Recall from Daniel's talk: $X \rightsquigarrow D(X) :=$ derived category of sheaves of $\mathbb{Q}$-vector spaces on X.

X complex variety $\rightsquigarrow Sh(X, \mathcal{A}b)$ abelian category of sheaves of abelian groups. (X has its complex / metric topology).

Given any abelian category $\mathcal{A} \rightsquigarrow D(\mathcal{A})$ derived category of $\mathcal{A}$.

(objects: complexes in $\mathcal{A}$
morphisms: morphisms of complexes $[\text{quasi-isomorphisms}]^{-1}$)

Scared? Try "Derived categories for the working mathematician" by Richard Thomas.

<u>Motivation:</u> 1) it is interesting and important to study sheaf cohomology etc.

<u>without $D^b(X)$:</u>

Recall: $\mathcal{F}/X$ $\quad H^\cdot(X,\mathcal{F})$

- take an injective resolution $\mathcal{F} \to I^\bullet$
- apply our functor $\Gamma(X, I^\bullet)$
- take cohomology: $H^i(X,\mathcal{F}) := H^i(\Gamma(X, I^\bullet))$

$\boxed{\text{why independent of resolution?}}$

<u>with $D(X)$:</u>

in the language of derived categories: $X \xrightarrow{\pi} pt$.

$\mathcal{F} \in D(X) = D(Sh(X, \mathcal{A}b))$.

$R\pi_*\mathcal{F} \in$ ~~$R\Gamma(X,\mathcal{F})$~~ $\in D(pt) = D(\mathcal{A}b)$.

bad! $\to H^i(R\Gamma(X,\mathcal{F})) = H^i(X)$.

$\boxed{\text{independence of resolution built in.}}$

2) ... spectral sequences for compositions of derived functors $\longleftrightarrow$ relations like $f_* \circ g_* = (f \circ g)_*$.

<u>Six operations</u>

Recall that we have bifunctors $\otimes$, $\mathcal{H}om$ on $Sh(X; \mathcal{A}b)$.

$\rightsquigarrow \; - \overset{L}{\otimes} -, \; R\mathcal{H}om$ on $D(X)$.

Given a map: $X \xrightarrow{f} Y$

<u>Daniel's talk:</u>

$$Sh(X) \underset{f^*}{\overset{f_*, f_!}{\rightleftarrows}} Sh(Y)$$

$\Downarrow$ derive

$$D(X) \underset{f^*, f^!}{\overset{f_*, f_!}{\rightleftarrows}} D(Y)$$

<u>Adjunctions:</u> (f^*, f_*), $(f_!, f^!)$.

In general $f^!$ is difficult to calculate. Two cases where it's "easy":

1) $f: Z \hookrightarrow X$ inclusion of a locally closed subvariety:
$f^!\mathcal{F} =$ sections with support on Z.

2) $f: X \to Y$ smooth fibration of rel. $\dim_{\mathbb{C}} d$: $\quad f^! \cong f^*[2d]$.

Don't forget standard triangles:

$U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X \underset{\text{closed}}{\overset{i}{\hookleftarrow}} Z$

$j_!j^! \to id \to i_*i^* \xrightarrow{+}$ $\quad$ ($\pi_{X!}$: long ex. seqn of H_c)

$i_!i^! \to id \to j_*j^* \xrightarrow{+}$ $\quad$ (π_{X*}: long exact sequence of $H^\cdot$)

Perverse sheaves on G/B

G connected reductive algebraic group, $B \subset G$ Borel subgroup.

$T \subset B$ Maximal torus

W = Weyl group

$$G/B = \bigsqcup_{w \in W} BwB/B \qquad BwB/B \cong \mathbb{A}^{\ell(w)}$$

Bruhat decomposition.

$i_w : BwB/B \hookrightarrow G/B$.

