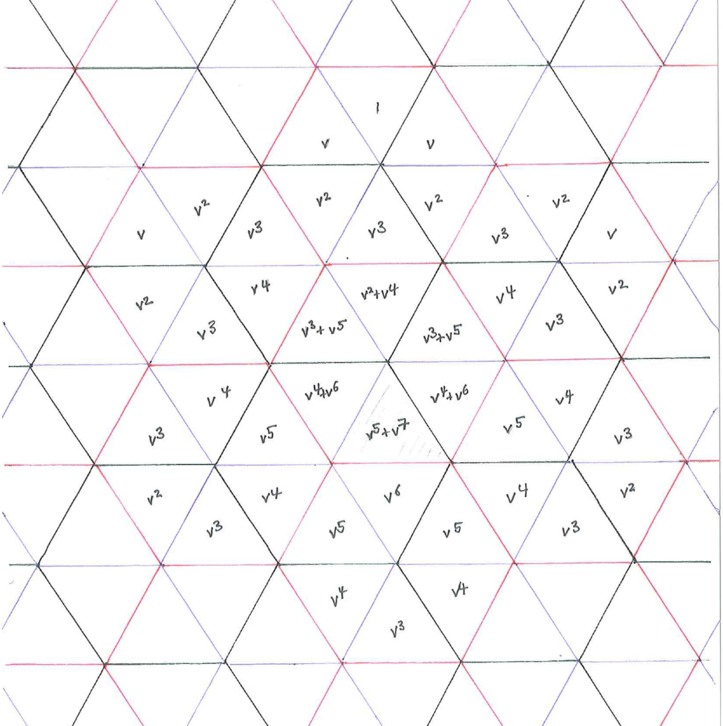
$\cdot \, \underline{H}_s =$



In 1979 Kazhdan and Lusztig defined a new basis for the Hecke algebra using the combinatorial structure of W. We denote this new basis by $\{\underline{H}_x \mid x \in W\}$. It satisfies

$$\underline{H}_x := H_x + \sum_{\substack{y \in W \\ \ell(y) < \ell(x)}} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$. These polynomials are the *Kazhdan-Lusztig polynomials*.



1
v
v
v
v^2
v^3
v^2
v^3
v^2
v^3
v^2
v
v^2
v^4
v^2+v^4
v^4
v^2
v^3
v^3+v^5
v^3+v^5
v^3
v^4
v^4+v^6
v^4+v^6
v^4
v^3
v^5
v^5+v^7
v^5
v^3
v^2
v^4
v^6
v^4
v^2
v^3
v^5
v^5
v^3
v^4
v^4
v^3

The definition is inductive. The first few Kazhdan-Lusztig basis elements are easily defined:

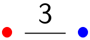
$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

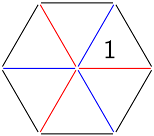
Now the work begins. Suppose that we have calculated $\underline{H}_y$ for all y with $\ell(y) \leqslant \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

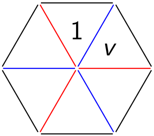
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

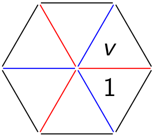
The formula for the action of $\underline{H}_s$ shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

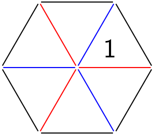


$\underline{H}_{id} =$ 

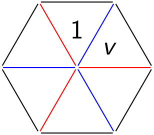
$\underline{H}_{\color{blue}t} =$ 

$\underline{H}_{\color{red}s} =$ 

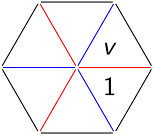
$\underline{H}_{id} =$



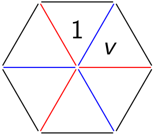
$\underline{H}_{t} =$



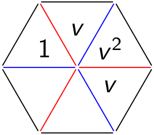
$\underline{H}_{s} =$



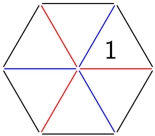
$\underline{H}_{t}\underline{H}_{s} =$

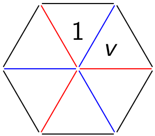


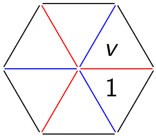
$\cdot\, \underline{H}_{s} =$

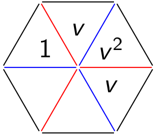


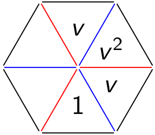
$= \underline{H}_{ts}$

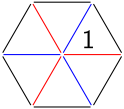
$\underline{H}_{id} =$ 

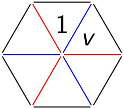
$\underline{H}_{t} =$ 

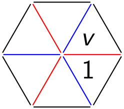
$\underline{H}_{s} =$ 

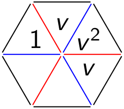
$\underline{H}_{ts} =$ 

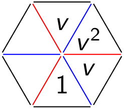
$\underline{H}_{st} =$ 

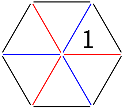
$\underline{H}_{id} =$ 

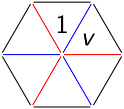
$\underline{H}_{t} =$ 

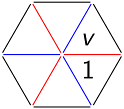
$\underline{H}_{s} =$ 

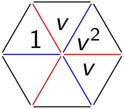
$\underline{H}_{ts} =$ 

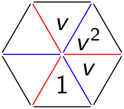
$\underline{H}_{st} =$ 

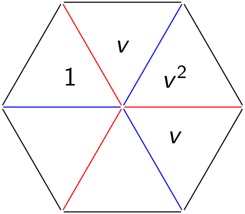
$\underline{H}_{id} =$ 

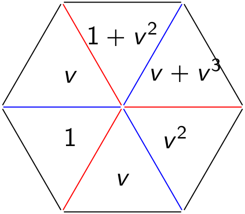
$\underline{H}_{t} =$ 

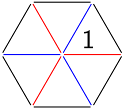
$\underline{H}_{s} =$ 

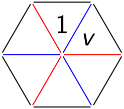
$\underline{H}_{ts} =$ 

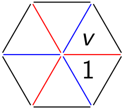
$\underline{H}_{st} =$ 

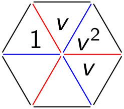
$\underline{H}_{ts}\underline{H}_{t} =$ 

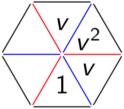
$\cdot\, \underline{H}_{t} =$ 

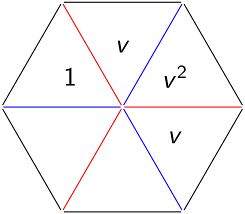
$\underline{H}_{id} =$ 

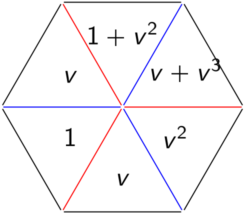
$\underline{H}_{t} =$ 

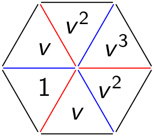
$\underline{H}_{s} =$ 

$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

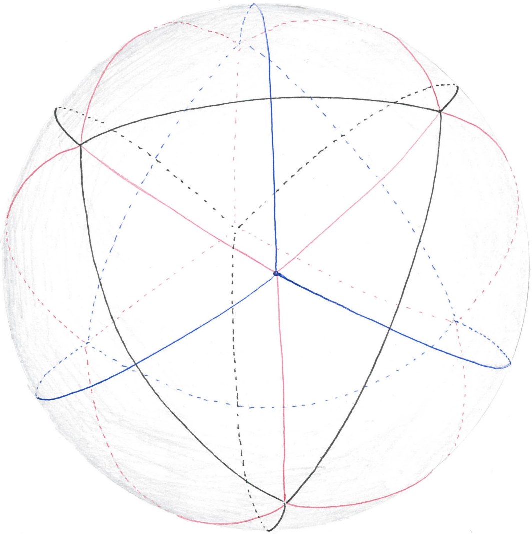
$\underline{H}_{ts}\underline{H}_{t} =$ 

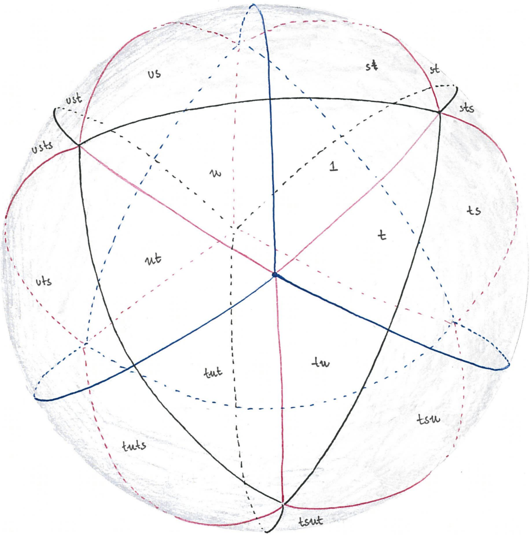
$\cdot\, \underline{H}_{t} =$ 

Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_{t} - \underline{H}_{t} =$ 

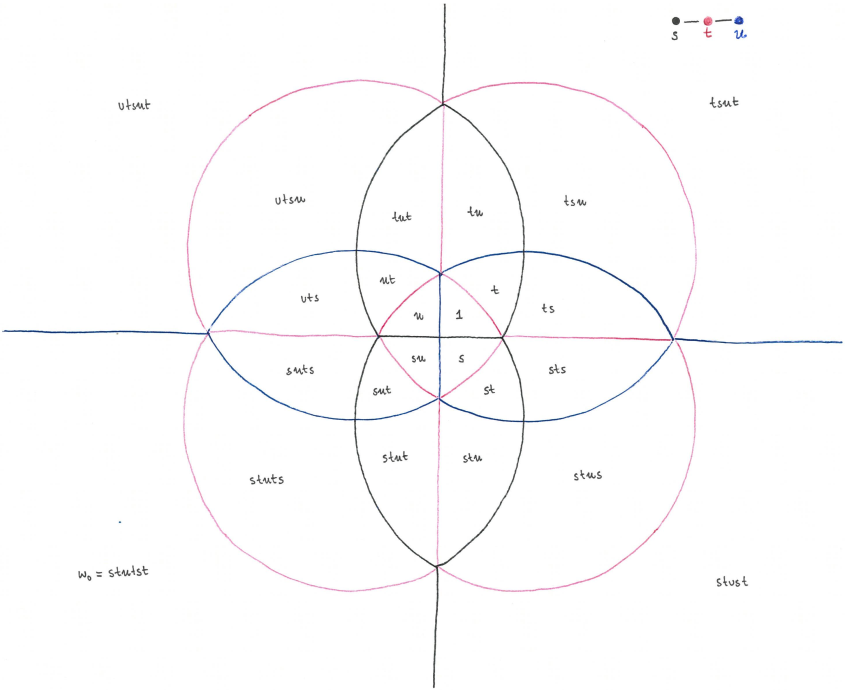
For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$ (Kazhdan-Lusztig basis elements are *smooth*.)

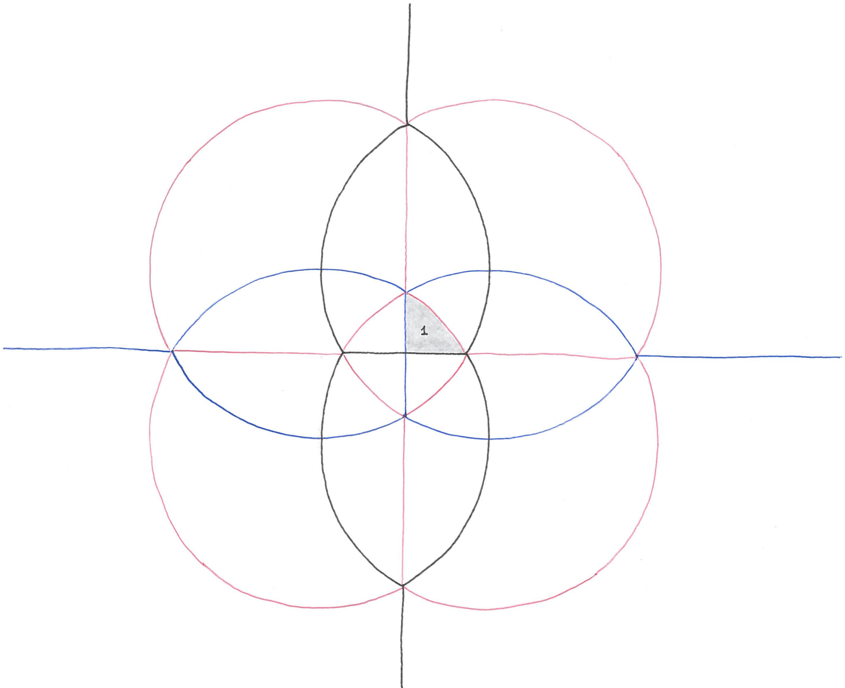
However in higher rank the situation quickly becomes more interesting...

