Kazhdan-Lusztig polynomials, and some Hodge theory II (jt. Ben Elias)

Lefschetz linear algebra
~~Hodge theory scaffolding~~

$H = H^\bullet$ f.d. graded $\mathbb{R}$-vector space

$\langle -,- \rangle : H^\bullet \times H^\bullet \to \mathbb{R}$ graded, non-degenerate symmetric form. (i.e. $\langle H^i, H^j \rangle = 0$ if $i \neq -j$).

$L : H^\bullet \to H^{\bullet+2}$ operator satisfying $\langle L\alpha, \beta \rangle = \langle \alpha, L\beta \rangle$ $\forall \alpha, \beta \in H$.

L <u>satisfies hard Lefschetz</u> (hL) if $L^i : H^{-i} \xrightarrow{\sim} H^i$ $\Longleftrightarrow$ L is the e in a unique $\mathfrak{sl}_2(\mathbb{R})$ $=\mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ action on H with $h\alpha = i\alpha$ for $\alpha \in H^i$.

Assume (hL): Set $P_L^{-i} := \ker L^{i+1} \subset H^{-i}$ (primitive subspace).

Then $H = \bigoplus_{\substack{i \geq 0 \\ i \geq j \geq 0}} L^j P_L^{-i}$ (decomp of $\mathfrak{sl}_2(\mathbb{R})$-rep.) $(*)$

Assume either $H^{odd} = 0$ or $H^{even} = 0$ (parity vanishing)

and let H^{min} be the smallest non-zero degree.

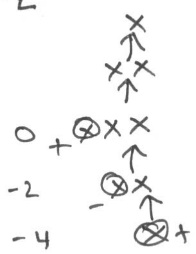
Lefschetz form on H^{-i}: $(\alpha, \beta)_L^{-i} := \langle \alpha, L^i \beta \rangle$.

<u>Hodge-Riemann relations</u>: for ~~m~~ $m \geq 0$ the restriction of $(-,-)_L^{-min+2m}$

(HR) to $P_L^{-min+2m}$ is $(-1)^m$-definite.

$\Updownarrow$ $(*)$ is orthogonal wrt $(-,-)_L^{-i}$

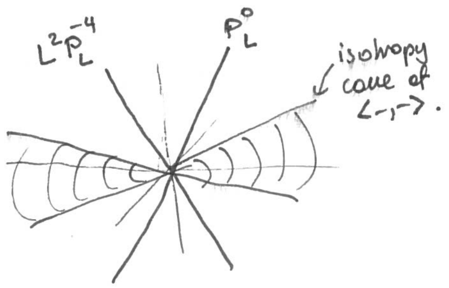
formula for the signatures of $(-,-)_L^{-i}$ on H^{-i} for all $i \geq 0$.



<u>Example</u>: $H^{odd} = 0$.

"all cohomology is seen in H^0" : $\tilde{P}_L^{i} := L^{i/2} P_L^{-i} \subset H^0$.

H^0:



subspaces move when we move L.

EARLIER

Main example: X smooth projective$/\mathbb{C}$

$H^\bullet(X) := H^{\dim_{\mathbb{C}} X + i}_{singular}(X; \mathbb{R})$.

$\langle -,- \rangle$ intersection form

$L = c_1(\mathcal{L})$ — ample class.

Two ~~tricks~~ general techniques for establishing (hL) and (HR):

~~Remark: In general (HR) is easy, (hL) is hard!~~

dCM: First establish (hL) (hard), then (HR) (easy).

Limit lemma: Let $y \mapsto L_y \in \mathrm{Hom}(H^\bullet, H^{\bullet+2})$ be a continuous family of Lefschetz operators satisfying hard Lefschetz for all $y \in [0, \infty)$.

If one L_{y_0} satisfies (HR) then they all do.