This is a nice stratification:

$$D^b_W(G/B).$$

$$C_w = \underline{\mathbb{Q}}_{BwB/B}(\ell(w))$$

Write: $\Delta_w := i_{w!}$ ~~$\underline{\mathbb{Q}}_{BwB/B}(\ell(w))$~~ $i_{w!}$

Given any $P = \sum a_i v^i \in \mathbb{N}[v^{\pm 1}]$ and $\mathcal{F} \in D^b$ set $P \cdot \mathcal{F} = \sum_i P[i]^{\oplus a_i}$.

$D^b_{const}(BwB/B)$ is ~~generated~~ semi-simple and generated by C_w.

Hence, $\mathcal{F} \in D^b_W(G/B)$ $\quad i_w^* \mathcal{F} \cong P_w \cdot C_w$ for some $P_w \in \mathbb{N}[v^{\pm 1}]$.

Def: recall $\mathcal{H}$ from last time.

$$ch : D^b_W(G/B) \longrightarrow \mathcal{H}.$$

$$\mathcal{F} \longmapsto \sum_{w \in W} P_w H_w.$$

Remark: $\mathcal{F} \in {}^pD^{\leq 0} \iff ch(\mathcal{F}) \in \bigoplus \mathbb{Z}[v] H_w$.

Hence $\mathcal{F}$ is perverse $\iff ch(\mathcal{F}), ch(\mathbb{D}\mathcal{F}) \in \bigoplus \mathbb{Z}[v] H_w$.

Thm: (KL)

$$ch(IC(\overline{BwB/B})) = \underline{H}_w.$$

Duality: ~~what does~~ how is Poincaré duality "upgraded" to $\mathcal{D}^b(X)$?

First $X = pt$: $\mathbb{D}(X) = \mathcal{D}^b(\mathrm{Vect}) \cong$ graded vector spaces. (explains why taking cohomology is not such a bad operation).

We would expect ~~duality~~ $\mathbb{D}$ = duality on graded vector spaces.

Only has nice properties if v.s. are finite dimensional in each degree.
$\rightsquigarrow \mathbb{D}^2 \cong \mathrm{id}$ etc.

Moral: we need to impose finiteness on ~~our vector sp~~ $\mathcal{D}(X)$.

Given $\mathcal{F} \in \mathcal{D}(X) \rightsquigarrow \mathcal{F} = (\dots \xrightarrow{d_{i-1}} \mathcal{F}^i \xrightarrow{d_i} \mathcal{F}^{i+1} \to \dots)$.

set $\mathcal{H}^i(\mathcal{F}) = \ker d_i / \operatorname{im} d_{i-1}$. (a sheaf).

Def: 1) An object $\mathcal{F} \in$ ~~D(X)~~ $\mathrm{Sh}(X; \mathbb{Q})$ is constructible if there exists a finite partition

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

on X into locally closed subsets such that $\mathcal{F}|_{X_\lambda}$ is (locally constant = local system) for all $\lambda \in \Lambda$.

2) $\mathcal{D}^b_c(X)$ = full subcategory of $\mathcal{D}(X)$ consisting of complexes $\mathcal{F}$ such that:
- $\mathcal{H}^i(\mathcal{F}) = 0$ for $|i| \gg 0$ "bounded"
- $\mathcal{H}^i(\mathcal{F})$ constructible for all i.

Eg: $\mathcal{D}^b_c(pt) \cong$ finite dim graded vector spaces.

Given $X \xrightarrow{\pi_X} pt$ set $\omega_X := \pi_X^! \underline{\mathbb{Q}}_{pt}$ "dualising sheaf".

Eg: X ~~manifold~~ (orientable) manifold $\dim_{\mathbb{R}} = n$

$$\omega_X \cong or[n] \quad (\cong \underline{\mathbb{Q}}_X).$$

In general stalks of $\omega_X \longleftrightarrow$ local cohomology of X at x.

~~Example~~ Remark:
All functors preserve $\mathcal{D}^b_c$ ξ: says something very strong about the topology of algebraic maps.
(Eg: find a map $\mathbb{R} \xrightarrow{\pi} S^1$ such that $\pi_* \underline{\mathbb{Q}}_{\mathbb{R}}$ is not constructible!)
Origin of the name.