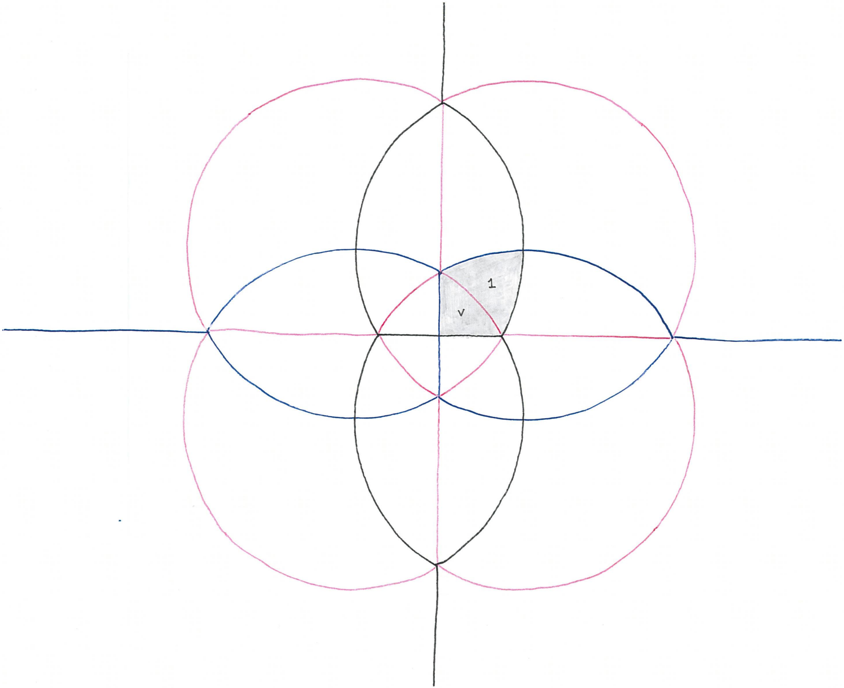


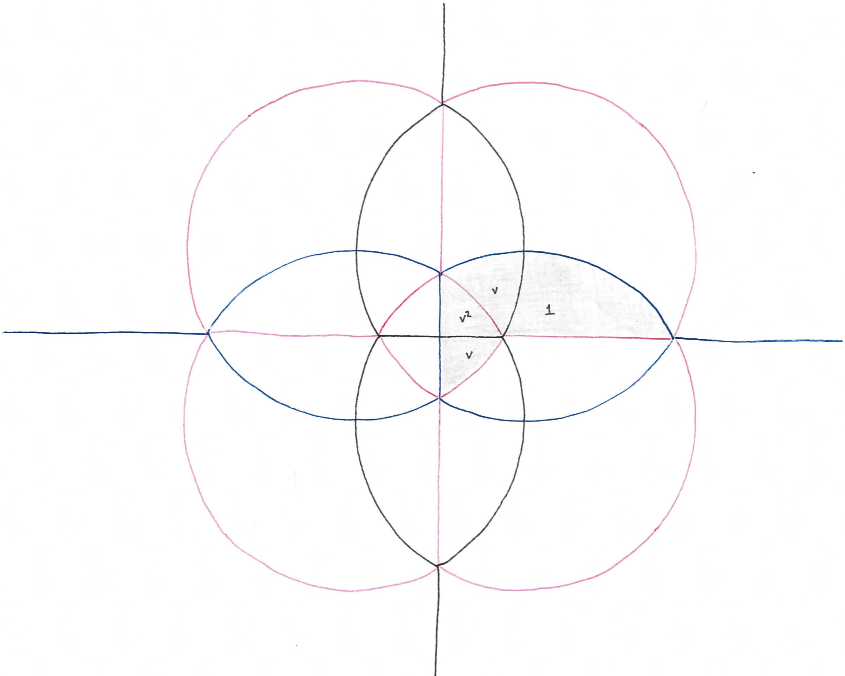


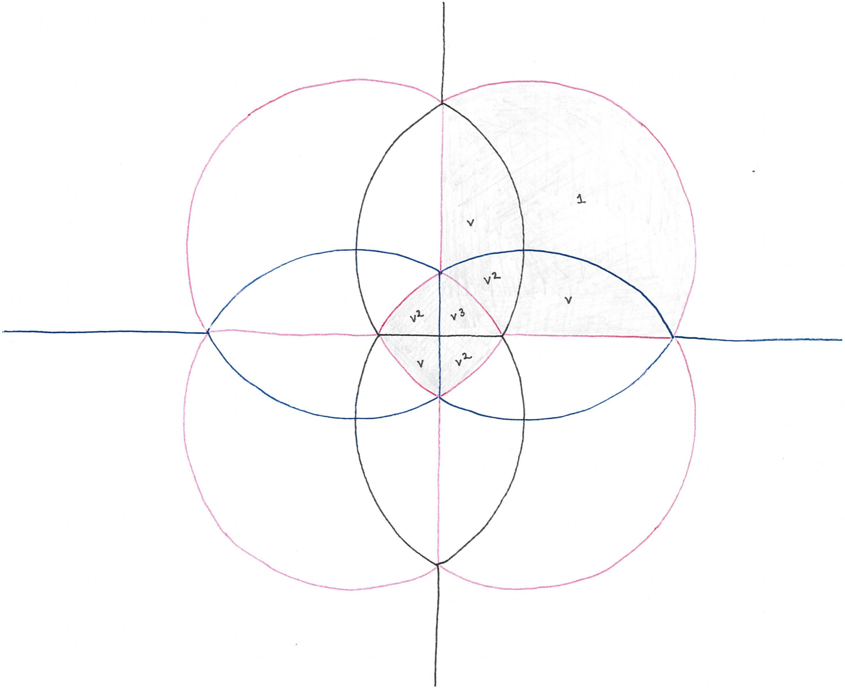
us
st
st
ust
sts
usts
u
1
ts
t
ut
uts
tut
tu
tsu
tuts
tsut


s t u
utsut
tsut
utsu
tut
tu
tsu
ut
t
uts
ts
u
1
su
s
suts
sts
sut
st
stut
stu
stuts
stus
w0 = stutst
stust

