s) 213 | $\begin{smallmatrix}1 & 3\\ 2\end{smallmatrix}$ $\begin{smallmatrix}1 & 3\\ 2\end{smallmatrix}$

st 231 | $\begin{smallmatrix}1 & 3\\ 2\end{smallmatrix}$ $\begin{smallmatrix}1 & 2\\ 3\end{smallmatrix}$

Same right cell.

Lecture 1: Hecke Algebras, the Kazhdan-Lusztig basis and cells.

1.1 Coxeter Systems

Definition: A Coxeter system is a pair (W, S) where W is a group and $S \subset W$ is a generating set such that W has a presentation:

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle \quad \text{where } m_{ss} = 1, \ m_{st} \in \{2, 3, \dots, \infty\} \text{ for } s \neq t.$$

~~If $m_{st} = \infty$ we mean st has infinite interpret $(st)^{m_{st}} = 1$ to see be the empty relation.~~

$$= \langle S \mid s^2 = 1, \ \underbrace{st\dots}_{m_{st}} = \underbrace{ts\dots}_{m_{st}\text{-terms}} \rangle$$

"braid relations"

$m_{st} = \infty \Rightarrow$ no relation!

Examples: 1) Let $W = \mathfrak{S}_n$ be the symmetric group of $\{1, \dots, n\}$.

Set $S = \{s_i = (i, i+1) \mid 1 \leq i < n\}$. Then

$$W = \langle S \mid s_i^2 = 1, \ s_i s_j = s_j s_i \text{ if } |i-j| > 2, \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle.$$

2) Let V be a Euclidean vector space 2-dimension. Let $\Gamma \subset O(V)$ be a finite subgroup generated by reflections.

Exercise: a) Let V be a Euclidean vector space. ~~and s and t two reflections.~~ ~~Show that $(st)^{m_{st}} = 1$ for some~~ Show that if $\Gamma \subset O(V)$ is finite then

$\Gamma = \langle s, t \mid s^2 = t^2 = (st)^{m_{st}} = 1 \rangle$. (A "dihedral group").

b) Why is 1) a special case of 2)?

> Oral: you can get between any two reduced expressions with braid relations

We will need the following concepts:

~~The length of $w \in W$ is the length~~ $w = s_1 s_2 \dots s_k$ with $s_i \in S$

- A **reduced expression** is an expression for $w \in W$ in the generating set S of minimal length. An important fact is
- The **length** $\ell(w)$ of $w \in W$ is the length of a reduced expression for W.
- ~~The Bruhat order on W is the partial order generated by~~
- ~~The reflections~~

~~The reflections $T \subset W$ are the elements $T = \bigcup_{w \in W} w S w^{-1}$.~~

- The **Bruhat order** on W is the partial order generated by

$$tx \leq x \quad \text{if } t \in T \text{ and } \ell(tx) < \ell(x).$$

Alternatively $x \leq y$ if x can be obtained from a reduced expression of y by "cancelling terms".

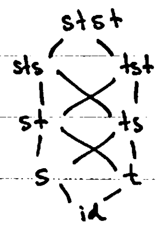
$$\boxed{\begin{array}{l} y = s_1 \dots s_k \\ x = s_{i_1} \dots s_{i_\ell} \text{ for } 1 \leq i_1 \leq \dots \leq i_\ell \leq k. \end{array}}$$

$y = s_{i_1} s_{i_2} \dots s_{i_k}$

if $y = s_{i_1} s_{i_2} \dots s_{i_k}$ $\quad x = s_{i_1} \cancel{s_{i_j}} s_{i_k} \Rightarrow x \leq y.$

E

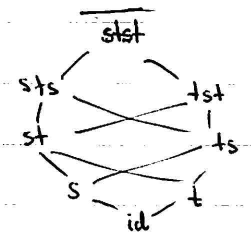
Examples: $W = \langle s,t \mid (st)^{4} = 1\rangle$.

