MIRROR SYMMETRY, LANGLANDS DUALITY AND THE HITCHIN SYSTEM

Abstract. Notes of talks by Tamas Hausel in Oxford, Trinity Term, 2010. Notes by Gergely Berczi, Michael Groechenig and Geordie Williamson.

1. Higgs bundles and the Hitchin system

1.1. **The moduli space of vector bundle on a curve.** Let C be a complex projective curve of genus $g > 1$. We fix integers $n > 0$ and $d \in \mathbb{Z}$. We asssume throughout that $(d, n) = 1$.

1.1.1. *GL_n.* A central object of study in these talks will be:

$$\mathcal{N}^d := \begin{array}{c} \text{moduli space of rank } n \text{ vector bundles on } C \\ \text{which are semi-stable of degree } d. \end{array}$$

This space can be constructed using geometric invariant theory (GIT) or gauge theory.

We recall that a vector bundle is called *stable* if ever subbundle F satisfies

$$\mu(F) = \frac{\deg F}{\operatorname{rk} F} \leq \mu(E) = \frac{\deg E}{\operatorname{rk} E}$$

A vector bundle is *stable* if one has strict inequality above for all proper subbundles.

Remark 1.1. In general one has to be careful in constructing such moduli spaces. However, as we assume $(d, n) = 1$ the notions of semi-stability and stability agree and geometric invariant theory allows us to conclude that $\mathcal{N}^d$ is a fine moduli space, which is smooth and projective.

1.1.2. *SL_n and PGL_n.* Consider the determinant morphism

$$\det : \mathcal{N}^d \to \operatorname{Jac}^d(C)$$

which sends a vector bundle of rank n to its highest exterior power $\Lambda^n E$.

Choose $\Lambda \in \operatorname{Jac}^d(C)$ and define $\check{\mathcal{N}}^\Lambda := \det^{-1}(\Lambda)$. We can think of points in $\check{\mathcal{N}}^\Lambda$ as "twisted SL_n-bundles". It is easy to see that $\check{\mathcal{N}}^\Lambda$ does not depend (up to isomorphism) on the choice of $\Lambda \in \operatorname{Jac}^d(C)$. We often abuse notation and write $\check{\mathcal{N}}^d$ instead of $\check{\mathcal{N}}^\Lambda$.

The abelian variety $\mathrm{Pic}^0(C) = \mathrm{Jac}^0(C)$ acts on $\mathcal{N}^d$ via

$$(L, E) \mapsto L \otimes E.$$

We define

$$\hat{\mathcal{N}}^d := \mathcal{N}^d / \mathrm{Pic}^0(C).$$

One can show that $\hat{\mathcal{N}}^d = \check{\mathcal{N}}^\Lambda / \Gamma$, where $\Gamma := \mathrm{Pic}^0(C)[n]$ denotes the n-torsion points of the Jacobian. Hence $\hat{\mathcal{N}}^d$ is a projective orbifold.

1.1.3. *Cohomology.* The cohomology of $\mathcal{N}^d$, $\check{\mathcal{N}}^d$ and $\hat{\mathcal{N}}^d$ is well understood.

Here we only comment on the additive structure. The first major breakthrough was make in 1975 by Harder and Narasimhan who obtained recursive formulae for the number of points of these varieties over finite fields. It is then possible to use the Weil conjectures (which had been proven the year before by Deligne) to obtain formulae for the Betti numbers. In 1981 Atiyah and Bott gave a completely different gauge-theoretic proof [1].

The central result of Harder and Narasimhan's paper is the following:

Theorem 1.2 (Harder-Narasimhan)**.** *The finite group* Γ *acts trivially on* $H^*(\check{\mathcal{N}}^d)$*. In particular, we have* $H^*(\check{\mathcal{N}}^d) = H^*(\hat{\mathcal{N}}^d)$*.*

Remark 1.3. This result is very difficult to prove and relies on showing that the spaces $\check{\mathcal{N}}^d$ and $\hat{\mathcal{N}}^d$) have the same number of points over finite fields. The analogue of this result is false in the context of character varieties. [links to Ngǒ ... ask Tamas]

1.2. **The Hitchin system.** We now consider the Hitchin system, which is a related to the the cotangent bundle to the moduli spaces considered in the previous section. As in the previous section, fix n and d and abbreviate $\mathcal{N} := \mathcal{N}^d$.