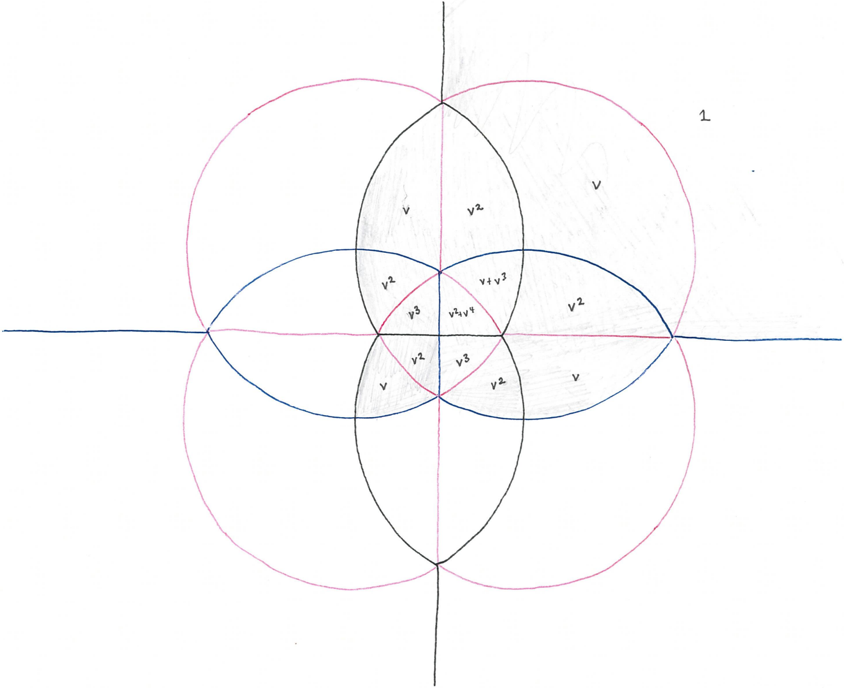

1


1
v

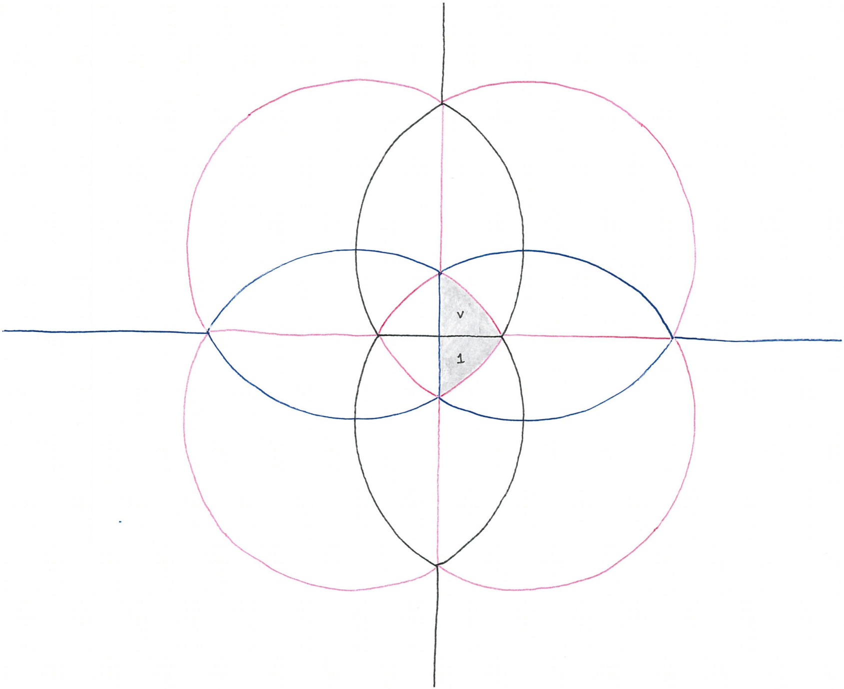

v
v²
1
v

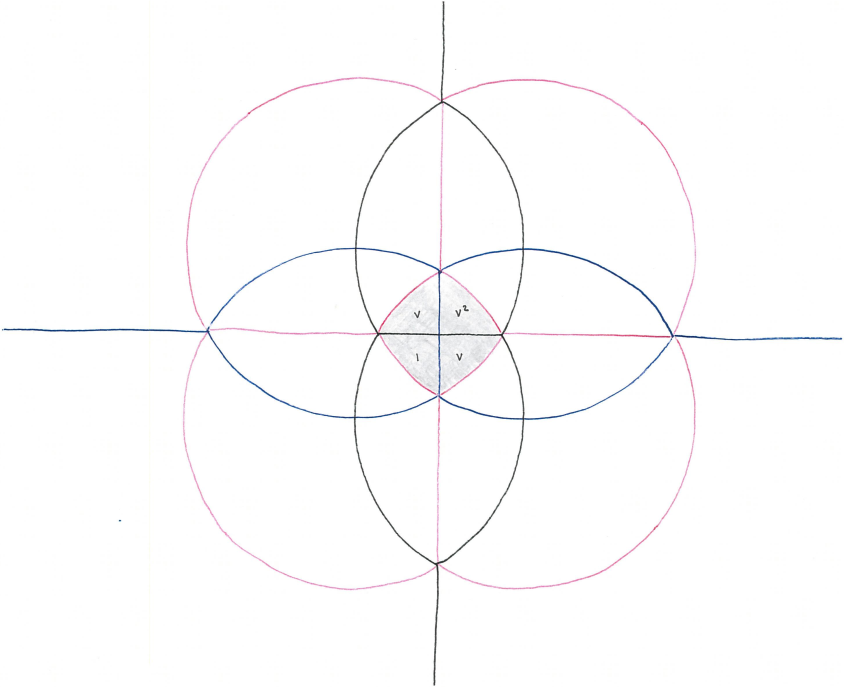


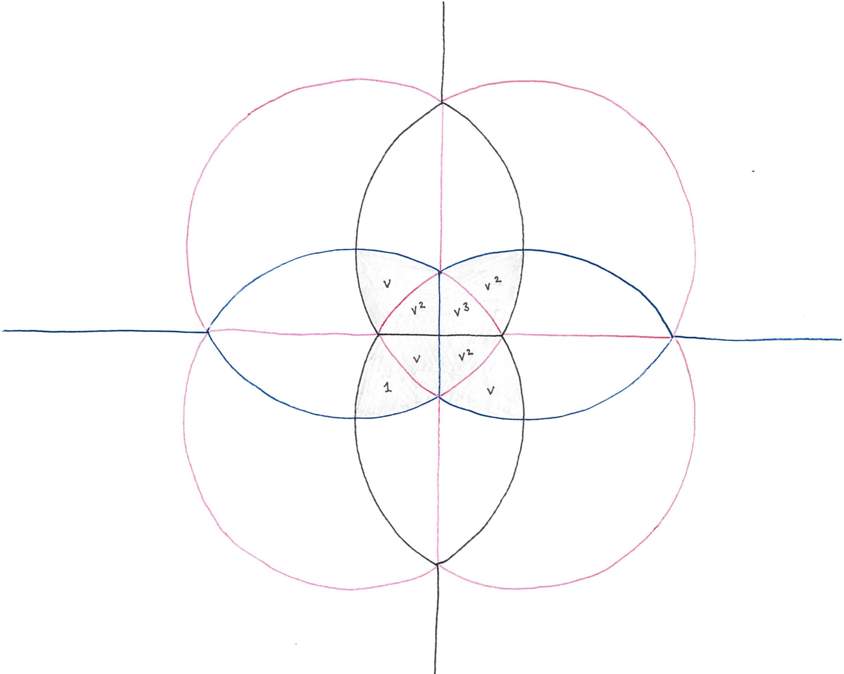
1
v
v2
v
v2
v3
v
v2

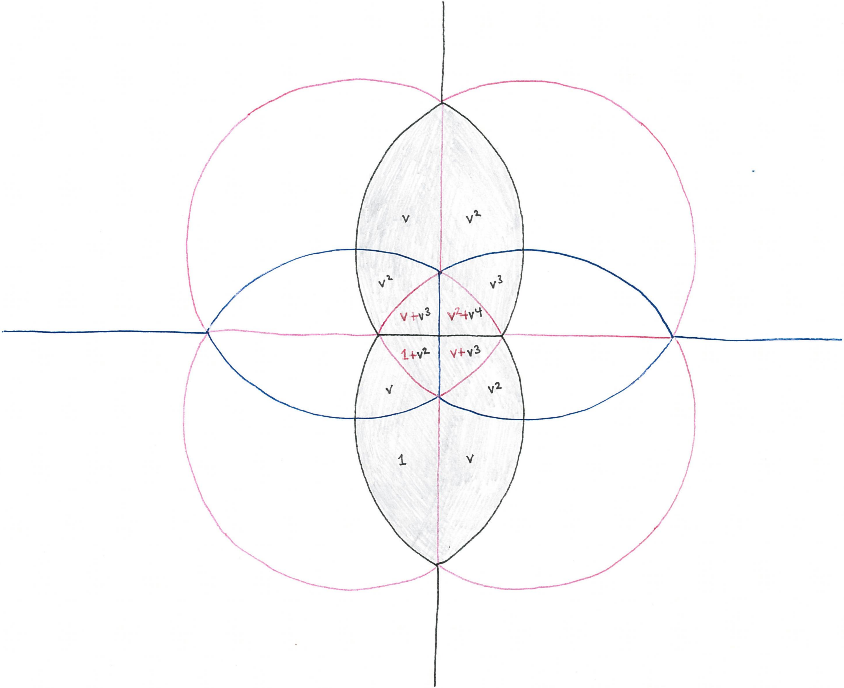


1
v
v
v2
v2
v+v3
v3
v2+v4
v2
v2
v3
v
v2
v

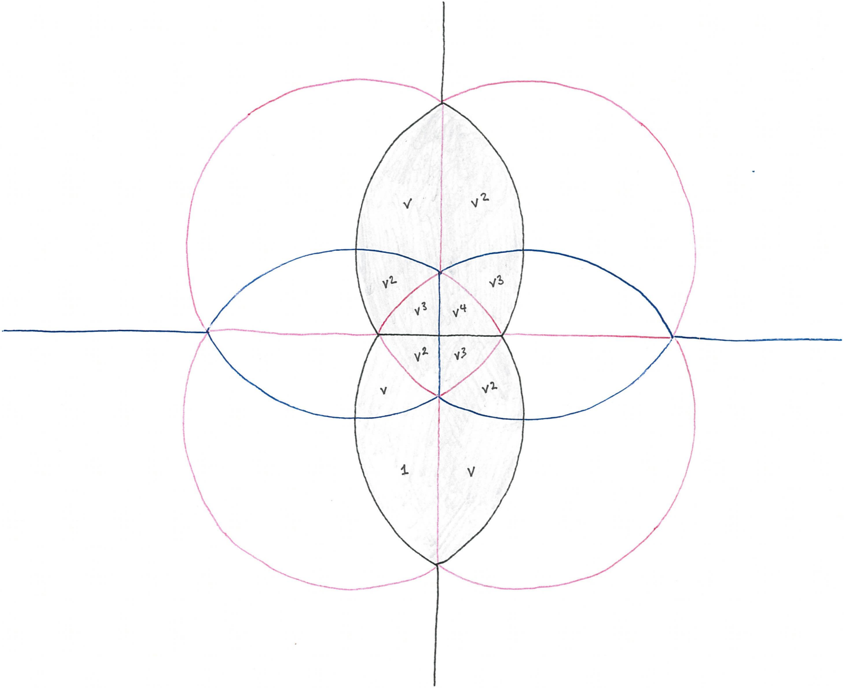

v
1

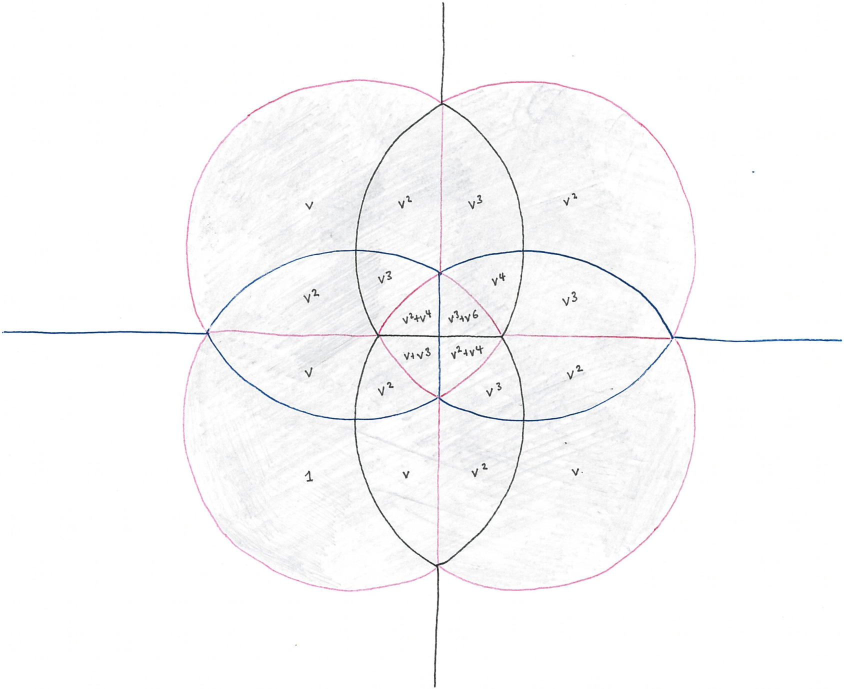

v
v²
1
v


v
v2
v2
v3
v
v2
1
v

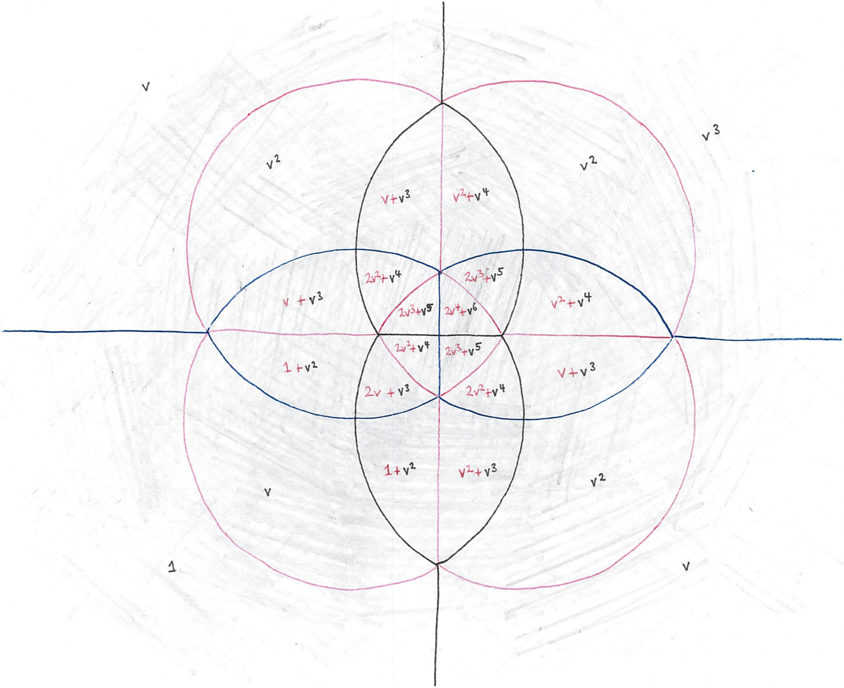


v
v²
v²
v³
v+v³
v²+v⁴
1+v²
v+v³
v
v²
1
v

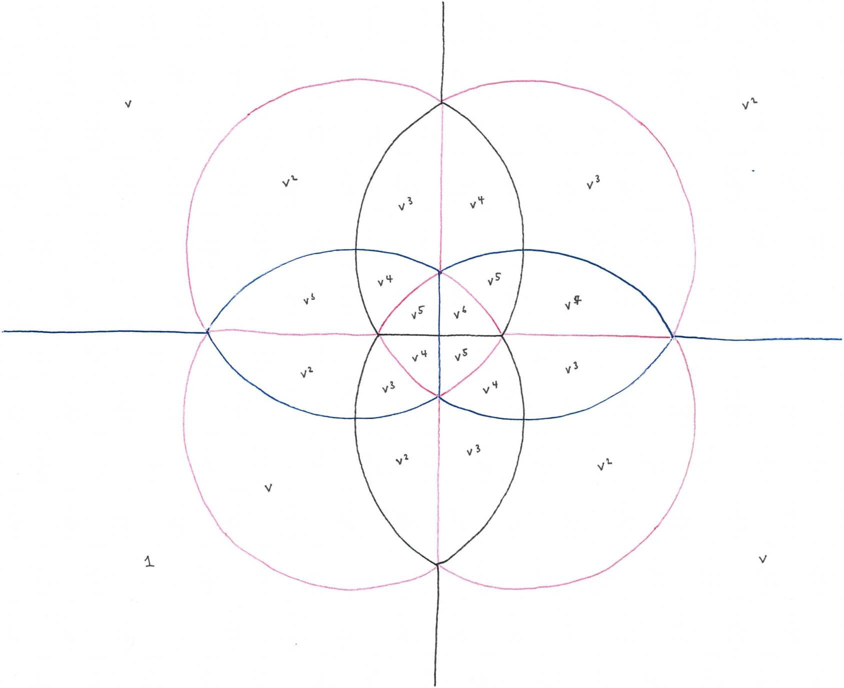

v
v2
v2
v3
v3
v4
v2
v3
v
v2
1
v

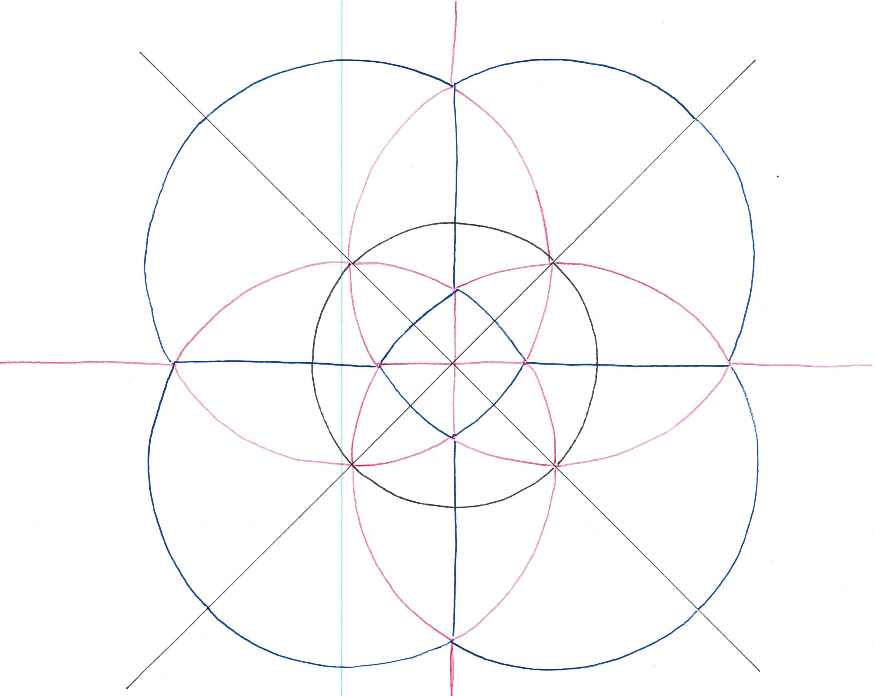


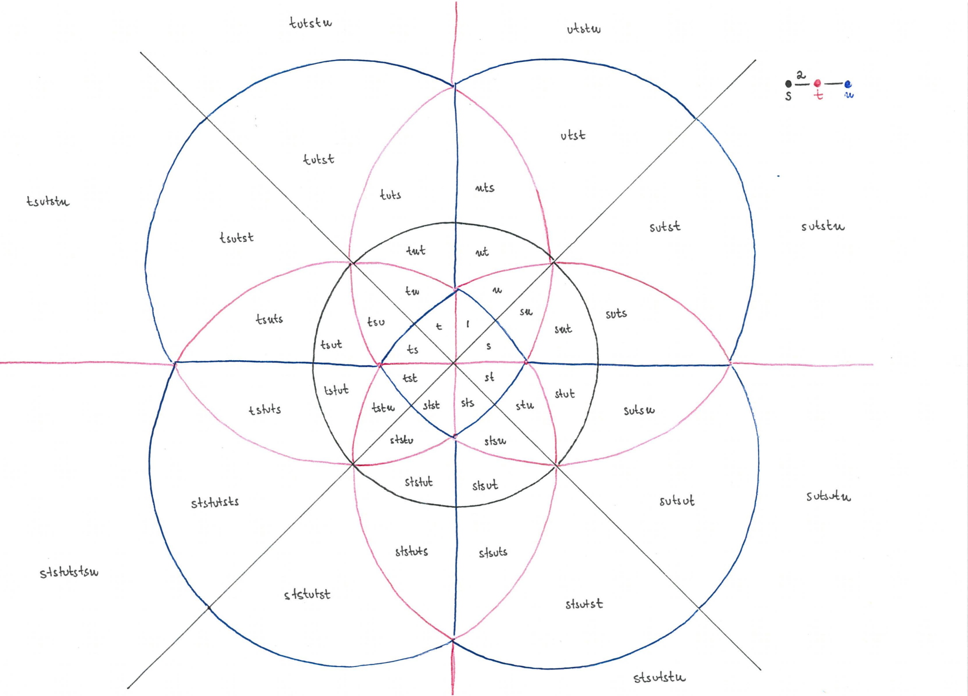
v
v2
v3
v2
v3
v4
v2
v3
v2+v4
v3+v6
v+v3
v2+v4
v
v2
v3
v2
1
