Constructible derived category, six operations

① **Sheaves** (Obvious morphisms)

Def: An abelian presheaf $\mathcal{F}$ on a top-space X is a functor $\mathrm{Open}(X)^{op} \longrightarrow \mathrm{ab}$

$\mathcal{F}: U \longmapsto \mathcal{F}(U) = \Gamma(U, \mathcal{F})$

$V \subset U \longmapsto \rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s_{|V}$

$\phi \mapsto 0$

s.t. $\rho^U_U = id$, $\rho^V_W \rho^U_V = \rho^U_W$.

Def: An ab-presheaf $\mathcal{F}$ on X is a sheaf if $\forall U \subset X$, $\forall (U_i)$ covering of U,

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{ij} \mathcal{F}(U_{ij})$$

$s \mapsto (s_{|U_i})_{i \in I}$

$(s_i)_{i \in I} \mapsto (s_{i|U_{ij}} - s_{j|U_{ij}})$

is exact, where $U_{ij} = U_i \cap U_j$.

i.e. sections are det. loc. and local sections can be glued.

Def: For $\mathcal{F} \in \mathrm{Presheaf}(X, \mathrm{ab})$ and $x \in X$, we define the stalk $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$.

Prop: $\forall \mathcal{F} \in \mathrm{Presheaf}(X, \mathrm{ab})$,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall \varphi"] \arrow[d, "\exists a_{\mathcal{F}}"'] & \mathcal{G} \in Sh(X, \mathrm{ab}) \\ \mathcal{F}^a \in Sh(X, \mathrm{ab}) \arrow[ur, dashed, "\exists ! \psi"'] & \end{tikzcd}$$

$\mathrm{Hom}_{\mathrm{Presheaf}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{Sh}(\mathcal{F}^a, \mathcal{G})$

$\imath : \mathrm{Sheaf} \hookrightarrow \mathrm{Presheaf}$

$(\ )^a$ left adjoint to $\imath$.

"sheaf associated to the presheaf"

$\boxed{\text{same stalks}}$

Ex: cst presheaf $\underline{k}_X$

$U \mapsto k$, $\rho^U_V = id$

$(\underline{k}_X)^a = (\underline{k}_X)$ cst sheaf

$U \mapsto k^{\pi_0(U)}$, obvious rs.

Ex: loc. syst. **Ex**: skyscraper

Prop: $\mathrm{Presheaf}(X, \mathrm{ab})$ is abelian with obvious Ker, Im ...

$Sh(X, \mathrm{ab})$ is abelian

$\mathrm{Im}\, \varphi = (\mathrm{Im}_{\mathrm{Pre}})^a$ ...

Prop: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact iff (in $Sh(X, \mathrm{ab})$) $\forall x \in X$, $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ is exact.

Rmk: surj. for sheaves means: $\forall x \in X$, $\forall U \ni x$, $\exists V \ni x$, $\varphi_U : \mathcal{F}(V) \twoheadrightarrow \mathcal{F}''(V)$

Ex: Poincaré lemma

X complex analytic manifold

$$0 \to \underline{\mathbb{C}}_X \to \mathcal{O}_X \to \Omega^1_X \to \dots \to \Omega^n_X \to 0$$

The holomorphic de Rham complex is a c-soft res. of the cst sheaf

(or for X real C^∞-manifold of dim n; C^∞_X is c-soft, any C^∞_X-mod

$$0 \to \underline{\mathbb{C}}_X \to C^{\infty,(0)} \to \dots \to C^{\infty,(m)}_X \to 0)$$

Def: For $\mathcal{F} \in Sh(X)$, $\mathrm{supp}\, \mathcal{F}$ = compl. of union of all $U \subset X$ s.t. $\mathcal{F}_{|U} = 0$

For $s \in \mathcal{F}(U)$, $\mathrm{supp}\, s$ = compl. of union of all $V \subset X$ s.t. $s_{|U} = 0$.