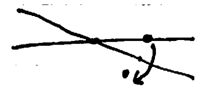


Bruhat order.

Exercise: (if these things are new!)

Bruhat order on S_3, S_4.





1.2 The Hecke algebra

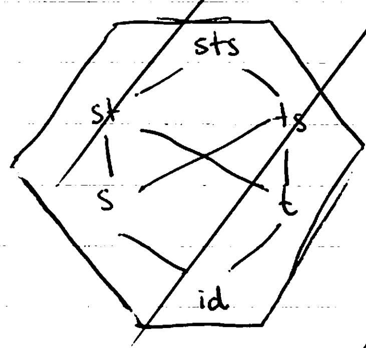
Consider $\mathcal{H}$ the unital associative algebra over $\mathbb{Z}[v^{\pm 1}]$ ~~subject to the relations~~ generated by $\{H_s \mid s \in S\}$ subject to the relations

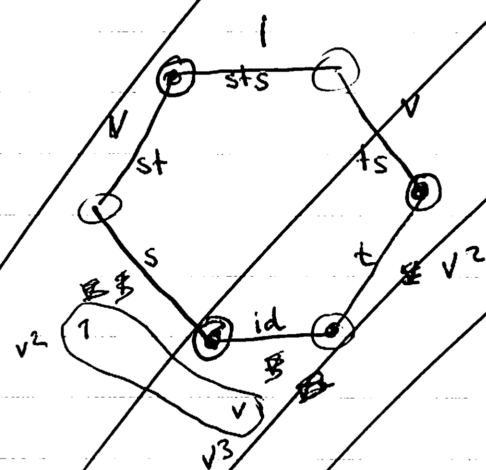
$$H_s^2 = (v^{-1}-v)H_s + 1$$

$$\underbrace{H_sH_t\cdots}_{m_{st}\text{-terms}} = \underbrace{H_tH_s\cdots}_{m_{st}\text{-terms}}$$

Thm (Bourbaki, Groupes et ...)

Example $W = \langle s,t \mid s^2 = t^2 = 1,\ sts = tst\rangle$.





$$\left(\frac{1}{\sqrt q}T_s\right)^2 = \frac{1}{q}(q-1)T_s + \frac{q}{q}\cdot 1$$

$$= (\sqrt q - \tfrac{1}{\sqrt q})(\tfrac{1}{\sqrt q}$$

Examples: • $W = \mathfrak{S}_n$ Symmetric group on $n\{1,\dots,n\}$.

$S = \{s_i = (i,i+1) \mid 1 \le i < n\}$.

$W = \langle s_i \mid s_i^2 = 1,\ s_is_j = s_js_i \text{ if } |i-j| \geq 2\quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}\ \}\rangle$.

• If V is a Euclidean vector space $/\mathbb{R}$.
Any finite subgroup $W \subset O(V)$ generated by reflections (Coxeter 1930)

• ~~Affine~~ Discrete s affine reflection groups, hyperbolic reflection groups, ...

1.2 The Hecke algebra

$= \mathcal{H}(W,S)$

$\mathcal{H}$ unital associative algebra over $\mathbb{Z}[v^{\pm 1}]$ generated by H_s subject to the rels:

$$\left.\begin{array}{ll} 1) & H_s^2 = (v^{-1}-v)H_s + 1 \\ 2) & \underbrace{H_sH_t\ldots}_{m_{st}\text{-terms}} = \underbrace{H_tH_s\ldots}_{m_{st}\text{-terms}} \end{array}\right\} \text{(Hgenerators)}$$

For any $x \in W$ choose a reduced expression $x = s_1 \ldots s_k$ and set

$$H_x := H_{s_1}H_{s_2}\cdots H_{s_k}.$$

(H_x does not depend on the reduced expression by Tits theorem.)

Thm (Bourbaki, Grp. alg. de Lie, Ch IV §2 Ex 23
Humphreys, Refl. Grps. Cox Groups, 7.1–7.3).

i) H_x does not depend on the choice of reduced expression

ii) $\mathcal{H}$ is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_x \mid x \in W\}$.