v
v2
v



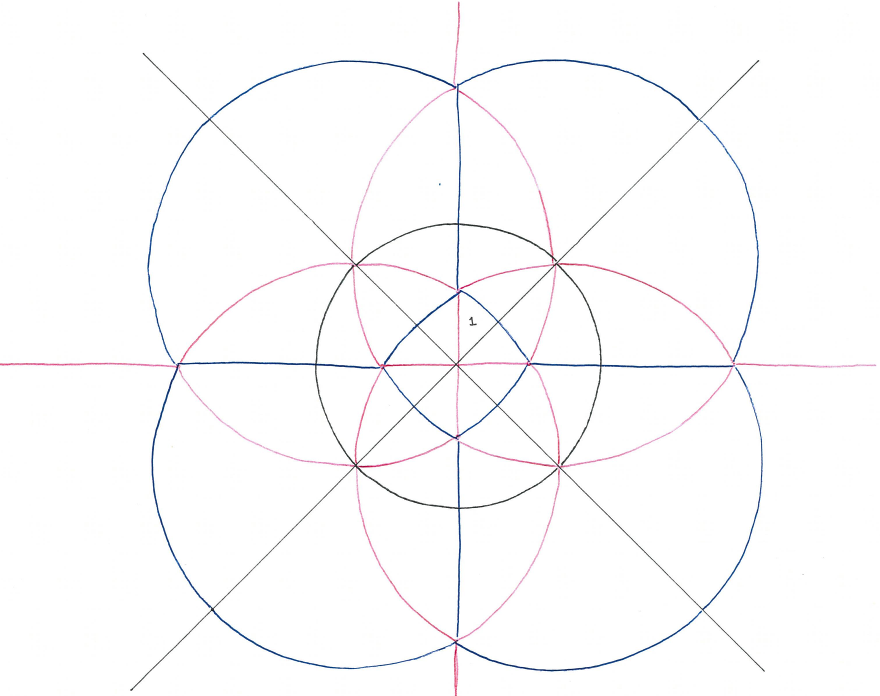
v
v 3
v 2
v 2
v + v 3
v 2 + v 4
2v 2 + v 4
2v 3 + v 5
v + v 3
2v 3 + v 5
2v 4 + v 6
v 2 + v 4
2v 2 + v 4
2v 3 + v 5
1 + v 2
v + v 3
2v + v 3
2v 2 + v 4
1 + v 2
v 2 + v 3
v
v 2
1
v

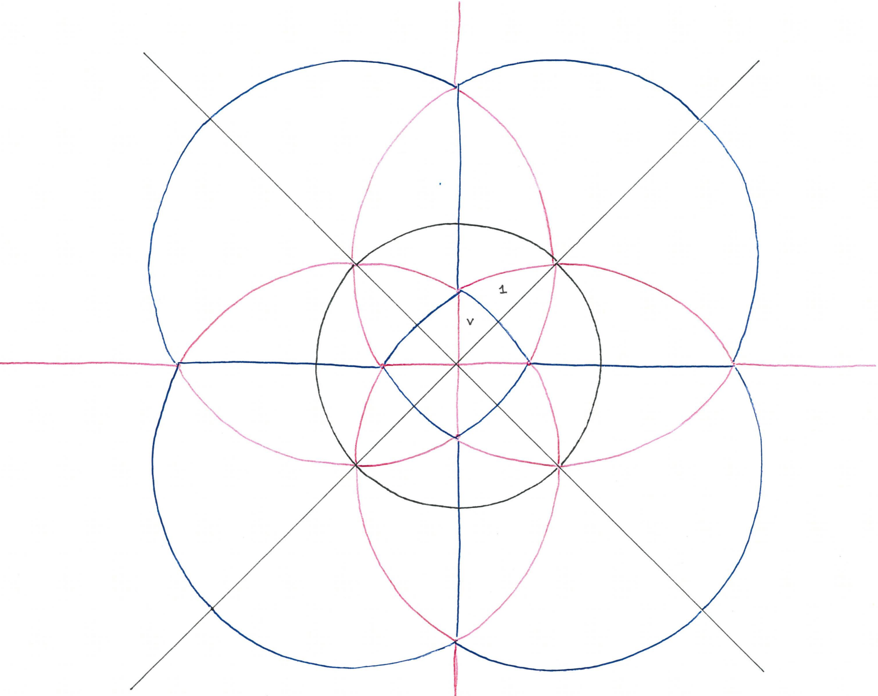

v
v^2
v^2
v^3
v^3
v^4
v^4
v^5
v^5
v^4
v^5
v^6
v^4
v^5
v^2
v^3
v^3
v^4
v^2
v^3
v^2
v
1
v

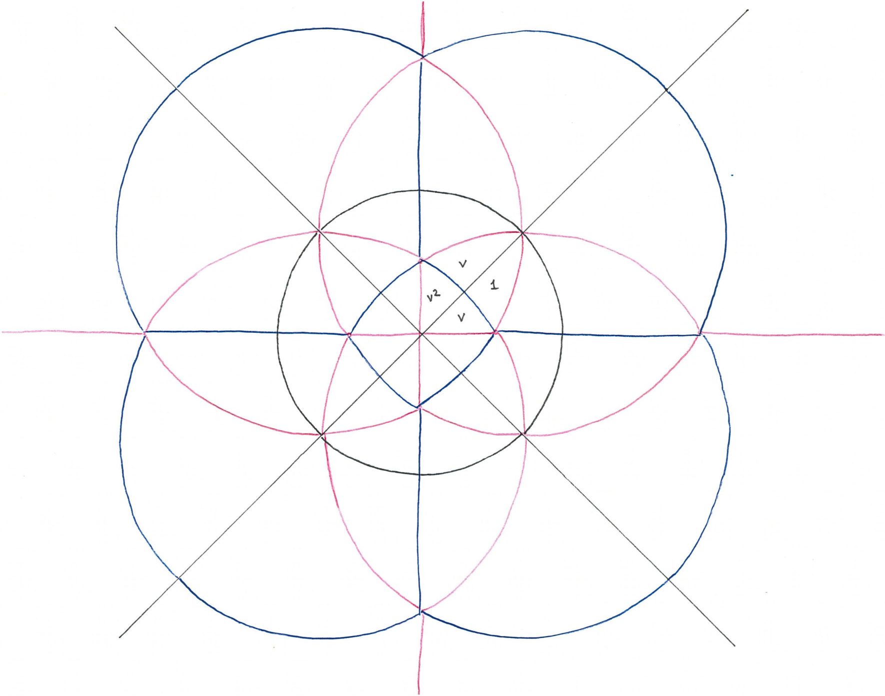




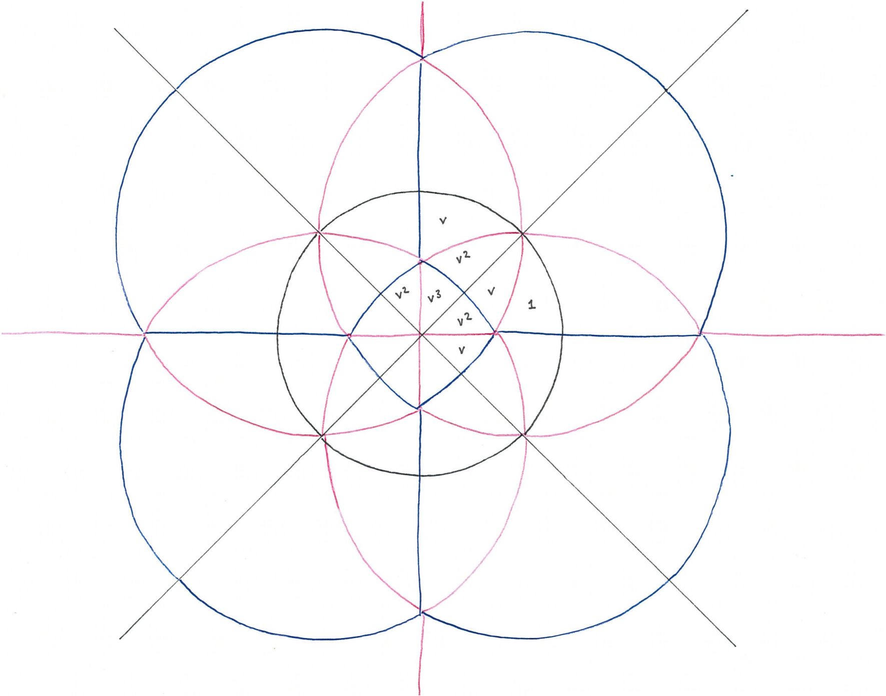
tutstu
utstu
s
t
u
2
tutst
utst
tsutstu
tuts
uts
sutstu
tsutst
sutst
tut
ut
tu
u
tsuts
tsu
t
1
su
sut
suts
tsut
ts
s
tst
st
tstut
tstuts
tstu
stst
sts
stu
stut
sutsu
ststu
stsu
ststut
stsut
ststutsts
sutsut
sutsutu
ststuts
stsuts
ststutstsu
ststutst
stsutst
stsutstu

