Categorifying the Hecke algebra

Recall: we fix a connected reductive algebraic group G

$T \subset B \subset G$ Borus

(W, S) Weyl group and simple reflections.

$\mathcal{H} = \mathcal{H}(W,S) \ni \underline{H}_x$ Kazhdan-Lusztig basis

This talk will be about three questions:

1) $\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y$ geometric interpretation of $h_{y,x}$.

(KL Thm: $ch(IC(\overline{BxB/B})) = \underline{H}_x$).

2) $\underline{H}_x \underline{H}_y = \sum_{z \in W} h_{xyz} \underline{H}_z$ for self-dual $h_{x,y,z} \in \mathbb{Z}[v,v^{-1}]$, ~~$h_{x,y,z} = h_{x,y,z}$~~.

Conjecture (KL): $h_{x,y,z} \in \mathbb{N}[v^{\pm 1}]$.

3) Can one say anything at all about $h_{x,y,z}$?

Conjecture of Lusztig: "leading terms" in 2-sided cells admit simple descriptions in terms of the K-theory of finite group actions.

(Not an arbitrary question: important in the classification of characters of finite reductive groups.)

The character of an intersection cohomology complex

Recall: $G/B = \bigsqcup_{w \in W} BwB/B$, $BwB/B \simeq \mathbb{C}^{\ell(w)}$.

Bruhat decomp ~~Bruhat decomposition~~ $i_w : BwB/B \hookrightarrow G/B$

$D^b_W(G/B)$ = derived category constructible wrt to B-orbits.

$\mathcal{F} \in D^b_W(G/B)$: $i_w^* \mathcal{F} \cong P_w \cdot$ ~~$\mathbb{Q}_{BwB/B}$~~ C_w. $\quad C_w = \underline{\mathbb{Q}}_{BwB/B}[\ell(w)]$.

Def: $ch(\mathcal{F}) = \sum_{w \in W} P_w H_w$.

Set $s \in S$ $\quad P_s = \overline{BsB}$, $\quad \pi_s : G/B \to G/P_s$.

Eg: 1) $W = S_2$. $\quad \underline{H}_e, \underline{H}_s$. $\quad \cancel{\underline{H}_e^2 = \underline{H}_e}$ $\underline{H}_e = 1$.

$$\underline{H}_s^2 = (v+v^{-1})\underline{H}_s.$$

2) $W = S_3$. $\quad \underline{H}_e = 1, \quad \underline{H}_s^2 = (v+v^{-1})\underline{H}_s, \quad \underline{H}_t^2 = (v+v^{-1})\underline{H}_s.$

$$\underline{H}_s \underline{H}_t = \underline{H}_{st}, \quad \underline{H}_t \underline{H}_s = \underline{H}_{ts}, \quad \underline{H}_s \underline{H}_{st} = (v+v^{-1})\underline{H}_{st}, \quad \underline{H}_t \underline{H}_{ts} = (v+v^{-1})\underline{H}_{ts}.$$

$$\underline{H}_{st} \underline{H}_s = \underline{H}_s + \underline{H}_{sts}, \quad \underline{H}_{ts} \underline{H}_t = \underline{H}_{sts} + \underline{H}_t,$$

$$\underline{H}_{sts} \underline{H}_{st} = (v+v^{-1})^2 \underline{H}_{sts}, \quad \underline{H}_{sts}^2 = (v+v^{-1})(v^2+1+v^{-2})\underline{H}_{sts}. \quad \text{etc.}$$

$$= (v^3 + 2v + 2v^{-1} + v^{-3})\underline{H}_{sts}$$

3) $W = S_6$: $\quad \underline{H}_{121343} \underline{H}_{123245} = (v^2 + 2 + v^{-2})\underline{H}_{12134325} + (v^3 + 3v + 3v^{-1} + v^{-3})\underline{H}_{121324325}.$

Nice to do B_2 as well. (didn't have time)

$$\underline{H}_{123454} \underline{H}_{123423} = \underline{H}_{12132543} + \underline{H}_{12143543} + \underline{H}_{12325432} +$$

$$\underline{H}_{13243543} + (v+v^{-1})\underline{H}_{121325432} + \underline{H}_{1213243543} +$$

$$\underline{H}_{1214325432} + \underline{H}_{1324325432} + \underline{H}_{121324325432}.$$

Our first goal is to prove:

Thm: (KL) $\quad ch(IC(\overline{BwB}/B)) \cong \underline{H}_w$.

