Kazhdan-Lusztig polynomials, Soergel bimodules and some Hodge theory I

(jt. Ben Elias)

Hecke algebra:

~~Background~~: G split finite reductive group $/\mathbb{F}_q$.

$G \supset B \supset T$ Borel, maximal torus.

in another colour

$H \otimes \mathbb{C}$

$v \mapsto 1/\sqrt{|\mathbb{F}_q|}$

~~Hecke algebra~~ $H = \mathrm{Fun}_{B\times B}(G, \mathbb{C})$ (Algebra under convolution).

Iwahori: $\mathrm{Fun}_{B\times B}(G,\mathbb{C})$ has a presentation "independent of q". using only (W,S), ~~the Coxeter system of~~ Weyl group of $G \supset T$.

$$\begin{cases} H \text{ free } \mathbb{Z}[v^{\pm 1}]\text{-module basis } \{H_x \mid x \in W\}. \\ H_x H_s = \begin{cases} H_{xs} & xs > x \\ (v^{-1}-v)H_x + H_{xs} \end{cases} \end{cases}$$

$\rightsquigarrow$ Hecke algebra of any Coxeter system.

Now let G be a complex reductive group $G \supset B \supset T$ as above.

$$\underset{\mathcal{H}}{\text{Hecke category}} = \text{semi-simple complexes} = \langle IC(\overline{BwB}) \mid w \in W \rangle_{\oplus, [1]} \subset D^b_{B\times B}(G;\mathbb{C})$$

monoidal category under convolution $*$.

(preserved by the decomposition theorem)

Soergel: $\mathbb{H}^* : \mathcal{H} \longrightarrow H^*_{B\times B}(pt)\text{-Mod}^{\mathbb{Z}} = R\text{-Mod}^{\mathbb{Z}}\text{-}R$

Hypercohomology

where $R = H^*_B(pt) = H^*_T(pt) = \mathcal{O}(\mathrm{Lie}\, T)$.

degree 2

Soergel: $\mathbb{H}^*$ is a fully-faithful monoidal functor.

The image of $\mathbb{H}^*$ is the category of Soergel bimodules $\mathcal{B}$.

Hence $\mathbb{H}^*$ gives a ~~monoidal equivalence~~ $\mathcal{H} \xrightarrow{\sim} \mathcal{B}$.
of monoidal additive categories

H is generated as an algebra by the indicator functions of $\overline{BsB} \subset G$.

$\mathcal{H}$ is generated by the extension by zero sheaves $\mathbb{C}_{\overline{BsB}}[1] \in D^b_{B\times B}(G)$.

One calculates:

$$\mathbb{H}^*(G, \mathbb{C}_{\overline{BsB}}[1]) = R \otimes_{R^s} R[1] =: B_s \in R\text{-Mod}^{\mathbb{Z}}\text{-}R.$$

Iwahori: Let (W,S) be the ~~Weyl group, simple reflections of~~ Coxeter system of $G \supset B \supset T$. $\leq$ Bruhat order

H free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_x \mid x \in W\}$ and multiplication $s \in S$, $w \in W$

$$H_x H_s = \begin{cases} H_{xs} & xs > x \\ (v^{-1}-v)H_x + H_{xs} & xs < x. \end{cases}$$

$\rightsquigarrow$ ~~$H(W,S)$~~ Hecke algebra for any Coxeter system.

This yields an elementary description of $\mathcal{B}$:

$$(*) \qquad \mathcal{B} = \begin{array}{l} \text{full additive subcategory of } R\text{-Mod}^{\mathbb{Z}}\text{-}R \text{ generated by} \\ \text{all direct summands of } BS(\underline{w}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u \\ \text{for all sequences } \underline{w} = st\ldots u. \end{array}$$

Hence to describe $\mathcal{B}$ (hence $\mathcal{H}$) one only needs Lie T, its W-action and commutative algebra.