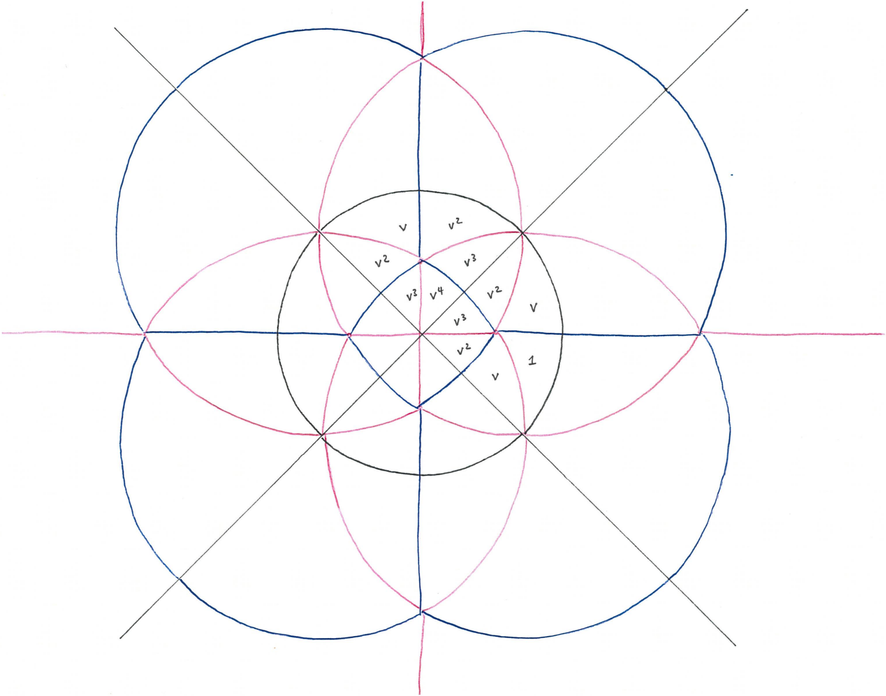

1


1
v

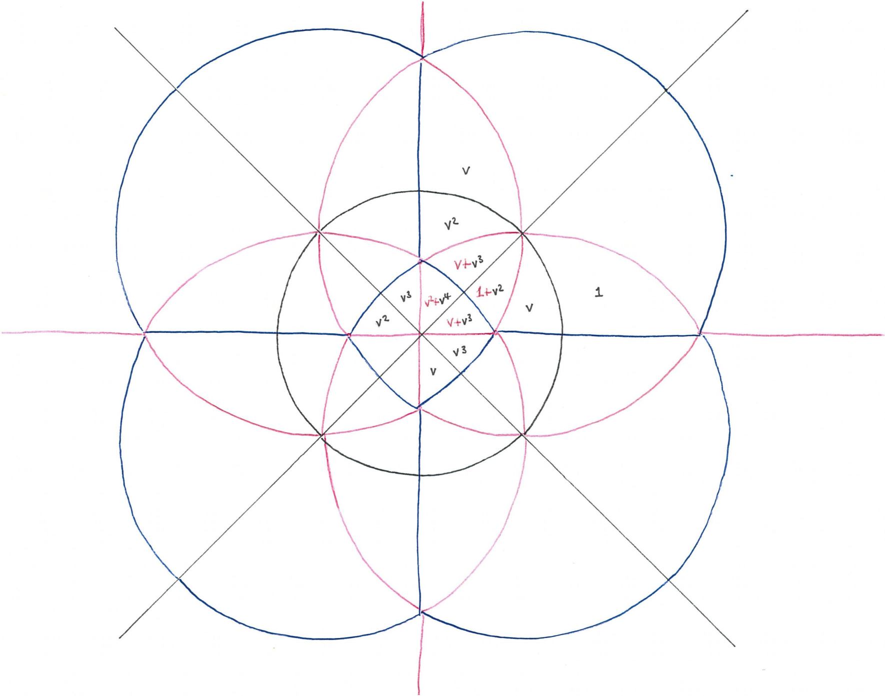


v
1
v²
v

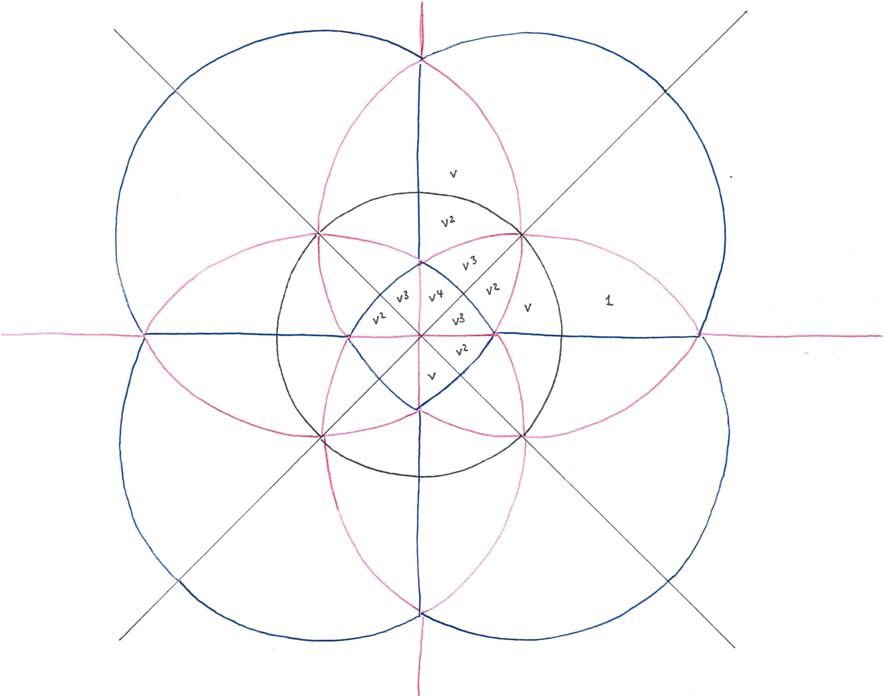

v
v2
v2
v3
v
1
v2
v



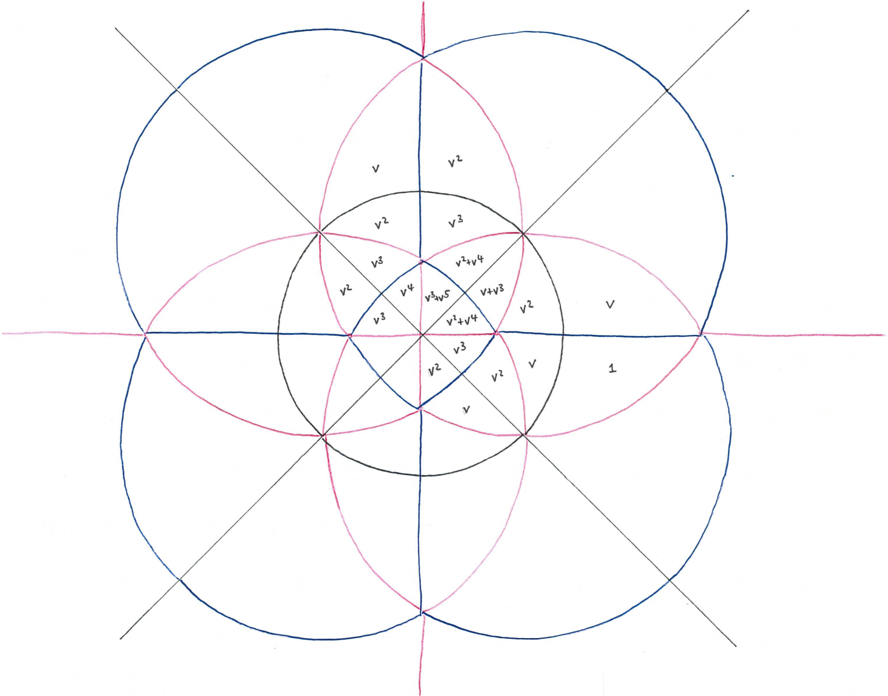
v
v2
v2
v3
v3
v4
v2
v
v3
v2
1
v



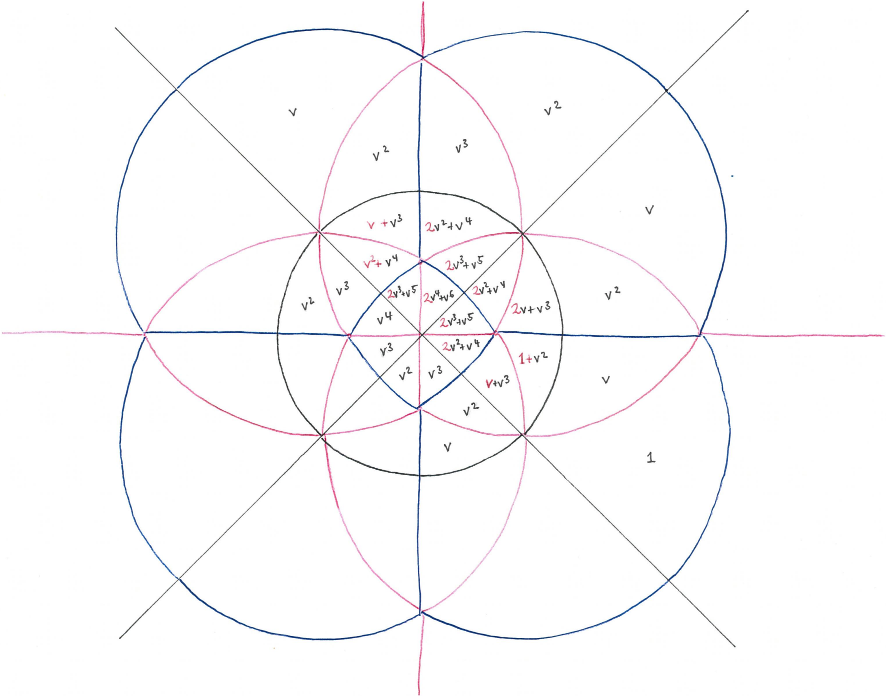
v
v2
v+v3
1+v2
1
v
v3
v2+v4
v2
v+v3
v3
v