Remark: $\mathcal{H}$ may also be defined as follows.

$$\mathcal{H} = \bigoplus \mathbb{Z}[v^{\pm 1}] H_x$$

and

$$H_xH_s = \begin{cases} H_{xs} & \text{if } xs > x \\ (v^{-1}-v)H_x + H_{xs} & \text{if } xs < x \end{cases} \quad \text{(Hbasis).}$$

~~(Very important)~~ Exercise (important)

a) Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$. Consider

$\mathcal{H}(G,B) =$ B-biinvariant $\mathbb{C}$-valued functions on G.

$$(f * g)(h) = \frac{1}{|B|} \sum_{\substack{g,g' \in G \\ gg'=h}} f(g)g(g').$$

Hint: Set $T_s = v^{-1}H_s$.
$T_s \mapsto$ indicator function of ...

Show that $\mathcal{H}(G,B) \cong \mathcal{H}(W,S)$ for $W = S_n$ specialised at $v = \frac{1}{\sqrt{q}} \in \mathbb{C}$. $\iff v^2 = \frac{1}{q}$

b). Deduce the theorem in this case.

$$T_s = \sqrt{q}\, H_s.$$

$$T_s^2 = q H_s^2 = v^{-2}\left((v^{-1}-v)H_s + v^{-2}\right)$$
$$= (v^{-2}-1)(v^{-1}H_s) + 1. \quad H_s = \frac{1}{\sqrt{q}}T_s$$

Hint:

consider

$\frac{1}{\sqrt{q}} \mathbb{1}_{BsB} \leftrightarrow H_s$

indicator function of $B(\ldots)B \subset G$.

1.3 The Kazhdan-Lusztig basis

Note that $H_s(H_s + (v - v^{-1})) = (v^{-1} - v)H_s + 1 + (v - v^{-1})H_s = 1$.
Hence H_s is invertible. Also to note that $w = s_1 \dots s_k$

$$H_w^{-1} = (H_{s_1} \cdots H_{s_k})^{-1} = H_{s_k}^{-1} \cdots H_{s_1}^{-1}$$

$$= \sum_{y \leq x} r_{x,w} H_w \quad (*) \qquad r_{w,w} = 1.$$

<u>Def/lemma:</u> ~~There is an involution~~ The endomorphism $h \mapsto \bar{h}$ on $\mathcal{H}$ defined by

$$\bar{v} = v^{-1}, \qquad \overline{H_x} = H_x^{-1}.$$

Defines an involution: $\overline{ab} = \bar{a}\,\bar{b}$ and $\overline{(\bar{a})} = a$.

<u>Proof:</u> Check the ~~defining~~ relations. (Hbasis).

<u>Def:</u> An element $h \in \mathcal{H}$ satisfying $\bar{h} = h$ is called <u>self-dual</u>.

<u>Thm</u> (Kazhdan-Lusztig 1979). There exists a unique basis $\{\underline{H}_y \mid y \in W\}$ for $\mathcal{H}$ such that

1) $\underline{H}_y$ is self-dual.

2) $\underline{H}_y = H_y + \bigoplus_{x<y} v\mathbb{Z}[v] H_x$.

<u>Proof:</u> <u>uniqueness:</u> $\underline{H}_y$ and $\underline{H}_y'$ ~~assume~~ be given.

$$d = \underline{H}_y - \underline{H}_y' = \sum_{x<y} g_x H_x. \qquad g_x \in v\mathbb{Z}[v].$$

Now; choose x maximal in the Bruhat order with $g_x \neq 0$.

$$\bar{d} = d \implies \underset{v\mathbb{Z}[v]}{\underset{\cap}{g_x}} = \underset{v^{-1}\mathbb{Z}[v^{-1}]}{\underset{\cap}{\overline{g_x}}} \quad (\text{by } (*)).$$

contradiction.