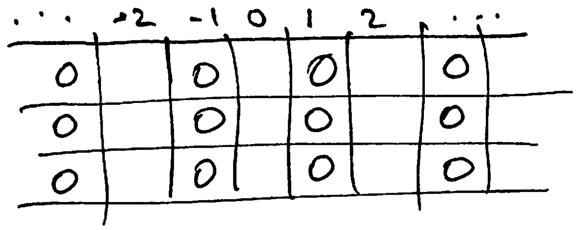
Def: $\mathcal{F}$ is called $*$-~~parity~~ even (odd) if ~~we can write~~ the ~~stalks~~ sheaves

$\cancel{\mathcal{F} \cong \mathcal{F}_{ev} \oplus \mathcal{F}_{odd}}$

Such ~~that the stalks of~~ $\mathcal{H}^i(\mathcal{F}) = 0$ for i even (resp. odd).

$\mathcal{F}$ is called $*$-parity if it can be decomposed as a direct sum $\mathcal{F} = \mathcal{F}_{ev} \oplus \mathcal{F}_{odd}$.

Remark: $\mathcal{F}$ even:



BxB/B

BwB/B

2) $\mathcal{F}$ even

$\Leftrightarrow ch(\mathcal{F}) = \sum P_w H_w$

and $v^{\ell(w)} P_w \in \mathbb{N}[v^2]$.

Lemma: Suppose we have a distinguished triangle in $D^b_W(G/B)$

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \xrightarrow{+}$$

with $\mathcal{F}'$, $\mathcal{F}''$ $*$-even. Then $\mathcal{F}$ is $*$-even and

$$ch(\mathcal{F}) = ch(\mathcal{F}') + ch(\mathcal{F}'').$$

Proof: Apply i_w^*: $\quad i_w^*\mathcal{F}' \to i_w^*\mathcal{F} \to i_w^*\mathcal{F}'' \xrightarrow{+}$

can omit.

Take cohomology:

$$\underbrace{\mathcal{H}^{2i}(i_w^*\mathcal{F}'')}_{=0} \to \mathcal{H}^{2i}(i_w^*\mathcal{F}') \to \mathcal{H}^{2i}(i_w^*\mathcal{F}) \to \mathcal{H}^{2i}(i_w^*\mathcal{F}'') \to \underbrace{\mathcal{H}^{2i+1}(i_w^*\mathcal{F}')}_{=0}$$

$\Rightarrow \mathcal{H}^{2i+1}(i_w^*\mathcal{F}) = 0$

and $\mathcal{H}^{2i}(i_w^*\mathcal{F}) = \mathcal{H}^{2i}(i_w^*\mathcal{F}') \oplus \mathcal{H}^{2i}(i_w^*\mathcal{F}'')$. $\square$

Def: Define $\Delta_w := i_{w!} C_w$.

Remark: $ch(\Delta_w) = H_w$.

Push-pull lemma: (Springer, Brylinski, MacPherson). (1/4 page)

Suppose $\mathcal{F}$ is $*$-parity:

$$ch(\pi_s^* \pi_{s*} \mathcal{F}[1]) \cong ch(\mathcal{F}) \underline{H}_s. \qquad (*)$$

Without loss of generality: $\mathcal{F}$ is $*$-even.