The cotangent bundle $T^*\mathcal{N}$ is a (non-projective) algebraic symplectic variety. The ring of regular functions $\mathbb{C}[T^*\mathcal{N}]$ is known to be finitely-generated. The affinization

$$\chi : T^*\mathcal{N} \to \mathcal{A} = \mathrm{Spec}(\mathbb{C}[T^*\mathcal{N}])$$

is the *Hitchin map.*

We now describe this map more explicitly. For a point $E \in \mathcal{N}$ standard deformation theory gives us an identification

$$T_E\mathcal{N} = H^1(C, \mathrm{End}(E)).$$

Applying Serre duality we obtain

$$T_E^*\mathcal{N} = H^0(C, \mathrm{End}(E) \otimes K)$$

where K denotes the canonical bundle of C. An element

$$\phi \in H^0(C, End(E) \otimes K)$$

is called a *Higgs field*. Morally it can be thought of as a matrix of one-forms on the curve.

For any $(E, \phi) \in T^*\mathcal{N}$ we can consider the characteristic polynomial of the Higgs field. It has the form

$$t^n + a_1 t^{n-1} + \cdots + a_n$$

where $a_i \in H^0(K^n)$. For example $a_n \in H^0(K^n)$ is the determinant of the Higgs field.

The Hitchin map then has the explicit description

$$\begin{array}{rl} \chi : T^*\mathcal{N} \to & \mathcal{A} := \bigoplus_{i=1}^n H^0(K^i) \\ (E, \phi) \mapsto & (a_1, a_2, \ldots, a_n) \end{array}$$

The affine space $\mathcal{A}$ is called the *Hitchin base.*

In the SL_n-case we have

$$T^*_E \check{\mathcal{N}}^\Lambda = H^0(End_0(E) \otimes K)$$

that is, a covector at E is given by a *trace free* Higgs field. Thus in this case the Hitchin base is

$$\check{\mathcal{A}} = \bigoplus_{i=2}^n H^0(C, K^i).$$

The determination of the Hitchin base for PGL_n is left as an exercise.

Recall that $T^*\mathcal{N}$ is an algebraic symplectic variety.

Theorem 1.4 (Hitchin, 1987)**.** *If ψ_i, ψ_j are two coordinate function, then they Poisson commute, i.e. $\{\psi_i, \psi_j\} = 0$. We have dim $\mathcal{A}$ = dim $\mathcal{N}$ and the generic fibres of the ψ are open subsets of abelian varieties. Therefore we have an algebraically completely integrable Hamiltonian system.*

As a next step we will projectivize $\chi : T^*\mathcal{N} \to \mathcal{A}$. Recall that a complex point in $T^*\mathcal{N}$ is given by a pair (E, ϕ). In order to projectivize we need to allow E to become unstable.

Definition 1.5. A *Higgs bundle* is a pair (E, ϕ) where E is a vector bundle and $\phi \in H^0(C, \mathrm{End}(E) \otimes K)$ is a Higgs field.

The definition for semi-stability and stability for Higgs-bundles is almost the sames as for vector bundles except we only consider ϕ-invariant subbundles. The moduli-space of semi-stable Higgs bundles is denoted by $\mathcal{M}^d$, it is a non-singular quasi-projective variety, having $T^*\mathcal{N}$ as an open subvariety.

Of course, we can extend $\chi : \mathcal{M}^d \to \mathcal{A}$. The following result shows that we have succeeded in projectivizing the Hitchen map:

Theorem 1.6 (Hitchin 1987, Nilsen 1991, Faltings 1993)**.** *χ is a proper algebraically completely integrable Hamiltonian system. Its generic fibres are abelian varieties.*

2. Topological mirror symmetry for Higgs bundles

The goal of this lecture (and perhaps the next) is to establish the global picture:

$$\begin{tikzcd} \check{\mathcal{M}} \arrow[dr, "\check{\chi}"'] & & \hat{\mathcal{M}} \arrow[dl, "\hat{\chi}"] \\ & \mathcal{A}^0 & \end{tikzcd}$$

(Here $\check{\chi}$ and $\hat{\chi}$ are the Hitchin maps). The important point is that the generic fibres are (torsors for) dual abelian varieties.