$$\implies d = 0 \qquad \underline{H}_y = \underline{H}_y'.$$

Existence: $\underline{H}_{id} = H_{id}$, $\underline{H}_s = H_s + v$.

$$\overline{\underline{H}_s} = H_s + (v - v^{-1}) + v^{-1} = H_s + v. \qquad \checkmark.$$

Note:

$$H_x \underline{H}_s = H_x(H_s + v) = \begin{cases} H_{xs} + vH_x & \text{if } xs > x \\ H_{xs} + v^{-1} H_x. & \text{if } xs < x \end{cases}$$

We assume for induction we have constructed $\underline{H}_x$ for all $x < w$.
Choose $s \in S$ with $ws < w$ and set

$$\underline{H}_{ws}\underline{H}_s = \sum_{z \leq w} g_z \underline{H}_z.$$

(Need $\{\leq ws\}s \subseteq \{\leq w\}$ $\cup \{\leq ws\} =$

(need a property of the Bruhat order here).

Clearly $\underline{H}_{ws}\underline{H}_s$ is self-dual. Also $g_z \in \mathbb{Z}[v]$ for all z.

Hence

$$\underline{H}_{ws}\underline{H}_s - \sum_{z<w} g_z(0)\underline{H}_z = H_w + \bigoplus_{z<w} v\mathbb{Z}[v] H_{z} \qquad \square$$

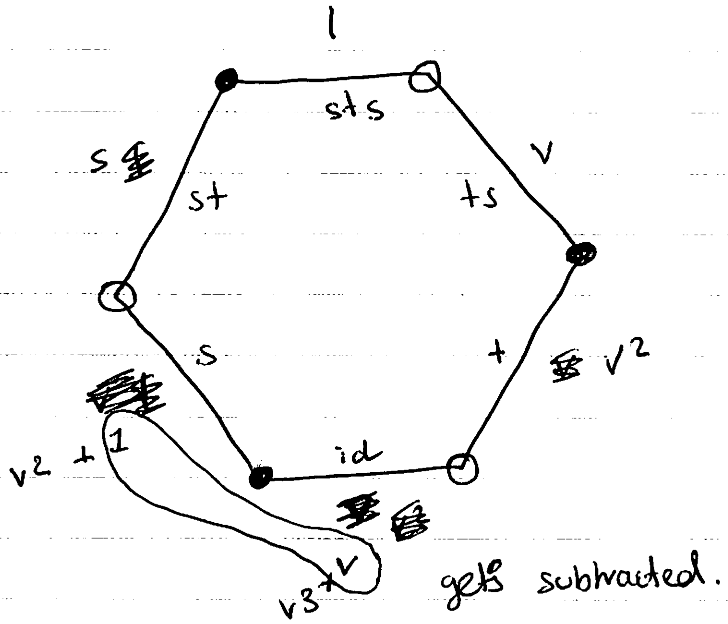
~~Self dual~~ satisfies the conditions.

Exercise:

$$\underline{H}_w \underline{H}_s = \begin{cases} \underline{H}_{ws} + \sum\limits_{\substack{x \leq w \\ xs<x}} \mu(x,w)\underline{H}_x. & \text{if } ws > w \\ (v+v^{-1})\underline{H}_w & \text{if } ws < w. \end{cases}$$

coeff. of v in $h_{x,w}$

Example:



Remarks: $\underline{H}_x = \sum_{y \leq x} h_{y,x} H_x$.

Conjecture (Kazhdan-Lusztig)

$h_{y,x} \in v\mathbb{Z}[v]$ are (almost) Kazhdan-Lusztig polynomials.

Conjecture (KL) $h_{y,x}$ has positive coefficients.

This course: $h_{y,x} \leftrightarrow$ singularities of Schubert varieties

$\Rightarrow$ KL conjecture for ~~these~~ Weyl groups.

Exercise: Calculate all KL polynomials for S_4.