Set

$$\mathbb{D} := R\mathcal{H}om(-, \omega_X).$$

<u>Perverse sheaves</u>: given a local system ~~on~~ $\mathcal{L}$ on X_λ

$$\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2d_\lambda] \qquad \text{where } d_\lambda = \dim_{\mathbb{C}} X_\lambda.$$

~~Want a cat~~ $\rightsquigarrow \quad \mathbb{D}(\mathcal{L}oc(X_\lambda)[d_\lambda]) = \mathcal{L}oc(X_\lambda)[d_\lambda].$

Would like a category ~~where~~ with a nice duality.

<u>Idea</u>: "glue" together $\mathcal{L}oc(X_\lambda)[d_\lambda]$ for each stratum.

Set:

$$^p\mathcal{D}^{\leq 0}(X) = \{ \mathcal{F} \in D^b_\Lambda(X) \mid \mathcal{H}^i(i_\lambda^* \mathcal{F}) = 0 \text{ for } i > -d_\lambda \}.$$

		$-d_\lambda$			0	
X_λ	$*$	$*$	0	0	0	
$\vdots$	$*$	$*$	$*$	0	0	
	$*$	$\vdots$	$\vdots$		0	
X_0	$*$				$*$	0

<u>Set</u>:

$$\mathrm{Perv}_\Lambda(X) := \{ \mathcal{F} \mid \mathcal{F}, \mathbb{D}\mathcal{F} \in {}^p\mathcal{D}^{\leq 0}(X) \}.$$

<u>Thm</u>: a) $\mathrm{Perv}_\Lambda(X)$ is an abelian category. ~~Every object~~

b) Every object $\mathcal{F} \in \mathrm{Perv}_\Lambda(X)$ is of finite length.

c) $\mathrm{Perv}_\Lambda(X)$ is preserved by duality.

d) One has a bijection

$$\left\{ \begin{matrix} \text{simple objects} \\ \text{in Perv} \end{matrix} \right\} / \cong \quad \longleftrightarrow \quad \text{pairs } (X_\lambda, \mathcal{L}) \text{ where } X_\lambda \text{ is a stratum and } \mathcal{L} \text{ is an irreducible local system on } X_\lambda.$$

$$IC(\overline{X_\lambda}, \mathcal{L}) \longleftarrow\!\shortmid \; (X_\lambda, \mathcal{L})$$

<u>Remarks</u>: 1) a) is a real miracle. Discovered thanks to D-modules.

2) b) Another way of motivating perverse sheaves: they are those sheaves for which the Lefschetz hyperplane theorem holds "universally".

$$\mathbb{D} : D^b_c(X) \longrightarrow D^b_c(X)^{op}.$$

Has the following properties:

$$\mathbb{D}^2 \simeq id \qquad \text{(duality)}.$$

$$\mathbb{D} f_* \simeq f_! \mathbb{D}$$

$$\mathbb{D} f^! \simeq f^! \mathbb{D}$$

~~base~~ base change:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "g"'] & Y \end{tikzcd}$$

$$g^* f_! \cong \tilde{f}_! \tilde{g}^*.$$

<u>Example:</u> X smooth, $\dim_{\mathbb{C}} = d$. Then

$$\mathbb{D}\pi_* \underline{\mathbb{Q}}_X \cong \pi_! \mathbb{D}\underline{\mathbb{Q}}_X \cong \pi_! \underline{\mathbb{Q}}_X [2n].$$

Take $\mathcal{H}^{-i}$:

$$H^i(X)^* \cong H_c^{2n-i}(X). \qquad \text{Poincare \underline{duality}.}$$

If X is singular, we can detect the failure of Poincare duality <u>locally</u>!

<u>The character of a constructible complex:</u> Fix a (nice) stratification

$$(*) \qquad X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

$\leftarrow \; (\mathbb{D} D^b_\Lambda(X) = D^b_\Lambda(X)).$

write $D^b_\Lambda(X) =$ full subcat. of $D^b_c(X)$ constr. wrt to $(*)$.