The reason we want to do this is the following. We have not yet mentioned this, but $\check{\mathcal{M}}$ and $\hat{\mathcal{M}}$ are *hyperkähler*, which means that they have an S^2 of complex structures. Hence we can change complex structure

$$\begin{tikzcd} \check{\mathcal{M}}_{DR} \arrow[dr, "\check{\chi}"'] & & \hat{\mathcal{M}}_{DR} \arrow[dl, "\hat{\chi}"] \\ & \mathcal{A}^0 & \end{tikzcd}$$

and in this model the fibres become (Lagrangian ???) tori. This is the setting proposed by SYZ for mirror symmetry.

To begin, we need to have a description of the generic fibres of the Hitchin map. It turns out that they will be related to the Jacobian of the "spectral curve", associated to a point in the Hitchin base.

The moduli space of SL_n-Higgs bundles of degree d is a non-singular quasi-projective variety, it is defined after the choice of a degree d line bundle Λ. $\check{\mathcal{M}}^d = \check{\mathcal{M}}^\Lambda$, the isomorphism class is independent of the chosen line bundle. It's complex points are isomorphism class of semi-stable (i.e. stable by coprime assumption) Higgs bundle (E, ϕ) of rank n, $\det E = \Lambda$ and $\phi \in H^0(End_0(E) \otimes K)$. As in the GL_n-case the Hitchin map $\check{\chi}$ is given by the coefficients of the characteristic polynomial, it is proper and a completely integrable system. We also have that $\mathrm{T}^* \check{\mathcal{N}} \subset \check{\mathcal{M}}^d$ open and dense.

Let us remember the two constructions of the moduli space of PGL_n-Higgs bundles. The cotangent bundle $\mathrm{T}^* \operatorname{Pic}^0(C) = \operatorname{Pic}^0(C) \times H^0(C, K)$

acts on $\mathcal{M}^d$ by

$$(L,\varphi)\dot{(}E,\phi) \mapsto (L \otimes E, \varphi + \phi)$$

This induces an action of $\Gamma = \operatorname{Pic}^0[n]$ on $\check{\mathcal{M}}^d$. Then we may either define the PGL_n-moduli space as

$$\hat{\mathcal{M}}^d = \mathcal{M}^d / \operatorname{T}^* \operatorname{Pic}^0(C) \cong \chi^{-1}(\mathcal{A}^0)/\operatorname{Pic}^0(C)$$

or as

$$\check{\mathcal{M}}/\Gamma$$

To avoid difficulties in forming the first quotient it is wise to quotient first in Higgs direction, as indicated. The second quotient tells us that we obtain an orbifold. Since $\check{\chi}$ is compatible with the Γ action, we obtain a well-defined Hitchin map

$$\hat{\chi}: \ \hat{\mathcal{M}}^d = \check{\mathcal{M}}^d/\Gamma \to \mathcal{A}^0 = \hat{\mathcal{A}}.$$

2.1. **Spectral curves.** Question: it seems that the spectral curve can be defined globally over the Hitchin base ... is this true?

The simple idea of describing a polynomial by its zeroes leads to spectral curves. Recall:

$$\begin{array}{ccccccccccc} a & \in & \mathcal{A} & & & H^0K & \times & \cdots & \times & H^0(K^n) \\ & & & a & = & t^n & + & a_1 t^{n-1} & + & \cdots & + & a_n \end{array}$$

What should be the spectrum of such a polynomial? Look at one point $p \in C$, there we get $\Phi_p : E_p \to E_p \otimes K_p$, we expect of an eigenvalue v_p of Φ_p to satisfy

$$\exists v \in E_p - 0 : \ \Phi_p(v) = v_p v.$$

To make any sense, we need $v_p \in K_p$. We do now consider all eigenvalues as a subset of the total space X of the bundle $K \to C$, and want to identify it with the complex points of a scheme. The resulting object will be called the spectral curve corresponding to $a \in \mathcal{A}$.