Proof: Consider the symmetric forms $(-,-)^{-i}_{L_y}$ on H^{-i}. ~~(HR) for some~~ L_y

(hL) $\Rightarrow$ all forms are non-degenerate.

(HR) for L_{y_0} $\Rightarrow$ we know the signature of $(-,-)^{-i}_{L_{y_0}}$.

$\Rightarrow$ Signature can't change in a family $\Rightarrow$ (HR) for all y. □

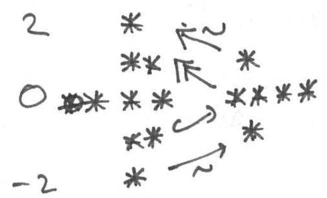
Weak Lefschetz: If $X_H \subset X$ is a smooth hyperplane section

then $i^* : H^i(X) \to H^{i+1}(X_H)$ is iso $\xrightarrow{\sim}$ for $i < -1$, injective $\hookrightarrow$ for $i = -1$.

dually: $i_* : H^i(X_H) \to H^{i+1}(X)$ is $\xrightarrow{\sim}$ for $i > 1$, $\twoheadrightarrow$ for $i = 1$.

Also $i_* i^* = L$.

$\Rightarrow$ hard Lefschetz except for degree 0. $L : H^{-1} \to H^1$.



But if $\alpha \in \ker L \Rightarrow i^*\alpha \in \ker L \subset H^0(X_H)$

~~Hence if $\Rightarrow i^*\alpha \neq 0 \overset{(HR)}{\Rightarrow} 0 \neq \langle i^*\alpha, i^*\alpha \rangle = \langle \alpha, L\alpha \rangle$.~~ #.

$$0 = \langle \alpha, L\alpha \rangle = \underset{\text{primitive}}{\langle i^*\alpha, i^*\alpha \rangle} \Rightarrow i^*\alpha = 0 \underset{(\text{wL.})}{\Rightarrow} \alpha = 0.$$

The following weakening is useful:
~~Difficult to imitate weak Lefschetz combinatorially~~.

Weak Lefschetz substitute: $\phi : H^\bullet \to W^{\bullet+1}$ degree one map.

L-equivariant, $\phi : H^{<0} \hookrightarrow W^{\leq 0}$, $\langle \alpha, L\beta \rangle = \langle \phi\alpha, \phi\beta \rangle$.

If W satisfies (HR) then $L^i : H^i \to H^{-i}$ is injective.

Remark: $X \Rightarrow$ (hL) if H has symmetric Betti numbers.

× Sometimes useful to ~~can~~ assume weaker statement: $L^i : W^{-i} \hookrightarrow W^i$, + (HR) on primitive classes.

We now return to bimodules:

Recall: $\mathfrak{h}/\mathbb{R}$ reflection rep of (W,S) (i.e. $s(v) = v - \langle \alpha_s, v\rangle \alpha_s^\vee$ $\forall s \in S$ for some $\alpha_s \in \mathfrak{h}^*$, $\alpha_s^\vee \in \mathfrak{h}$.)

Assume: $\{\alpha_s^\vee \mid s \in V\} \subset V$ linearly indep.

Choose $\rho \in V^*$ with $\langle \alpha_s^\vee, \rho\rangle > 0$ $\forall s \in S$ ("ample class")

$R = \mathcal{O}(V)$, $\deg V^* = 2$, $W \curvearrowright R$, $R^s \subset R$ s-invariants.

$\mathcal{B}$ = full additive graded Karoubian monoidal subcat of $R\text{-Mod-}R^{\mathbb{Z}}$ generated by $B_s := B \otimes_{R^s} B(1)$.
($\oplus$, $[m]$, $\overset{|}{\oplus}$, $\otimes$)

B_w := unique biggest summand of $BS(\underline{w}) := B_s B_t \cdots B_u$ for $\underline{w} = st \ldots u$ an red. exp. for w. $\Big\} \Rightarrow$ induces a form $\langle -,- \rangle$ on B_w

Soergel: $ch : [\mathcal{B}]_{\text{split}} \xrightarrow{\sim} H$. Soergel's conjecture: $ch(B_x) = \underline{H}_x$. $S(x)$.