[VERBAL: Soergel made the first step in freeing $\mathcal{H}$ from a concrete realisation.

Soergel's set up: let $\mathfrak{h}$ denote ~~a reflection rep~~ the geometric realisation of (W, S) with coroots $\{\alpha_s^\vee \mid s \in S\} \subset \mathfrak{h}$ and roots $\{\alpha_s \mid s \in S\} \subset \mathfrak{h}^*$.

~~(There is some flex~~

Note: there is some flexibility and subtlety in the choice of $\mathfrak{h}$. Soergel usually wants $\{\alpha_s^\vee \mid \cancel{s \in S}\} \subset \mathfrak{h}$ and $\{\alpha_s\}$ to be linearly independent "reflection faithful". All we need is that $\{\alpha_s^\vee\}$ be linearly independent, and that $\mathfrak{h} / \mathbb{R}$. More generally one can work with any reflection representation.

$\vdash$

Set $R = \mathcal{O}(\overset{\text{degree } 2}{\mathfrak{h}})$ and repeat the definition of $\mathcal{B}$ as in $(*)$.

$\rightsquigarrow$ category of Soergel bimodules assoc. to (W, S) (and $\mathfrak{h}$).

Elias-W: $\mathcal{B}$ has a presentation by generators and relations.

[VERBAL: I find it ~~essentially impossible~~ very difficult to do any calculations in Soergel bimodules. Generators and relations allow one to perform difficult calculations quite easily.]

$[\mathcal{B}]$ = split Grothendieck group of $[\mathcal{B}]$, i.e. $[B] = [B'] + [B'']$ if $B \cong B' \oplus B''$.

$[\mathcal{B}]$ is a $\mathbb{Z}[v, v^{-1}]$-algebra via $[B][B'] = [B \otimes_R B']$, $v[B] = [B(1)]$.

<u>Soergel's categorification thm:</u> $\exists !$ isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$c : H \xrightarrow{\sim} [\mathcal{B}] \qquad \text{sending } \underline{H}_s \mapsto [B_s].$$

where $\underline{H}_s = H_s + vH_{id}$.

Remark 1: $\boxed{\text{Suppose that } \mathfrak{h} = \mathrm{Lie}\, T \circlearrowleft W.}$

$\boxed{\text{Then } \mathcal{H} \cong \mathcal{B} \text{ and hence the indecomposable Soergel bimodules}}$

Let $\{\underline{H}_x \mid x \in W\}$ denote the Kazhdan-Lusztig basis of $\mathcal{H}$.

$$\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y \quad \text{with} \quad h_{y,x} \begin{cases} = 1 & \text{if } y = x \\ \in v\mathbb{Z}[v] & \text{if } y < x. \end{cases}$$

KL polys.

Then $\mathcal{H} \cong \mathcal{B}$ and the isomorphism sends $\underline{H}_x \longmapsto \mathbb{H}(IC(\overline{BwB}))$.

In this case Soergel's theorem follows from results of Kazhdan-Lusztig, Springer, MacPherson...

Remark 2: Here is an outline of Soergel's proof:

(2) $\underline{H}_s^2 = (v+v^{-1})\underline{H}_s$. $B_s \otimes_R B_s = R \otimes_{R^s} R \otimes_{R^s} R = B_s(1) \oplus B_s(-1)$ easy.

1) show that $\underline{H}_s \mapsto [B_s]$ is an algebra homomorphism. ~~Construct c~~ (involves proving relations for dihedral groups,... already quite tricky). braid relations quite tricky.

2) Construct a left inverse to c. i.e. what does it mean to "take the stalk" of a Soergel bimodule? Any Soergel bimodule B is an f.g. $R \otimes R$-Module $\to$ ~~quasi~~ coherent sheaf on $\mathfrak{h} \times \mathfrak{h}$.

$w \in W$ set $Gr_w = \{(x\lambda, \lambda) \mid \lambda \in \mathfrak{h}\}$. For any $I \subset W$ set $\Gamma_I B = \{b \in B \mid \mathrm{supp}\, b \in \bigcup_{x \in I} Gr_x\}$.