$$\begin{tikzcd} C_a \arrow[r, phantom, "\subset"] \arrow[dr, "\pi_a"] & X \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

To construct this geometrical structure, note that there exists a tautological section $\lambda \in H^0(X, \pi^* K)$ satisfying $\lambda(x) = x$. We can now pullback the sections a_i to X and obtain a section

$$s_a \in H^0(X, \pi^* K^n)$$

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

Clearly C_a equals the zero set of this section, i.e. $C_a = s_a^{-1}(0)$, which comes naturally with a scheme structure.

2.2. **Generic fibres of the Hitchin map.** [1] The fibers of the Hitchin map can get very complicated, reducible, non-reduced, etc. But for generic $a \in \mathcal{A}$ the spectral curve C_a is smooth. We want to get a correspondence between line bundles on C_a and Higgs bundle on C with characteristic polynomial a. Given a line bundle L on C we do at least know that $\pi_*(L)$ is a torsion free sheaf on C, but since C is a non-singular curve this means that it is actually a rank n vector bundle[2]. Remember the canonical section $\lambda \in H^0(X, \pi^*K)$, this gives us

$$L \xrightarrow{\lambda} L \otimes \pi_a^* K$$

we can now push this forward to the curve C to obtain

$$E = \pi_*(L) \xrightarrow{\pi_*(\lambda)} \pi_*(L \otimes \pi_a^* K) = \pi_*(L) \otimes K$$

where we used the *pull-push formula* for the last step. Hence we actually get a Higgs bundle $E \to E \otimes K$.

If C_a is moreover connected, (E, φ) cannot have any subobjects, hence it is automatically stable. This way we define a map:

$$\begin{gathered} \mathrm{Pic}^d(C_a) \to \mathcal{M}^d \\ L \mapsto (\pi_*(L) \otimes \det(\pi_*(\mathcal{O}))^{-1}, \pi_*(\lambda)) \end{gathered}$$

But it is not obvious that we actually have defined something lying over a this way. Since $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$ holds over C_a we can push-forward this equation to C, to obtain $\pi_*(\lambda)^n + a_{n-1}\pi_*(\lambda)^{n-1} + \cdots + a_0 = 0$. Densitiy and the Cayley-Hamilton theorem imply the assertion.

Theorem 2.1 (Hitchin 1987, Beauville-Narasimhan-Ramanan 1989)**.** *For $a \in \mathcal{A}_{reg}$ (i.e. having a non-singular and connected spectral curve C_a) we have $\chi^{-1}(a) \cong \mathrm{Pic}^d(C_a)$.*

We need some modifications for SL_n and PGL_n.

In the SL_n-case we have $a \in \mathcal{A}_0$, we need to find the line bundles L, s.t. $\pi_*(L)$ has the right determinant. Define $\mathrm{Prym}^d(C) \subset \mathrm{Jac}^d(C_a)$.

$$L \in \mathrm{Prym}^d(C_a) \Leftrightarrow \det \pi_*(L) \otimes \det(\pi_*(\mathcal{O}))^{-1} = \Lambda$$

$\forall a \in \mathcal{A}^0_{reg}$, $\check{\chi}^{-1}(a) \cong \mathrm{Prym}^d(C_a)$.

[1] Both notes say generic fibres for X - but X is the notation used for the total space of the canonical bundle

[2] Since the local rings of a curve are discrete valuation rings, torsion free and finitely generated modules are free.

For PGL_n we have $\check{\chi}^{-1}(a) \cong \mathrm{Prym}^d(C_a)/\Gamma$. This makes sense since for $L_\gamma \in \mathrm{Pic}(C)[n]$ we do have $\det(\pi_*(\pi^*(L_\gamma)\otimes L)) = \det(L_\gamma \otimes \pi_*(L)) = L_\gamma^n \otimes \det(\pi_* L) = \det(\pi_* L)$.