1.4 **Cells:** For all $x \in W$ and $s \in S$ write

$$\underline{H}_s \underline{H}_z = \sum \lambda^s_{x,z} \underline{H}_x \qquad \underline{H}_z \underline{H}_s = \sum \rho^s_{x,z} \underline{H}_x .$$

Def:

$\underset{L}{\leq}$: preorder generated by $x \underset{L}{\leq} z$ if $\exists\, s$ with $\lambda^s_{x,z} \neq 0$.

$\underset{R}{\leq}$: p " $\rho^s_{x,z} \neq 0$.

$\underset{LR}{\leq}$: preorder generated by $\underset{L}{\leq}$, $\underset{R}{\leq}$.

The ~~equiv~~ $x \underset{L}{\sim} y \Leftrightarrow x \underset{L}{\leq} y,\ y \underset{L}{\leq} x$. (Sim: $\underset{R}{\sim}$, $\underset{LR}{\sim}$)

Remarks: Because $\{\underline{H}_s\}$ generates $\mathcal{H}$ for any $x \in W$ the submodule

$$\bigoplus_{y \underset{L}{\leq} x} \mathbb{Z}[v^{\pm 1}] \underline{H}_y \quad \text{is a left ideal.}$$

In fact, any ideal of $\mathcal{H}$ spanned by KL basis elts is a union of left cells. (Sim. right ideals, two-sided ideals).

~~Exer~~ Lemma: i) $x \underset{L}{\leq} z \Rightarrow \mathcal{R}(x) \supset \mathcal{R}(y)$.

ii) $x \underset{R}{\leq} z \Rightarrow \mathcal{L}(x) \supset \mathcal{L}(y)$.

$\mathcal{L}(x) = \{ s \in S \mid sx < x \}$, $\mathcal{R}(x) = \{ s \in S \mid$

Proof: If $zs' < x \Rightarrow \underline{H}_z \underline{H}_s = (v+v^{-1}) \underline{H}_z$ (Exercise).

$\Rightarrow \underline{H}_z \in \{ h \in H \mid h \underline{H}_{s'} = (v+v^{-1}) h \}$.

WLOG: $\lambda^s_{x,z} \neq 0$. $\Rightarrow \underline{H}_x$

$\Rightarrow \underline{H}_s \underline{H}_z = \ldots + \underline{H}_x + \ldots$

Lemma: $x \leq_L z \Rightarrow \mathcal{R}(x) \supset \mathcal{R}(z)$
$x \leq_R z \Rightarrow \mathcal{L}(x) \supset \mathcal{L}(z)$.

Proof: $zt < z \Rightarrow \underline{H}_z \underline{H}_t = (v+v^{-1}) \underline{H}_z$.

$$\Rightarrow \underline{H}_z \in \{ h \in \mathcal{H} \mid h \underline{H}_t = (v+v^{-1}) h \}.$$

Now: wlog $\lambda^t_{x,z} \neq 0$.

$$\Rightarrow \underline{H}_t \underline{H}_z = \sum \lambda^t_{x,z} \underline{H}_x$$

$$\underline{H}_t \underline{H}_z \underline{H}_s = (v+v^{-1}) \underline{H}_t \underline{H}_z = \sum (v+v^{-1}) \lambda^t_{x,z} \underline{H}_x.$$

$$\parallel$$

$$\sum \lambda^t_{x,z} \underline{H}_x \underline{H}_s. \qquad \text{(Induction over the Bruhat order).}$$

$$\Rightarrow \underline{H}_x \underline{H}_s = (v+v^{-1}) \underline{H}_x \quad \text{for all } x \text{ with } \lambda^t_{x,z} \neq 0.$$

$\square$

Proof: $zt < z \Rightarrow \underline{H}_z \underline{H}_t = (v+v^{-1}) \underline{H}_z$.

$$\Rightarrow \underline{H}_z \in \{ h \in \mathcal{H} \mid h \underline{H}_t = (v+v^{-1}) h \}.$$