Proof: Set $N(\mathcal{F}) = \{ w \in W \mid i_w^* \mathcal{F} \neq 0 \}$. We induct on $|N(\mathcal{F})|$.

$|N(\mathcal{F})| = 1$: Then $\mathcal{F} \cong P \cdot \Delta_w$ for some $w \in W$ and $P \in \mathbb{N}[v^{\pm 1}]$.

Then it is enough to verify $(*)$ for Δ_w.

We do this below.

By induction on $|\{ w \mid i_w^* \mathcal{F} \neq 0 \}|$ reduces to Δ_w. Then an explicit calculation.

$|N(\mathcal{F})| > 1$: Choose $w \in W$ maximal with $i_w^* \mathcal{F} \neq 0$. Then standard d.t: $|N(i_* i^* \mathcal{F})| = N(\mathcal{F}) - 1$.

$$\underbrace{i_{w!} i_w^! \mathcal{F}}_{\cong\, P \cdot \Delta_w} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{+1} \qquad ch(\mathcal{F}) = ch\, P \cdot H_w + ch(i_* i^* \mathcal{F}).$$

where $i : \overline{\mathrm{Supp}\,\mathcal{F}} \setminus BwB/B \hookrightarrow G/B$ is the inclusion. ~~Now~~ Hence

we ~~clean~~ apply the lemma and conclude that $\pi_s^* \pi_{s*}(\ )$

$$\pi_s^* \pi_{s*} (P \cdot \Delta_w)[1] \to \pi_s^* \pi_{s*} \mathcal{F} \to i_* i^* \mathcal{F}[1] \xrightarrow{+1}$$

even. (induction). even.

Now apply the above lemma. $\square$.

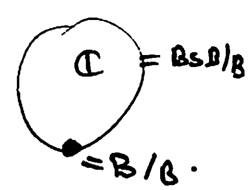
Lemma: We have dts:

$$\Delta_{ws} \to \pi_s^* \pi_{s*} \Delta_w[1] \to \Delta_w[1] \xrightarrow{+1} \qquad ws > w$$

$$\Delta_w[-1] \to \pi_s^* \pi_{s*} \Delta_w[1] \to \Delta_{ws} \xrightarrow{+1} \qquad ws < w$$

In particular:

$$ch(H_w \underline{H}_s) = \begin{cases} H_{ws} + v H_w & \text{if } ws > w, \\ H_{ws} + v^{-1} H_w & \text{if } ws < w. \end{cases}$$

$\mathbb{P} =$ 

Proof: reduce to $\mathbb{P}^1$: $\pi_s : \mathbb{P}^1 \to pt$.

- $\Delta_s \to \pi_s^* \pi_{s*} \underline{\mathbb{Q}}_{id}[1] = \underline{\mathbb{Q}}_{\mathbb{P}^1}[1] \to \Delta_{id}[1] \xrightarrow{+1}$

$\pi_{s*} i_{s!} \underline{\mathbb{Q}}_{\mathbb{C}}[1] = \pi_{s!} i_{s!} \underline{\mathbb{Q}}_{\mathbb{C}}[1] = \mathbb{Q}[-1]$

$$\Delta_s[-1] \to \underline{\mathbb{Q}}_{\mathbb{P}^1} \to \underline{\mathbb{Q}}_{id} \xrightarrow{+1} \qquad \square.$$

Thm: $\quad \mathrm{ch}(\mathrm{IC}(\overline{BwB/B})) = \underline{H}_w.$

Proof: ~~Recall that we have~~ ~~The proofs~~

Choose a reduced expression $\quad w = s_1 \dots s_m$:

$$\begin{tikzcd}
Bs_1B \times^B Bs_2B \times^B \dots \times^B Bs_mB/B \arrow[r, "\sim"] \arrow[d, hook] & BwB/B \arrow[d, hook] \\
BS(s_1, \dots, s_m) := P_{s_1} \times^B P_{s_2} \times^B \dots \times^B P_{s_m} \arrow[r, "\pi"] & \overline{BwB/B}
\end{tikzcd}$$

Resolution of singularities: (Bott-Samelson resolution).