Summary: Fibres of the Hitchin map

(1) For GL_n: By Thm 2.1, for $a \in \mathcal{A}_{reg}$

$$\mathcal{A}_a := \chi^{-1}(a) \simeq \mathrm{Jac}^d(C_a).$$

(2) For SL_n: following the definitions it is straighforward that for $a \in \mathcal{A}^0_{reg}$

$$\check{\mathcal{A}}_a := \check{\chi}^{-1}(a) \simeq \mathrm{Prym}^d(C_a).$$

(3) For PGL_n: There are two ways of thinking of the Hitchin fibre:

$$\hat{\mathcal{A}}_a := \mathrm{Prym}^d(C)/\Gamma \simeq \mathrm{Jac}^d(C_a)/\,\mathrm{Pic}^0(C),$$

where $\mathrm{Pic}^0(C)$ acts on $\mathrm{Jac}^d(C_a)$ by tensoring with the pull-back line bundle. A short computation shows that Γ indeed acts on $\mathrm{Prym}^d(C)$.

2.3. **Symmetries of the Hitchin fibration (Ngo's terminology).** We will see in this subsection how natural Abelian varieties act on these fibers, giving a torsor structure. Again, we study the GL_n, SL_n, PGL_n cases separately.

(1) For GL_n: Fixing $a \in \mathcal{A}_{reg}$, tensor product gives a simply transitive action of $\mathrm{Pic}^0(C_a)$ on $\mathrm{Jac}^d(C_a)$, and therefore $\mathcal{M}_a$ is a torsor for $P_a := \mathrm{Pic}^0(C_a)$.

(2) For SL_n: Fixing $a \in \mathcal{A}^0_{reg}$, we have a (ramified) cover map $\pi : C_a \to C$.

Definition 2.2. The *norm map*

$$Nm_{C_a/C} : \mathrm{Pic}^0(C_a) \to \mathrm{Pic}^0(C)$$

is defined in any of the following three equivalent ways:

(a) Using divisors. For any divisor D, $Nm_{C_a/C}(\mathcal{O}(D)) = \mathcal{O}(\pi_* D)$, where $\pi : C_a \to C$ is the projection. This definition points out why Nm is a group homomorphism.

(b) For $L \in \mathrm{Pic}^0(C_a)$ define $Nm_{C_a/C}(L) = det(\pi_*(L))\otimes det^{-1}(\pi_*\mathcal{O}_{C_a})$.

(c) Using the fact that $\mathrm{Pic}^0(C), \mathrm{Pic}^0(C_a)$ are Abelian varieties, we can define the norm map as the dual of the pull-back map $\pi^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C_a)$, that is

$$Nm_{C_a/C} = \check{\pi} : \mathrm{Pic}^0(C_a) = \check{\mathrm{Pic}}^0(C_a) \to \check{\mathrm{Pic}}^0(C) \simeq \mathrm{Pic}^0(C).$$

Let $\mathrm{Prym}^0(C_a) := \ker(Nm_{C_a/C})$ denote the kernel of the norm map. Then $\mathrm{Prym}^0(C_a)$ acts on $\mathrm{Prym}^d(C_a) = \check{\mathcal{M}}_a$, and $\check{\mathcal{M}}_a$ is a torsor for $\check{P}_a := \mathrm{Prym}^0(C_a)$.

(3) For PGL_n: In this case

$$\hat{\mathcal{M}}_a = \check{\mathcal{M}}_a/\Gamma = \mathcal{M}_a/\operatorname{Pic}^0(C)$$

is a torsor for $\hat{P}_a := \check{P}_a/\Gamma = P_a/\operatorname{Pic}^0(C)$.

As a next step, we will show, that $\check{P}_a$ and $\hat{P}_a$ are dual abelian varieties.

2.4. **Duality of the Hitchin fibres.** Take the short exact sequence

$$0 \quad\to\quad \mathrm{Prym}^0(C_a) \quad\hookrightarrow\quad \mathrm{Pic}^0(C_a) \quad\overset{Nm_{C_a/C}}{\twoheadrightarrow}\quad \mathrm{Pic}(C) \quad\to\quad 0$$

and dualize. Since $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^0(C_a)$ are isomorphic to their duals, we get

$$0 \quad\leftarrow\quad \check{\mathrm{Prym}}^0(C_a) \quad\twoheadleftarrow\quad \mathrm{Pic}^0(C_a) \quad\overset{\pi^*}{\hookleftarrow}\quad \mathrm{Pic}(C) \quad\leftarrow\quad 0\ ,$$

and therefore

$$\check{\check{P}}_a = \mathrm{Pic}^0(C_a)/\operatorname{Pic}(C) = \hat{P}_a,$$

that is $\check{P}_a$ and $\hat{P}_a$ are duals.