Def: $H^n_{X} = R^n\Gamma(X, -)$, $H^n_c = R^n\Gamma_c(X, -)$ ...

② Operations
Hom and $\otimes$

Prop: $\mathcal{F}, \mathcal{G} \in Sh(X, Ab)$ then the presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf

$U \mapsto Hom(\mathcal{F}|_U, \mathcal{G}|_U)$

Def: $\mathcal{F}, \mathcal{G} \in Sh(X, Ab)$

$$\mathcal{F} \otimes \mathcal{G} = (\mathcal{F} \overset{Psh}{\otimes} \mathcal{G})^a$$

Prop: $\mathcal{H}om(\mathcal{G} \otimes \mathcal{F}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{G}, \mathcal{H}om(\mathcal{F}, \mathcal{H}))$

Prop: $\mathcal{H}om(-,-)$ left exact

$- \otimes -$ right exact

Direct and inverse images

$f : X \to Y$ continuous

. $\mathcal{F} \in Sh(X)$

$f_* \mathcal{F} : V \mapsto \mathcal{F}(f^{-1}(V))$

is a sheaf (direct image)

. $\mathcal{G} \in Sh(Y)$

$f^{-1}\mathcal{G} : U \mapsto \left(\varinjlim_{V \supset f(U)} \mathcal{G}(V)\right)^a$

The can. morphism

$$(f^{-1}\mathcal{G})_x \longrightarrow \mathcal{G}_{f(x)}$$

is an isom ($\Rightarrow f^{-1}$ exact).

~~Rmk: $\underline{k}_X$-modules $f^* = f^{-1}$ exact (~~ $f^* = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}$ ~~— only right exact for (q)coh sheaves here $f^{-1}\underline{k}_Y = \underline{k}_X$)~~

Adj: (f^*, f_*)

$Hom(f^*\mathcal{F}, \mathcal{G}) = Hom(\mathcal{F}, f_*\mathcal{G})$

Exactness: f^* exact

f_* left exact

Direct image with proper support:

(From now on: loc. comp. top. spaces)

$f_! \mathcal{F}(V) = \{ s \in \mathcal{F}(f^{-1}(V)) \mid f|_{supp\, s} : supp\, s \to V$ is proper.

subsheaf of $f_* \mathcal{F}$.

Would be nice to have adj: $(f_!, f^!)$ ...

need to go to der. cat.

Extremely useful: $(g \circ f)_* = g_* \circ f_*$, $(g \circ f)_! = g_! \circ f_!$

$(g \circ f)^* = f^* \circ g^*$.

Part. case: $Y = pt$.

$f_* = \Gamma(X, -)$, $f_! = \Gamma_c(X, -)$

$f^* M = \underline{M}_X$ | f proper: $f_* = f_!$
i, j ...

~~Leray-Serre spectral sequence:~~

~~$R^p g_* \circ R^q f_* \Rightarrow R^{p+q}(g \circ f)_*$~~

~~$H^p(Y, R^q f_* -) \Rightarrow H^{p+q}(X, -)$~~

Derived categories

$$\boxed{D_c^{(b)}(X, k):}$$

$\mathcal{F} = (\mathcal{F}^i, d^i)$

Obj = complexes of $\underline{k}_X$-modules

s.t. $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $|i| \gg 0$.

(b = bounded)

. $\mathcal{H}^i(\mathcal{F})$ are constructible:

$\exists X = \sqcup X_\lambda$ s.t. $\mathcal{F}|_{X_\lambda}$ loc. syst.

(c = constructible)

. morphisms: morph. of complexes

+ add formal inverses to quasi-isomorphisms.

$i^! = R\Gamma_Z$

sections with support in Z

$\Gamma_Z : Sh X \to Sh_Z$.