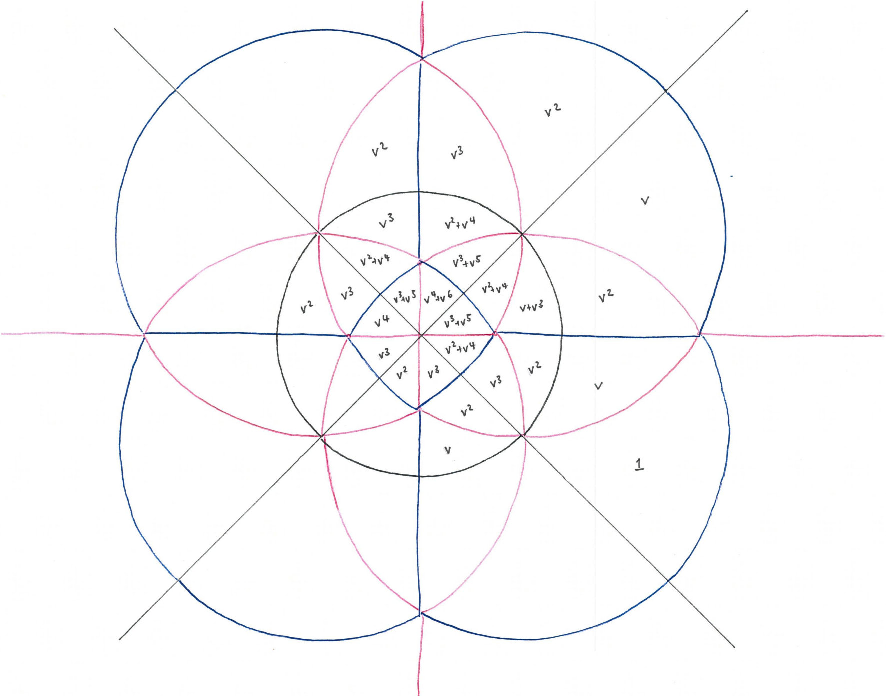
v
v2
v3
v3
v4
v2
v2
v
1
v3
v2
v



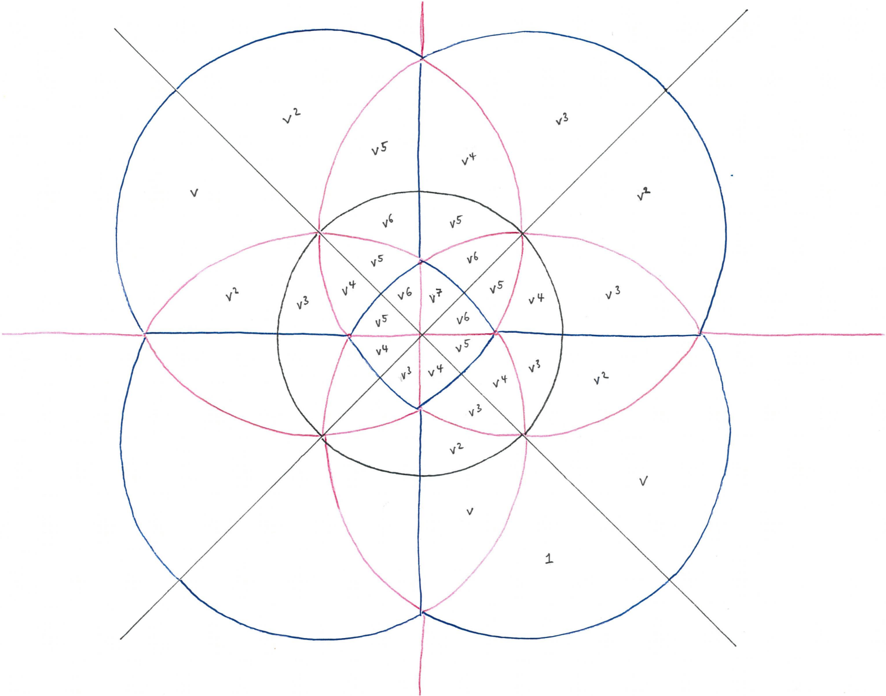
v
v2
v2
v3
v3
v2+v4
v2
v4
v3+v5
v+v3
v2
v
v3
v2+v4
v3
v2
v2
v
v
1

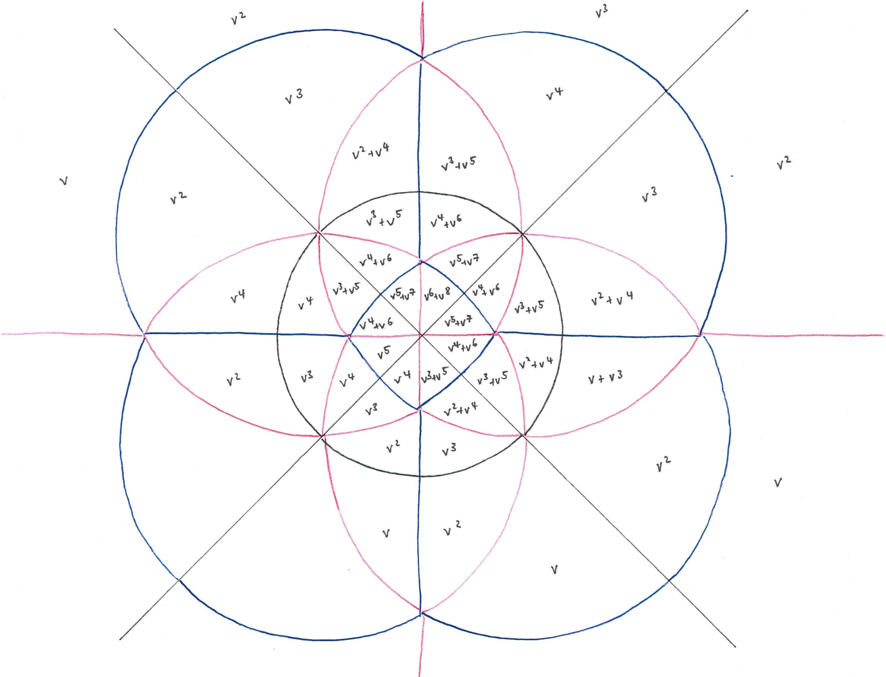


v
v 2
v 2
v 3
v
v + v3
2v 2+v 4
v2+ v4
2v3+ v5
v 2
v 3
2v3+v5
2v4+v6
2v2+v4
2v+v3
v 2
v4
2v3+v5
v3
2v2+v4
1+v2
v2
v 3
v+v3
v
v2
v
1

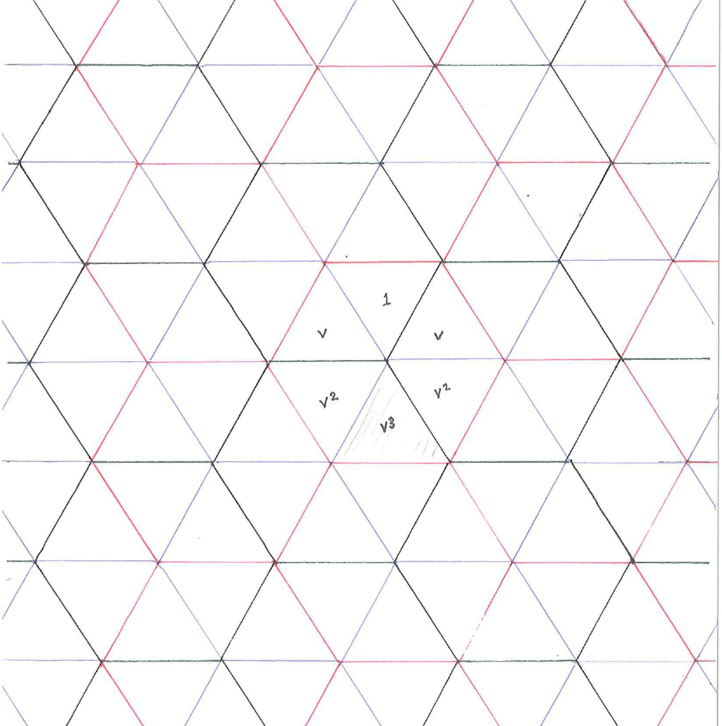


v2
v2
v3
v
v3
v2+v4
v2+v4
v3+v5
v2
v3
v3+v5
v4+v6
v3+v4
v+v3
v2
v4
v3+v5
v3
v2+v4
v2
v3
v3
v2
v
v2
v
1

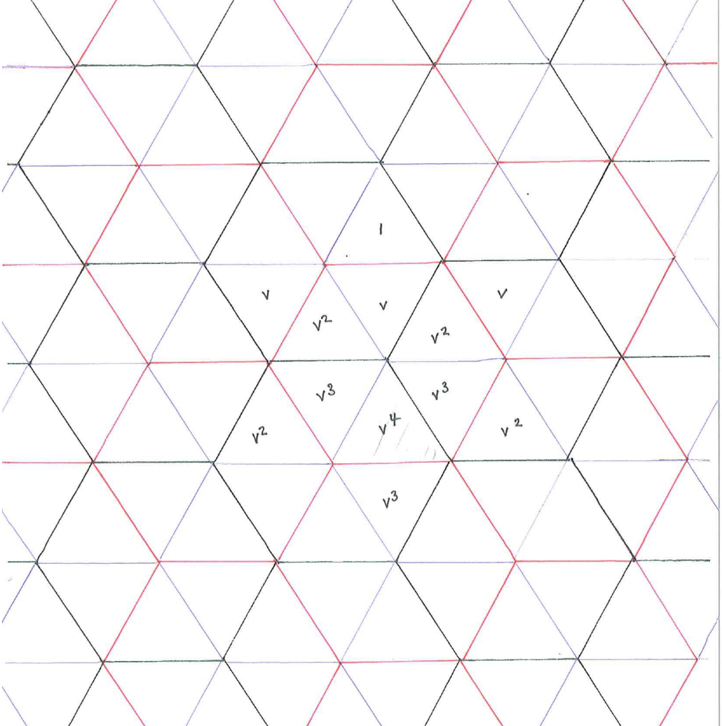

v2
v3
v5
v4
v
v2
v6
v5
v5
v6
v2
v4
v3
v6
v7
v5
v3
v4
v5
v6
v4
v3
v4
v5
v3
v4
v3
v4
v3
v2
v2
v
v
1