Decomposition Thm:

$$\pi_* \underline{\mathbb{Q}}_{BS(s_1,\dots,s_m)}[m] \cong \cancel{\bigoplus_{x \in W} V_x \otimes IC_x} \; IC_w \oplus \bigoplus_{x<w} V_x \otimes IC_x$$

Using the big Cartesian diagram appearing in Laurent's talk:

$$\rightsquigarrow \quad \mathrm{ch}(\pi_* \underline{\mathbb{Q}}_{BS(s_1,\dots,s_m)}) = \underline{H}_{s_1} \cdots \underline{H}_{s_m} \quad \text{self-dual.}$$

Hence:

$$\mathrm{ch}(IC_w) + \underbrace{\sum_{x<w} \mathrm{ch}(V_x)\,\mathrm{ch}(IC_x)}_{\text{self-dual by induction.}} = \underline{H}_{s_1} \cdots \underline{H}_{s_m}.$$

$\Rightarrow$ $\mathrm{ch}(IC_w)$ self-dual. $\quad$ IC-conds $\Rightarrow$ $\mathrm{ch}(IC_w) = \underline{H}_w.$ $\quad \square$

The Hecke Category

We have found a geometric meaning for $\underline{H}_w$. Can we multiply them?

Remember where the Hecke algebra "came from":

$$\mathcal{H}(W,S) = \mathrm{Fun}_{B\times B}(G;\mathbb{C}). \qquad G, B/\mathbb{F}_q.$$

We can describe convolution of functions as follows (problem with dividing by $|B|$). Common problem in categorification

$$G\times G \xrightarrow{q} G\times^B G \xrightarrow{m} G.$$

$$\begin{array}{cc} \swarrow p_1 & \downarrow p_2 \\ G & G \end{array}$$

Given $B\times B$-invariant ~~functions~~ sheaves f, g on G. where $h \in \mathrm{Fun}(G\times^B G)$ is such

$$f * g := m_! h \qquad \text{where } h \in \mathrm{Fun}(G\times^B G) \text{ is such that}$$

$$q^* h = p_1^* f \otimes p_2^* g.$$

$$(m_! h)(g) = \sum_{g'\in m^{-1}(g)} h(g').$$

"integration over the fibres".

$$\mathcal{H} = p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$$

When we categorify we want to be able to find $\mathcal{H}$.
Moreover this should be a ~~functorial~~ functor.

The solution is given by the equivariant derived category (Bernstein–Lunts).
One has the six functors for G-equivariant maps, + ~~operations of~~ restriction and induction functors changing group + quotient equivalence: $G \curvearrowright X$, $N \trianglelefteq G$ normal, $N \curvearrowright X$ free

$$D^b_G(X) \xrightarrow[q]{\sim} D^b_{G/N}(X/N).$$

Def:

$$D^b_{B\times B}(G) \times D^b_{B\times B}(G) \longrightarrow D^b_{B\times B}(G)$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} * \mathcal{G}.$$

$$m_* \, q_* \, \mathrm{res}^{B^3}_{B^4}(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}) =: \mathcal{F} * \mathcal{G}$$

~~Let $\mathcal{C}$~~

$\mathcal{HC}$ = full additive subcat of $D^b_{B\times B}(G)$ generated by shifts intersection cohomology complexes.

<u>Thm</u>: 1) $\mathcal{HC}$ is preserved under $*$ (by the decomp. thm)

$$D^b_{B\times B}(G) \xrightarrow{q} D^b_B(G/B) \xrightarrow{For} D^b_W(G/B)$$

2) ~~ch~~ given $\mathcal{F}, \mathcal{G} \in \mathcal{HC}$,

$$ch(\mathcal{F} * \mathcal{G}) \cong ch(\mathcal{F})\, ch(\mathcal{G}).$$

3) $\mathcal{HC}$ is a rigid additive tensor category.