Can. $i^! \to i^*$

$R\Gamma_Z$

$j_! j^* F \to F \to i_* i^* F \rightsquigarrow$

$i_* i^! F \to F \to j_* j^* F \rightsquigarrow$

X Hausdorff, paracompact, countable at ∞

inj $\Rightarrow$ flabby $\Rightarrow$ c-soft $\Leftrightarrow$ soft $\Leftarrow$ fine
($\Leftarrow$ field coeffs) (open) (compact) (closed)

$\forall s \in \Gamma(U,F)$, $U \subset Y$
$\forall U = \bigcup U_i$, $\exists (s_i) \in \Gamma(U_i,F)$
s.t. their supp is loc. fin.
add up to s,
and each s_i vanishes outside U_i.

f_* preserves inj & flabby
$f_!$ preserves c-soft.

- inj, flabby, c-soft and fine sheaves are acyclic for $\Gamma(Y,-)$ and compute $H(Y,-)$
- inj, flabby and fine are f_*-acyclic
- inj, flabby and c-soft are $\Gamma_c(Y,-)$ acyclic and compute $H_c(Y,-)$
- inj, flabby, c-soft and fc-soft are $f_!$-acyclic and compute $Rf_!$ (c-soft on fibers $f^{-1}(y)$)

$Rf_*, Rf_! : D^b_c(X,k) \to D^b_c(Y,k)$

(preserve constructibility ...)

complex whose $\mathcal{H}^n$ gives R^nf_*, $R^nf_!$

$f^*, f^! : D^b_c(Y,k) \to D^b_c(X,k)$

Adj: (f^*, Rf_*)

$(Rf_!, f^!)$

$f : X \to pt$ $\quad f^* k = k_X$

$f^! k = \omega_X$ dual. ~~...~~ complex

$Rf_* = R\Gamma(X,-)$ giving $H^*(X,-)$

$Rf_! = R\Gamma_c(X,-)$ giving $H^*_c(X,-)$

Very nice:

$R(g\circ f)_* = Rg_* \circ Rf_*$

$R(g\circ f)_! = Rg_! \circ Rf_!$

$(R^nf_*\mathcal{F})_y = \varinjlim_{V \ni y} H^n(f^{-1}V, \mathcal{F})$

$(R^nf_!\mathcal{F})_y = \varinjlim_{V\ni y} H^n_c(f^{-1}V, \mathcal{F})$

Duality: $\mathbb{D}_X = R\mathcal{H}om(-, \omega_X)$

$D^b_c(X,k)^{op} \to D^b_c(X,k)$

$\mathbb{D}_X^2 \simeq \mathrm{Id}$

$\boxed{\mathbb{D}_Y Rf_* \simeq Rf_! \mathbb{D}_X}$ $(*)$

$\mathbb{D}_X f^* \simeq f^! \mathbb{D}_Y$

X smooth ~~oriented (e.g.~~ complex manifold) dim d

then $\omega_X = \underline{or}_X[\cancel{\dots}] = \underline{k}_X[2d]$

if $f : X \to pt$

$(*)$ Poincaré duality

$R\Gamma(X,k) = (R\Gamma_c(-, k[2d]))^*$

$H^i(X,k) \simeq H^{2d-i}_c(X,k)^*$

f proper $\Rightarrow f_* = f_!$ / f smooth rel dim d complex ... then $f^! = f^*[2d]$

Inclusions

Borel's

$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$

open (⇒ smooth) closed (⇒ proper)

$(j_!, j^*, j_*)$

$j^* = j^!$

$(i^*, i_*, i^!)$

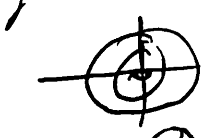
$i_* = i_!$

$i_!$ and $j_!$: extension by 0.

i^*, j^* : restrictions.

j_* : direct image

e.g. $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$



$H^0 = k$

$H^1 = k$

$(H^p j_* k_{\mathbb{C}^\times})_0 = \varinjlim_{U\ni 0} H^p(U\setminus 0, k)$ then 0.