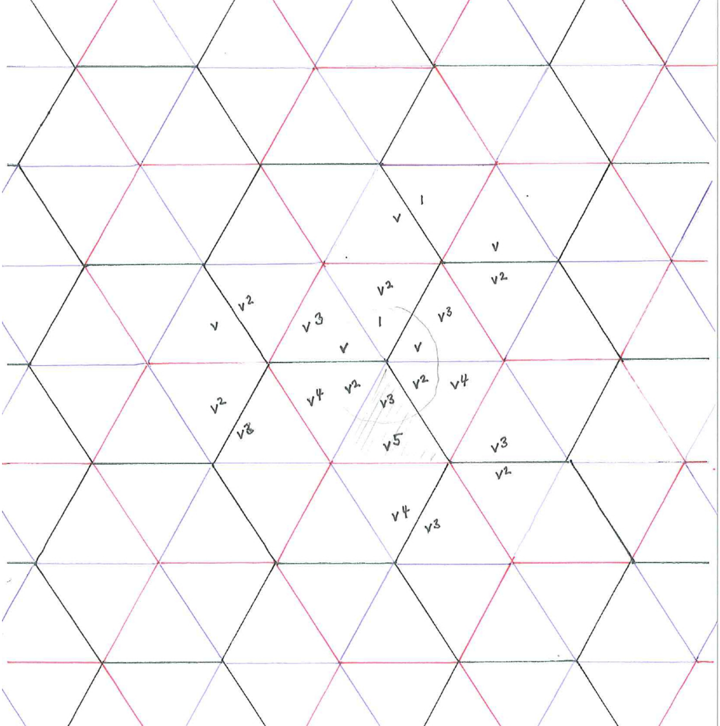


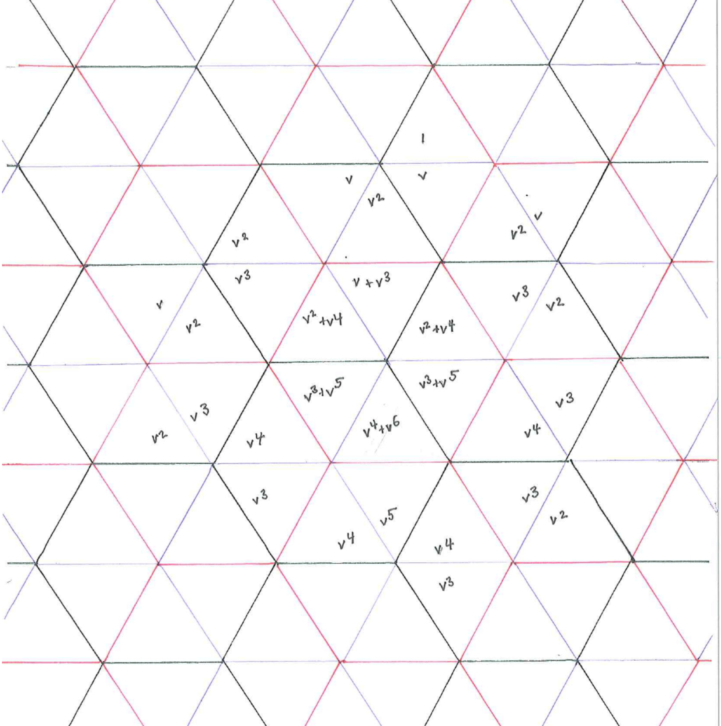
v^2
v^3
$v\,3$
v^4
v^2+v^4
v^3+v^5
v
v^2
v^2
$v\,3$
v^3+v^5
v^4+v^6
v^4+v^6
v^5+v^7
v^4
$v\,4$
v^3+v^5
v^5+v^7
v^6+v^8
v^4+v^6
v^3+v^5
$v^2+v\,4$
$v^4+v\,6$
v^5+v^7
$v\,5$
v^4+v^6
v^2
$v3$
$v\,4$
$v\,4$
v^3+v^5
v^3+v^5
$v^2+v\,4$
$v+v\,3$
v^3
v^2+v^4
v^2
$v\,3$
v^2
v
v
v^2
v

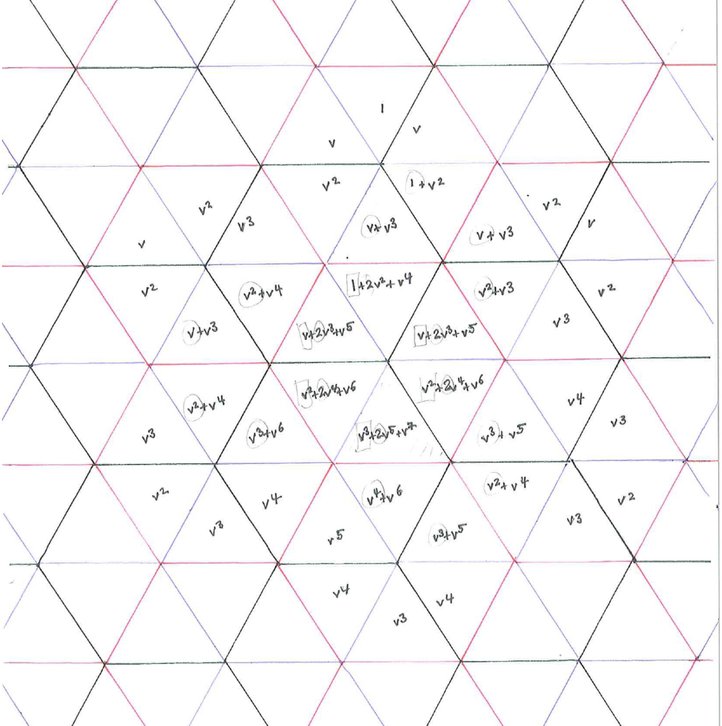


1
v
v
v2
v2
v3

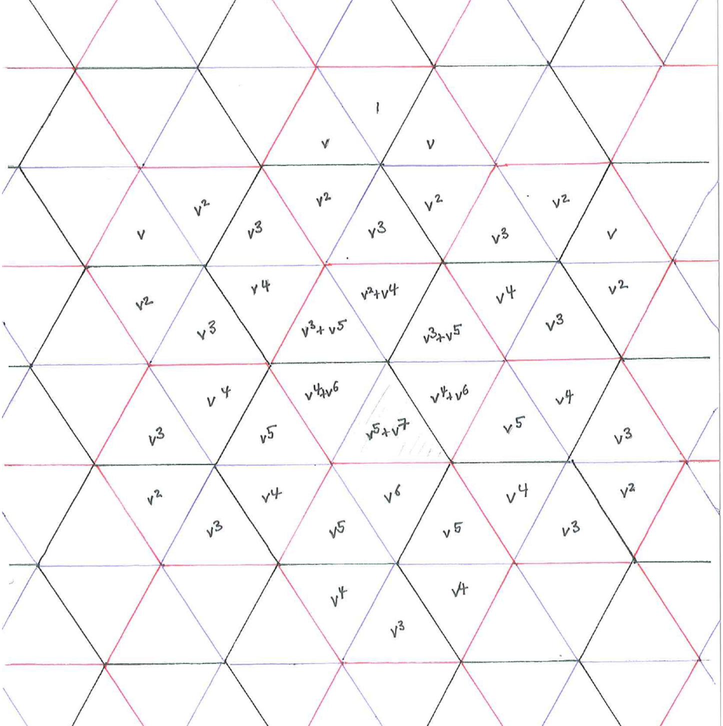

1
v
v^2
v
v
v^2
v^3
v^3
v^2
v^4
v^2
v^3

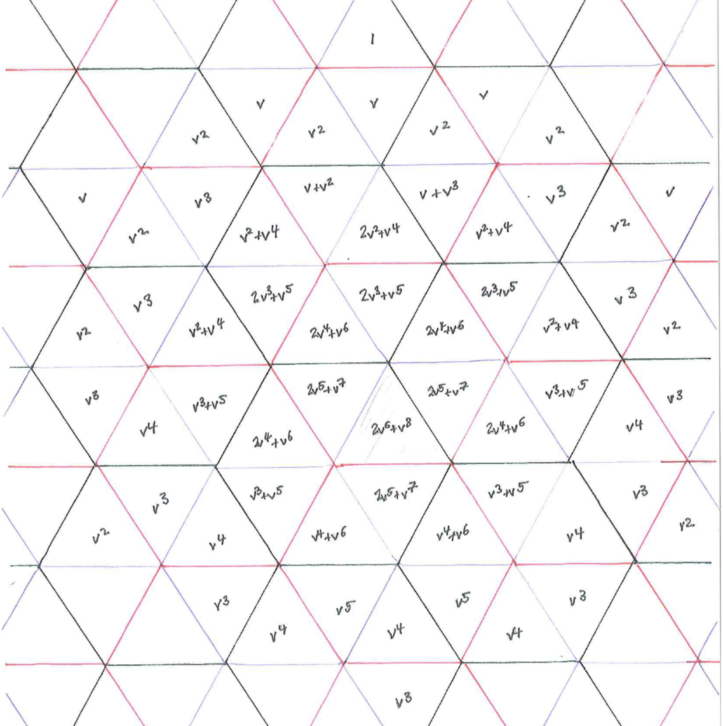

v
1
v
v2
v2
v
v2
v3
1
v3
v
v
v2
v4
v2
v4
v2
v3
v3
v5
v3
v2
v4
v3


1
v
v
v^2
v^2
v^2
v
v^3
$v+v^3$
v
v^2
v^2+v^4
v^2+v^4
v^3
v^2
v^3+v^5
v^3+v^5
v^3
v^2
v^3
v^4
v^4+v^6
v^4
v^3
v^5
v^4
v^4
v^3
v^3
v^2



1
v
v
v2
1+v2
v2
v3
v+v3
v+v3
v2
v
v
v2
v2+v4
1+2v2+v4
v2+v3
v2
v+v3
v+2v3+v5
v+2v3+v5
v3
v2+2v4+v6
v2+2v4+v6
v4
v2+v4
v3+v6
v3+2v5+v7
v3+v5
v3
v3
v2
v4
v4+v6
v2+v4
v2
v3
v5
v3+v5
v3
v4
v4
v3


1
v
v
v2
v2
v2
v2
v
v3
v3
v3
v
v4
v2+v4
v4
v2
v2
v3
v3+v5
v3+v5
v3
v4
v4+v6
v4+v6
v4
v3
v5
v5+v7
v5
v3
v2
v4
v6
v4
v2
v3
v5
v5
v3
v4
v4
v3



1
v
v
v
v^2
v^2
v^2
v^2
v
v^3
$v+v^2$
$v+v^3$
v^3
v
v^2
v^2+v^4
$2v^2+v^4$
v^2+v^4
v^2
v^3
$2v^3+v^5$
$2v^3+v^5$
$2v^3+v^5$
v^3
v^2
v^2+v^4
$2v^4+v^6$
$2v^4+v^6$
v^2+v^4
v^2
v^3
v^3+v^5
$2v^5+v^7$
$2v^5+v^7$
v^3+v^5
v^3
v^4
$2v^4+v^6$
$2v^6+v^8$
$2v^4+v^6$
v^4
v^3
v^3+v^5
$2v^5+v^7$
v^3+v^5
v^3
v^2
v^4
v^4+v^6
v^4+v^6
v^4
v^2
v^3
v^5
v^5
v^3
v^4
v^4
v^4
v^3

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$
\begin{aligned}
152q^{22} + {} & 3\,472q^{21} + 38\,791q^{20} + 293\,021q^{19} + 1\,370\,892q^{18} + \\
& + 4\,067\,059q^{17} + 7\,964\,012q^{16} + 11\,159\,003q^{15} + \\
& + 11\,808\,808q^{14} + 9\,859\,915q^{13} + 6\,778\,956q^{12} + \\
& + 3\,964\,369q^{11} + 2\,015\,441q^{10} + 906\,567q^{9} + \\
& + 363\,611q^{8} + 129\,820q^{7} + 41\,239q^{6} + \\
& + 11\,426q^{5} + 2\,677q^{4} + 492q^{3} + 61q^{2} + 3q
\end{aligned}
$$

(This polynomial is associated to the reflection group of type E_8. See `www.liegroups.org`.)

Why are Kazhdan-Lusztig polynomials useful?

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig conjecture (1979):

$$\mathrm{ch} L(x \cdot 0) = \sum_{y \geqslant x} (-1)^{\ell(x)-\ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch} \Delta(y \cdot 0).$$

Infinite dimensional highest weight representations of semi-simple Lie algebras.

Kazhdan-Lusztig conjecture (1979):

$$\mathrm{ch}L(x \cdot 0) = \sum_{y \geqslant x} (-1)^{\ell(x)-\ell(y)} h_{w_0 y, w_0 x}(1) \mathrm{ch}\Delta(y \cdot 0).$$

(A major generalisation of the Weyl character formula.)

Implications for representations of real Lie groups.

The Kazhdan-Lusztig conjecture was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

The Kazhdan-Lusztig conjecture was proved 1981 by Beilinson-Bernstein and Brylinski-Kashiwara using every trick in the book: algebraic differential equations "D-modules"; the Riemann-Hilbert correspondence (monodromy of differential equations); the theory of perverse sheaves (algebraic topology of singular varieties); Deligne's theory of weights (arithmetic geometry):

"The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne's method of weight filtrations which is capable to solve it.

So have a seat; it is going to be a long journey."

– Joseph Bernstein.