Wlog: $\lambda^t_{x,z} \neq 0$.

$$\underline{H}_s \underline{H}_z = \sum \lambda^s_{x,z} \underline{H}_x.$$

$$\underline{H}_s \underline{H}_z \underline{H}_t = \sum \lambda^s_{x,z} \underline{H}_x \underline{H}_t$$

$$\parallel$$

$$\sum \lambda^s_{x,z} (v+v^{-1}) \underline{H}_x.$$

Induction over the Bruhat order $\Rightarrow$ $(\lambda^s_{x,z} \neq 0 \Rightarrow \underline{H}_x \underline{H}_t = (v+v^{-1}) \underline{H}_x)$

$$\Rightarrow t \in \mathcal{R}(x) \ \forall x \text{ with } \lambda^s_{x,z} \neq 0. \quad \square$$

Thm (~~Robinson-Schensted~~ Kazhdan-Lusztig)

$$x \underset{L}{\sim} y \iff Q(x) = Q(y).$$

$$x \underset{LR}{\sim} y \iff \mathrm{Shape}(x) = \mathrm{Shape}(y).$$

1.5 Cells in S_n

Recall the Robinson-Schensted correspondence

$$S_n \longleftrightarrow \bigsqcup_{\lambda \vdash n} \left(\begin{array}{c}\text{pairs of standard} \\ \text{tableaux of} \\ \text{shape } \lambda\end{array} \right)$$

$$w \longmapsto (P(w), Q(w)).$$

Review of the formalism of perverse sheaves:

(all coefficients are $\mathbb{Q}$!)

X a variety equipped with a stratification $\mathfrak{S}$ $X = \bigsqcup_{\lambda \in \Lambda} \mathfrak{S} X_\lambda$ with each X_λ smooth.

$D^b_{c}(X)$ = full subcategory of $D^+(X)$ (bounded below sheaves of $\mathbb{Q}$-vector spaces on X) of complexes whose cohomology sheaves are

- $\mathcal{H}^i(\mathcal{F}) = 0$ for $|i| \gg 0$,
- $\mathcal{H}^i(\mathcal{F})$ is constructible for all i.

$f : X \longrightarrow Y$

$$D^b_c(X) \underset{f^*,\, f^!}{\overset{f_*,\, f_!}{\rightleftarrows}} D^b_c(X)$$

$\mathbb{D} : D^b_c(X) \to D^b_c(X)^{op}$ standard triangles.

Isomorphisms of functors:

$f_! \mathbb{D} \xrightarrow{\sim} \mathbb{D} f_*$ $\qquad$ $f^! \mathbb{D} \xrightarrow{\sim} \mathbb{D} f^*$ $\qquad$ $\mathbb{D}^2 \xrightarrow{\sim} id$

Eg: (Poincaré duality). X smooth.

$\mathcal{H}^i(\pi_* \underline{\mathbb{Q}}_X) = H^i(X; \mathbb{Q})$. $\qquad$ $\mathbb{D}\pi_* \underline{\mathbb{Q}}_X \cong \pi_! \mathbb{D}\underline{\mathbb{Q}}_X \cong \pi_! \underline{\mathbb{Q}}_X[\cancel{+}2n]$.

$\Rightarrow$ $\qquad$ Take i-th cohomology: $\mathbb{D}(\pi_* \underline{\mathbb{Q}}_X) \cong \pi_! \underline{\mathbb{Q}}_X[-2n]$.

$\rightsquigarrow$ $H^{-i}(X; \mathbb{Q})^* \cong H_c^{i-2n}(X; \mathbb{Q})$.

$$\boxed{\text{On a smooth (complex) variety of dimension } n,\ \mathcal{L} \in Loc(X) \quad \mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2n]}$$

$(2n-i)$.

$i : 2n-i$.

$H^{2n-i}(X; \mathbb{Q})^* \cong$