$(\mathcal{HC}, *)$

$\rightsquigarrow K_0$ "Categorification".

$(\mathcal{H}, \cdot)$.

<u>Proof</u>: 1) decomp. thm. 2) very similar to proof $ch(IC_w) = \underline{H}_w$.

3) straightforward. (observe that $(\underline{\mathbb{Q}}_{P_s}[1], \underline{\mathbb{Q}}_{P_s}[1])$ is a dual pair).

Fix a 2-sided cell: $\mathcal{C} \rightsquigarrow$ semi-simple tensor category.

<u>Tensor categories associated to cells:</u> ~~Pa: $\mathcal{HC}$ is not~~ $\le$

Let $\mathcal{C}$ be a two sided cell. The to $\mathcal{C}$ one can associate a

~~number~~ $a(z) = \max_{x,y} \deg h_{x,y,z}$ for $z \in \mathcal{C}$ $h_{x,y,z} = \gamma_{x,y,z} v^{a(z)} + \dots$

Then: given $Perv_{\le c} / P_{<c}$

$Perv_{\le c} / Perv_{<c}$.

Let $\mathcal{HC}_0 = \mathcal{HC} \cap Perv$

$\mathcal{HC}_{\le c} = \langle IC_x \mid x \in \{\le c\} \rangle_{\oplus}$.

$\mathcal{HC}_{<c}$ $\mathcal{P}^{ss}_{\le c}$

<u>Thm</u>: ~~Fs~~ (Lusztig)

~~$\mathcal{HC}_{\le c} / \mathcal{HC}_{<0}$~~

The product ~~$(\mathcal{P}^{ss}_{\le c} / \mathcal{P}^{ss}_{<c}, *_c)$~~

$$A *_c B = {}^p\mathcal{H}^{a(c)}(A * B)$$

makes $\mathcal{P}^{ss}_{\le c} / \mathcal{P}^{ss}_{<c}$ into a semi-simple tensor category.
rigid

Tensor categories associated to cells:

Fix a two-sided cell $\underline{c}$ $\rightsquigarrow$ semi-simple tensor category.

Let $z \in \underline{c}$ and define

$$a(\underline{c}) = \max_{x,y} \deg h_{x,y,z}.$$

where $z \in \underline{c}$ is fixed (indep. of $z \in \underline{c}$).

Set:

$$\mathcal{D}^{ss}_{\leq c} = \langle IC_x \mid x \in \{\leq c\} \rangle_{\oplus} \subset \mathrm{Perv}.$$

$$\mathcal{D}^{ss}_{< c} = \quad \dots$$

Thm: (Lusztig) Given $A, B \in \mathcal{D}^{ss}_{\leq c}$ if we define

$$A \ast_c B := {}^p\mathcal{H}^{a(c)} (A \ast B).$$

Then $\mathcal{D}^{ss}_{\leq c} / \mathcal{D}^{ss}_{< c}$ is a semi-simple rigid tensor category.

$$\mathcal{D}^{ss}_{\leq c} / \mathcal{D}^{ss}_{< c} = I_c$$

Thm: (Lusztig, Bezrukavnikov – Finkelberg – Ostrik)

If c is not exceptional then

$$I_c \xrightarrow{\sim} \mathrm{Coh}_{P(c)} (Y(c), Y(c)).$$

$Y(c)$: finite $P(c)$-set.

Only 3 exceptional cells, (in E_7, E_8).

Thm (BFO)

character sheaves in family c $\longleftrightarrow$ simple objects in $Z(I_c)$ (Drinfeld centre.) $\cancel{Z(\mathrm{Coh}_{P(c)}(Y(c), Y(c)))}$

$\rightsquigarrow$ more or less explicit description of character sheaves.