Kazhdan-Lusztig polynomials also play an important role in:

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);
iii) character formulae for simple modules for affine Lie algebras and quantum groups at roots of unity (conformal field theory);

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);
iii) character formulae for simple modules for affine Lie algebras and quantum groups at roots of unity (conformal field theory);
iv) the geometric Langlands correspondence (geometric Satake);

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);
iii) character formulae for simple modules for affine Lie algebras and quantum groups at roots of unity (conformal field theory);
iv) the geometric Langlands correspondence (geometric Satake);
v) symmetric polynomials, Macdonald polynomials, Littelwood-Richardson coefficients;

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);
iii) character formulae for simple modules for affine Lie algebras and quantum groups at roots of unity (conformal field theory);
iv) the geometric Langlands correspondence (geometric Satake);
v) symmetric polynomials, Macdonald polynomials, Littelwood-Richardson coefficients;
vi) many connections to combinatorics;

Kazhdan-Lusztig polynomials also play an important role in:

i) Lusztig's description of the character table of a finite group of Lie type.
ii) rational representations of reductive algebraic groups in positive characteristic (Lusztig's conjecture);
iii) character formulae for simple modules for affine Lie algebras and quantum groups at roots of unity (conformal field theory);
iv) the geometric Langlands correspondence (geometric Satake);
v) symmetric polynomials, Macdonald polynomials, Littelwood-Richardson coefficients;
vi) many connections to combinatorics;
vii) Kazhdan-Lusztig polynomials might end up helping us understand the HOMFLYPT polynomial of a link...

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Our proof makes heavy use of techniques developed by Wolfgang Soergel. We also use Hodge theoretic ideas of de Cataldo and Migliorini in a crucial way.

Theorem (Elias-W)

The Kazhdan-Lusztig positivity conjecture holds.

Our proof makes heavy use of techniques developed by Wolfgang Soergel. We also use Hodge theoretic ideas of de Cataldo and Migliorini in a crucial way.

We also obtain an algebraic proof of the Kazhdan-Lusztig conjecture, as well as many of the results mentioned on the previous slide.

Our goal is to understand the Kazhdan-Lusztig positivity conjectures:

$$\underline{H}_x = \sum h_{y,x} H_y \quad h_{y,x} \in \mathbb{Z}_{\geqslant 0}[v]$$

$$\underline{H}_x \underline{H}_y = \sum \mu^z_{x,y} \underline{H}_z \quad \mu^z_{x,y} \in \mathbb{Z}_{\geqslant 0}[v^{\pm 1}].$$

Our goal is to understand the Kazhdan-Lusztig positivity conjectures:

$$\underline{H}_x = \sum h_{y,x} H_y \quad h_{y,x} \in \mathbb{Z}_{\geqslant 0}[v]$$
$$\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z \quad \mu_{x,y}^z \in \mathbb{Z}_{\geqslant 0}[v^{\pm 1}].$$

A basic principle in combinatorics to show that a number is positive is to show that it is the cardinality of a set or the dimension of a vector space.

Our goal is to understand the Kazhdan-Lusztig positivity conjectures:

$$\underline{H}_x = \sum h_{y,x} H_y \quad h_{y,x} \in \mathbb{Z}_{\geqslant 0}[v]$$
$$\underline{H}_x \underline{H}_y = \sum \mu_{x,y}^z \underline{H}_z \quad \mu_{x,y}^z \in \mathbb{Z}_{\geqslant 0}[v^{\pm 1}].$$

A basic principle in combinatorics to show that a number is positive is to show that it is the cardinality of a set or the dimension of a vector space.

This is a baby example of *categorification*. One upgrades a number to an object in a category (in this example a set or vector space).

Given an abelian category $\mathcal{A}$ its *Grothendieck group* is

$$K_0(\mathcal{A}) = \bigoplus_{M \in \mathcal{A}} [M] / \left(\begin{array}{c} [M] = [M'] + [M''] \\ \text{for all short exact sequences} \\ 0 \to M' \to M \to M'' \to 0 \end{array} \right).$$

Given an additive category $\mathcal{B}$ its *split Grothendieck group* is

$$K_0^{\text{split}}(\mathcal{B}) = \bigoplus_{B \in \mathcal{B}} [B] / \left(\begin{array}{c} [B] = [B'] + [B''] \\ \text{whenever } B \cong B' \oplus B'' \end{array} \right).$$

Given an abelian category $\mathcal{A}$ its *Grothendieck group* is

$$K_0(\mathcal{A}) = \bigoplus_{M \in \mathcal{A}} [M] \Big/ \left(\begin{array}{c} [M] = [M'] + [M''] \\ \text{for all short exact sequences} \\ 0 \to M' \to M \to M'' \to 0 \end{array} \right).$$

Given an additive category $\mathcal{B}$ its *split Grothendieck group* is

$$K_0^{\text{split}}(\mathcal{B}) = \bigoplus_{B \in \mathcal{B}} [B] \Big/ \left(\begin{array}{c} [B] = [B'] + [B''] \\ \text{whenever } B \cong B' \oplus B'' \end{array} \right).$$

The passage from a category to its (split) Grothendieck group is the process of *decategorification*. Finding **interesting** inverses to this procedure is the process of *categorification*.

An example of categorification:

If G is a finite group, and Rep G denotes its category of finite dimensional complex representations, then character theory gives an isomorphism

$$K_0(\text{Rep } G) \otimes_{\mathbb{Z}} \mathbb{C} = Fun_{G_\Delta}(G, \mathbb{C})$$

where $Fun_{G_\Delta}(G, \mathbb{C})$ denotes the algebra of convolution invariant functions on G.

An example of categorification:

If G is a finite group, and Rep G denotes its category of finite dimensional complex representations, then character theory gives an isomorphism

$$K_0(\text{Rep } G) \otimes_{\mathbb{Z}} \mathbb{C} = Fun_{G_\Delta}(G, \mathbb{C})$$

where $Fun_{G_\Delta}(G, \mathbb{C})$ denotes the algebra of convolution invariant functions on G. This isomorphism gives the existence of an interesting basis for the function space (the basis of irreducible characters) which otherwise would be invisible.

Categorifying the Hecke algebra:

For simplicity assume that $W \subset O(V)$ is a finite reflection group. Set

$$R = \text{polynomial functions on } V$$

graded such that $V^* \subset R$ has degree 2. Because W acts on V, it also acts on R. For any simple reflection $s \in S$ consider $R^s \subset R$ the subalgebra of s-invariants.

Let $R - \text{biMod}$ denote the monoidal category of graded R-bimodules: $MM' := M \otimes_R M'$.

We denote by (1) the grading shift operator: $M(1)^i = M^{i+1}$.

For $s \in S$ let $B_s := R \otimes_{R^s} R$. Consider

$$\mathcal{B} = \begin{array}{c} \text{full Karoubian subcategory of } R-\text{biMod} \\ \text{generated by } B_s(m) \text{ for all } s \in S,\ m \in \mathbb{Z}. \end{array}$$

In other words, the objects $\mathcal{B}$ are the graded R-bimodule direct summands of bimodules of the form

$$B_s B_t \dots B_u = R \otimes_{R^s} R \otimes_{R^u} R \otimes \cdots \otimes_{R^u} R(m)$$

for arbitrary sequences $st \dots u$ and $m \in \mathbb{Z}$.

Let $K_0^{\text{split}}(\mathcal{B})$ denote the split Grothendieck group of $\mathcal{B}$. It is an algebra over $\mathbb{Z}[v^{\pm 1}]$ via $v[B] := B(1)$ and $[B'][B''] = [B'B'']$.

Soergel's categorification theorem (2005):

The split Grothendieck group of $\mathcal{B}$ is isomorphic to the Hecke algebra:

$$\mathrm{ch} : K_0^{\mathrm{split}}(\mathcal{B}) \xrightarrow{\sim} H$$

Soergel's categorification theorem (2005):

The split Grothendieck group of $\mathcal{B}$ is isomorphic to the Hecke algebra:

$$\mathrm{ch} : K_0^{\mathrm{split}}(\mathcal{B}) \xrightarrow{\sim} H$$

Soergel proves the existence of indecomposable bimodules B_x whose classes give a basis for $K_0^{\mathrm{split}}(\mathcal{B})$ and conjectures:

Soergel's conjecture:

$$\mathrm{ch}(B_x) = \underline{H}_x$$

Soergel's categorification theorem (2005):

The split Grothendieck group of $\mathcal{B}$ is isomorphic to the Hecke algebra:

$$\mathrm{ch} : K_0^{\mathrm{split}}(\mathcal{B}) \xrightarrow{\sim} H$$

Soergel proves the existence of indecomposable bimodules B_x whose classes give a basis for $K_0^{\mathrm{split}}(\mathcal{B})$ and conjectures:

Soergel's conjecture:

$$\mathrm{ch}(B_x) = \underline{H}_x$$

$\Rightarrow$Kazhdan-Lusztig positivity conjectures.

An important role in our inductive proof of Soergel's conjecture is played by certain much stronger statements about the bimodules B_x.

Set $\overline{B_x} := B_x \otimes_R \mathbb{R}$. This is a finite dimensional graded vector with left R action on the left. It is equipped with a non-degenerate symmetric form $\langle -, - \rangle$.

Example: if $w_0 \in W$ denotes the longest element then $B_{w_0} = R \otimes_{R^W} R$ and hence

$$\overline{B_{w_0}} = R/(R^W)^+$$

is the "coinvariant algebra": $(R^W)^+$ denotes the ideal of R generated by W-invariant polynomials of positive degree.

We show that $\overline{B_x}$ "looks like the cohomology of a smooth projective variety".

For any $\rho \in V^*$ in the interior of the fundamental alcove we have:

i) (Hard-Lefschetz theorem) $(\rho^i \cdot)$ gives an isomorphism

$$(\overline{B_x})^{\ell(x)-i} \to (\overline{B_x})^{\ell(x)+i}$$

1. (Hodge-Riemann bilinear relations) The restriction of the form $(\alpha, \beta) := \langle \alpha, \rho^i \beta \rangle$ to the kernel of ρ^{i+1} in $(\overline{B_x})^{\ell(x)-i}$ is definite.

What does this say for the coinvariant algebra?

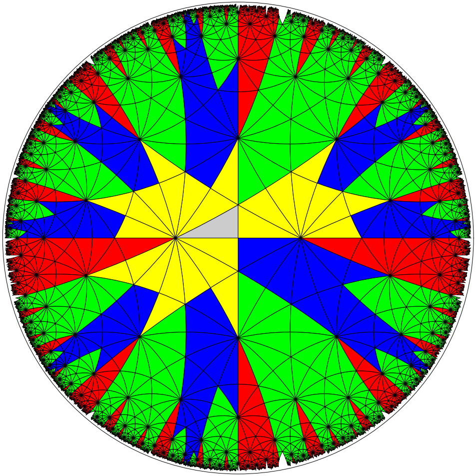
If W is a Weyl group then $R/(R^W)^+$ is isomorphic to the cohomology ring of the flag variety. Flag varieties are smooth projective varieties and these properties follow from classical Hodge theory.

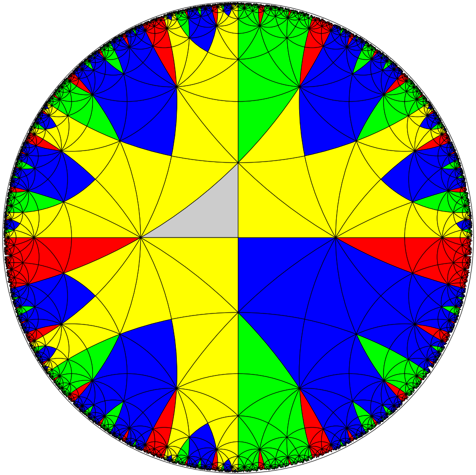
If W is not a Weyl group (for example the symmetries of the icosahedron), there is no algebraic variety with the coinvariant algebra as cohomology.

To any element of any Coxeter group W one has a space which looks like the cohomology of a smooth projective variety!

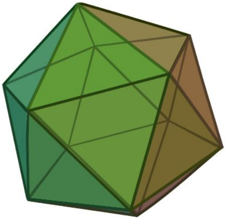
I will finish with two questions:

i) Is there any geometric interpretation of these spaces? (One can ask a similar question for the intersection cohomology of non-rational polytopes.)
ii) What does Kazhdan-Lusztig theory mean in the non-crystallographic case?





For more images of two-sided cells in hyperbolic groups see Paul Gunnell's web page.



Thanks for listening!