For all $w \in W$, $\Gamma_{\leq w} B / \Gamma_{< w} B = \bigoplus$ shifts of $\mathcal{O}(Gr_w)$.

OMIT

replace with notes below

taking multiplicities yields $ch(B) \in \bigoplus \mathbb{Z}_{\geq 0}[v^{\pm 1}] H_x \subset \mathcal{H}$.

Ex: B_s fits into an exact sequence $\mathcal{O}(Gr_{id}) \hookrightarrow B_s \twoheadrightarrow \mathcal{O}(Gr_s)$.

3) Classify the indecomposable objects in $\mathcal{B}$: ~~any two indecomposable~~ if B, B' ~~with support~~ are indecomposable and supported on $\bigcup_{x \leq w} Gr_x$ and are non-trivial along Gr_w then they are isomorphic up to a shift. (idempotent lifting argument. Not constructive!)

[Elias-W: alternative proof of this theorem ~~using~~ avoiding commutative algebra.

Remark 2: To prove this first one shows that $\underline{H}_s \mapsto [B_s]$ is an algebra homomorphism (braid relations are tricky).

Then it follows from:

Thm: For any reduced expression $\underline{w}$ for w there exists a unique summand $B_w \overset{\oplus}{\subset} BS(\underline{w})$ which does not occur as a summand of any smaller $BS(\underline{w}')$. Any indecomposable $B \in \mathcal{B}$ is isomorphic to a shift of B_w for some w. Hence $[\mathcal{B}] = \bigoplus_{w \in W} \mathbb{Z}[v^{\pm 1}][B_w]$.

Soergel also constructs explicit positive inverse $ch: [\mathcal{B}] \to \mathcal{H}$. (count multiplicities in "standard filtrations".)

Soergel's conjecture: $ch(B_x) = \underline{H}_x$: $S(x)$

$\Rightarrow$ $h_{y,x} \in \mathbb{Z}_{\geq 0}[v]$ (Kazhdan-Lusztig positivity conjecture)

$\Rightarrow$ Kazhdan-Lusztig conjecture { NB: Soergel's 1990 JAMS paper reduces KL conjecture to $\dim B_x \otimes_R \mathbb{C} = \sum_y h_{y,x}(1)$.

$\Rightarrow$ several other positivity conjectures and conjectures in Lie theory.

Thm (Elias-W) Soergel's conjecture holds.

Idea of proof: ~~in the geometric setting~~ work over $\mathbb{R}$

$$B_x = \mathbb{IH}^*(IC(\overline{BxB})) = \cancel{\mathbb{IH}^*_T(IC(\overline{BxB}/B))}$$

$$= \mathbb{IH}^*_T(\overline{BxB}/B)$$

$$\uparrow$$

$\Rightarrow$ $\overline{B_x} := B_x \otimes_R \mathbb{R} = \mathbb{IH}^*(\overline{BxB}/B)$ equivariant intersection cohomology of a ~~smooth~~ projective variety. (equivariantly formal)

Hence if we ~~take coefficients in $\mathbb{R}$~~ expect $\overline{B_x}$ to have non-trivial Hodge theoretic properties (hard Lefschetz, Hodge-Riemann bilinear relations).

We adapt ingenious arguments due to de Cataldo and Migliorini to prove that these Hodge theoretic properties always hold, regardless of whether (W,S) has a flag variety or not.

Remark: An essential assumption (not present in Soergel's original conjecture) is that ~~V~~ $\mathfrak{h}$ be defined over $\mathbb{R}$. Two days ago Ben and I discovered that Soergel's conjecture fails for certain ~~unusual~~ reflection representations of $\widetilde{A_n}$ which do not admit real forms! (Related to reps of quantum groups at a root of unity.)

There is a similar story for the "intersection cohomology" of non-rational polytopes.