$i_\lambda : X_\lambda \hookrightarrow X$ inclusion.

Then we ~~can~~ ~~given~~ can consider $\mathcal{H}^i(i_\lambda^* \mathcal{F})$

	-2	-1	0	1	2	
X_{λ_1}			$\mathcal{L}_2$	$\mathcal{L}_1^1$	$\mathcal{L}_1^2$	
X_{λ_2}						
X_{λ_3}						

(Does not determine complex up to isomorphism, but is still very useful!)

Eg: $X = \mathbb{P}^1\mathbb{C} = \mathbb{C} \sqcup \{pt\}.$

$\underline{h}_{\mathbb{P}^1}$

		0	
pt	0	$\mathbb{Q}_{pt}$	0
$\mathbb{C}$	0	$\mathbb{Q}_{\mathbb{C}}$	0

$j_* \underline{\mathbb{Q}}_{\mathbb{C}}$

	-1	0	1
pt		$\mathbb{Q}$	$\mathbb{Q}$
$\mathbb{C}$		$\mathbb{Q}$	

(<u>Exercise</u>!).

Intersection cohomology complexes:

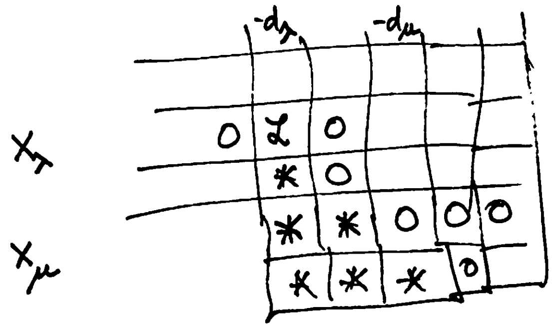
Thm: $IC(\overline{X}_\lambda, \mathcal{L})$ is uniquely determined by 1) $i_\lambda^* IC(\overline{X}_\lambda, \mathcal{L}) \cong \mathcal{L}$

2) $\mathcal{H}^i(i_\mu^* \, \mathcal{F}) = 0$ for $i \geq -d_\lambda$. $\mathcal{F} = IC(\overline{X}_\lambda, \mathcal{L})$.

2) ~~$\mathcal{H}^i(i_\mu^* \, \mathbb{D}IC(\overline{X}_\lambda, \mathcal{L})) = 0$ for $i \geq -d_\lambda$.~~

$\mathcal{H}^i(i_\mu^* \mathcal{F}) = \mathcal{H}^i(i_\mu^* \mathbb{D}\mathcal{F}) = 0$ for $i \geq -d_\lambda$. for $\mu \neq \lambda$.

Picture: IC, and $\mathbb{D}IC$.



Decomposition theorem

We can treat $Perv_\Lambda(X)$ $\subset D_c^b(X)$ "as if" it ~~were~~ where $Sh(X)$. (t-structures)

$\rightsquigarrow$ ~~cohomology~~ any $\mathcal{F} \in D_c^b(\mathcal{F}) \rightsquigarrow {}^p\mathcal{H}^i(\mathcal{F})$ "perverse cohomology".

Eg: easy on a stratum.

Call a $\mathcal{F} \in D_c^b(X)$ semi-simple if $\mathcal{F} \cong \bigoplus$

1) $\mathcal{F} \cong \bigoplus {}^p\mathcal{H}^i(\mathcal{F})[-i]$.

2) each ${}^p\mathcal{H}^i(\mathcal{F})$ is semi-simple.

Decomposition Thm: $f : \widetilde{X} \to X$ proper, $\widetilde{X}$ - smooth.

Then $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}]$ is semi-simple.

That is:
$$f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}] = \bigoplus_{X_\lambda, \mathcal{L}} V^\circ_{\lambda, \mathcal{L}} \otimes IC(X, \mathcal{L}).$$

Why amazing?

1) class of $\{IC(X, \mathcal{L})\}$ preserved under all proper maps.

2) ~~exposes very~~ applies many deep topological properties of algebraic maps. (I would be very happy to rave on about this!)