Proof by induction: Fix $x \in W$, $xs > x$ and assume $S(<xs)$.

Master formula: $$\underline{H}_x \underline{H}_s = \underline{H}_{xs} + \sum_{\substack{y<x\\ ys<y}} \mu(y,x)\, \underline{H}_y$$

$$S(xs) \iff B_x B_s = B_{xs} \oplus \bigoplus_{\substack{y<x\\ ys<y}} B_y^{\oplus \mu(y,x)}$$

As in any additive Krull-Schmidt cat:

$$\begin{pmatrix} \text{mult. of} \\ B_y \overset{|}{\oplus} B_x B_s \end{pmatrix} = \text{rank of } (-,-)_y^{x,s} : \mathrm{Hom}(B_y, B_x B_s) \times \mathrm{Hom}(B_x B_s, B_y) \to \underset{\Downarrow\, S(y)}{\mathrm{End}(B_y) = \mathbb{R}}$$

Soergel's hom formula gives $(y < xs)$: $\mathrm{Hom}(B_y, B_x B_s) = \begin{cases} \mu(y,x) & \text{if } y<x, ys<y \\ 0 & \text{otherwise.} \end{cases}$

Hence: $S(x) \iff (-,-)_y^{x,s}$ non-degenerate.

$S(<xs) \Rightarrow$ each B_y and $B_x B_s$ is equipped with a canonical non-degenerate form.

$\rightsquigarrow$ forms $\langle -,- \rangle$ on $\overline{B_x B_s} := B_x B_s \otimes_R \mathbb{R}$. $\leftarrow$ carries Lefschetz operator given by left mult. by ρ.

Embedding thm:

$(-,-)_s^{\dots}$ is definite.

$$\mathrm{Hom}(B_y, B_{xs}) \overset{i}{\hookrightarrow} P_s^{-\ell(y)} \subset \left(\overline{B_x B_s}\right)^{-\ell(y)}$$

image of generator $c_{bot} \in B_y^{-\ell(y)}$

Moreover, i is an isometry wrt $(-,-)_y^{x,s}$ and $(-,-)_s^{-\ell(y)}$. (up to a >0 scalar).

Hence (restriction of definite is definite)

$$\begin{pmatrix} (HR) \text{ for } \overline{B_x B_s} \\ \text{and } \langle -,- \rangle, s \end{pmatrix} \implies S(xs).$$

Rmk: It seems very difficult to establish (HR) directly.

For $0 \leq y \in \mathbb{R}$ consider the Lefschetz operator:

$$L_y := (s-) \otimes \mathrm{id}_{B_s} + \mathrm{id}_{B_x} \otimes (ys-).$$

So $\cancel{L_y = \rho-}$). $L_0 = L$.

Thm: If $\overline{B_x}$ satisfies (HR) then $\overline{B_x B_s}$ satisfies (HR) wrt L_y for $y \gg 0$.

Proof: idea: $\overline{B_x B_s}$ looks like $\overline{B_x} \otimes H^\bullet(\mathbb{P}^1_{\mathbb{C}})$ as $y \to \infty$.

Limit lemma: $\implies$ we only need (HL) for L_y on $B_x B_s$ for $y \gg 0$.

Idea for $y = 0$:

1

$$\begin{array}{ccc} BS(\underline{xs}) & \xrightarrow[\text{degree } 1]{\Phi} & \bigoplus_{i=1}^{\ell(xs)} BS(\underline{xs}\hat{i}). \\ \cup & & \cup \\ B_x B_s & \xrightarrow{\Phi} & W \end{array}$$

2 (Rouquier complex).

wts ≥ 0.

Now use induction and weak Lefschetz substitute.