C2.1a Lie algebras

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Michaelmas Term 2010

Problem Sheet 4: the $\mathfrak{sl}_2$ sheet

In this exercise sheet we will classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ *and show that any finite dimensional representation is completely reducible. This is a hard exercise, but is worth the effort!*

Recall that if we let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then e, f and h give a basis of $\mathfrak{sl}_2$ with relations

$$[h, e] = 2e, [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

Hence, a representation of $\mathfrak{sl}_2(\mathbb{C})$ consists of a vector space V over $\mathbb{C}$ together with three endomorphisms E, F and H satisfying

$$HE - EH = 2E, HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

(We recover the representation $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ by setting $\phi(e) = E$, $\phi(f) = F$ and $\phi(h) = H$.)

In this exercise we always assume that V is *finite dimensional.*

1. a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^k E = E(H - \lambda)^k.$$

(Here $\lambda \in \mathbb{C}$ and we write λ instead of $\lambda \cdot \mathrm{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ-eigenspace of H, then Ev belongs to the generalised $(\lambda + 2)$-eigenspace.

b) Deduce a similar statement for the action of F on the generalised eigenspaces of H.

c) Let λ be an eigenvalue for H which has maximal real part among all the eigenvalues of H. Use a) to show that $EV_\lambda = 0$.

d) Use b) to deduce that if $v \in V$ is arbitrary, then $F^n v = 0$ for large enough n.

2. a) Show the relation (for $n \geq 1$)

$$HF^n = F^n H - 2nF^n.$$

b) Show ($n \geq 1$ as before)

$$EF^n = F^n E + nF^{n-1}H - n(n-1)F^{n-1}.$$

c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^n F^n v = nE^{n-1}F^{n-1}(H - (n-1))v = n! \prod_{i=1}^{n} (H - (i-1))v.$$

3. Let λ be an eigenvalue of H with maximal real part (as in 1(c)) and let V_λ denote the generalised λ-eigenspace. Use 1(d) and 2(c) to deduce that H acts diagonalisably on V_λ and that λ is a non-negative integer.

4. a) Let λ be as in Question 3 and choose a non-zero vector $v \in V_\lambda$. We know by Questions 1 and 3 that $Ev = 0$ and that λ is an non-negative integer. Show the relations:

$$HF^k v = (\lambda - 2k)F^k v,$$
$$EF^k v = k(\lambda - (k-1))F^{k-1} v.$$

Deduce that $F^{\lambda+1} v = 0$ and that the $F^i v$ for $0 \leq i \leq \lambda$ are linearly independent and span a simple submodule of V.

b) Check that the above relations define an $\mathfrak{sl}_2(\mathbb{C})$-module for any non-negative integer λ. Deduce that there is (up to isomorphism) a unique simple module $V(\lambda)$ of dimension $\lambda+1$ for all non-negative integers λ.

5. *(Optional harder exercise)* Let V be an arbitrary finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.

a) Let $\lambda \in \mathbb{Z}$ be maximal amongst the eigenvalues of H, and let $V_\lambda \subset V$ denote the λ-eigenspace. Suppose that V has the property that $Ev = 0$ implies that $v \in V_\lambda$. Show that V is completely reducible.

b) Consider the endomorphism $c = EF + FE + \frac{1}{2}H^2$. Show that c commutes with E, F and H. (c is called the *Casimir* element.) Deduce from Schur's lemma that c acts as a scalar on any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Compute the scalar with which c acts on $V(m)$.

c) Show that V is completely reducible.