This is the first reflection of mirror symmetry. To summarize, we state

Theorem 2.3. *(Hausel-Thaddeus, 2003)*
For a regular $a \in \mathcal{A}^0_{reg}$ $\check{\mathcal{M}}_a$ and $\hat{\mathcal{M}}_a$ are torsors for dual Abelian varieties. (namely $\check{P}_a$ and $\hat{P}_a$.)

We can state this theorem more precisely using the language of gerbes. To this end here is a short summary.

2.5. **Gerbes on $\check{\mathcal{M}}$ and $\hat{\mathcal{M}}$.** Let A be a sheaf of Abelian groups on a variety X. The typical examples are $\mathcal{O}_X^*, \zeta_m, U(1)$, where ζ_m is the group of mth roots of unity. Note that $\zeta_m \subset \mathcal{O}_X^*$ and $\zeta \subset U(1)$.

Definition 2.4. (Rough definition of a torsor)
An A-torsor is a sheaf F of sets on X locally isomorphic to A, in particular $\Gamma(F, U)$ is a torsor for $\Gamma(A.U)$ for all $U \subset X$ open.

Examples:

- $\mathcal{O}_X^*$-torsor = line bundle
- $U(1)$-torsor = flat unitary line bundle
- ζ_m-torsor = ζ_m-Galois etale cover ????

Note that the natural tensor category structure on the set of torsors $Tors_A(U)$ is a group. Moreover, the automorphism of an A-torsor is an element of $\Gamma(A)$.

Definition 2.5. An A-gerbe B is a sheaf of categories so that $B|U$ is a torsor for the group $Tors_A(U)$.

Let $(\mathbb{E}, \phi)$ be the universal Higgs-bundle on $\check{\mathcal{M}} \times C$, where $\phi \in H^0(End_0\mathbb{E} \otimes \pi^*(K_C)$, and $\mathbb{E}_c$ be the fiber over $c \in C$. that is

$$\mathbb{E}_c = \mathbb{E}|_{\check{\mathcal{M}} \times c}.$$

Then $c_1(\mathbb{E}_c) \in H^2(\mathcal{M}, \mathbf{Z}) \simeq \mathbf{Z}$ is a generator. Note that $\mathbb{E}$ is not unique: it can be tensored by $L \in \mathrm{Pic}(\mathcal{M})$, but this property always holds.

Let $P\mathbb{E}_P \to \check{\mathcal{M}}$ be the corresponding PGL_n-bundle. Let B be the ζ_m-gerbe of liftings of $P\mathbb{E}_0$ as an SL_n-bundle. As there is no global lifting, B is not a trivial gerbe.

But for $a \in \mathcal{A}_{reg}$, $c_1(\mathbb{E}_P)|_{\check{\mathcal{M}}_a}$ is 0 mod n, and $\mathbb{E}_P)|_{\check{\mathcal{M}}_a}$ can be lifted as an SL_n-bundle, therefore $B|_{\check{\mathcal{M}}_a}$ is a trivial gerbe.

Theorem 2.6. *(Hausel-Thaddeus, 2003) Let B denote the corresponding $\zeta_m \subset U(1)$-gerbe. Then*

$$Triv^{U(1)}(\check{B}^e|_{\check{\mathcal{M}}_a^d}) \simeq \check{\mathcal{M}}_a^e$$

as $\check{P}_a$-torsors. Similarly,

$$Triv^{U(1)}(\hat{B}^e|_{\hat{\mathcal{M}}_a^d}) \simeq \hat{\mathcal{M}}_a^e.$$

2.6. **Topological consequences, The stringy E-polynomial of an orbifold.** Hope: Some analog of the Hodge diamond reflects the mirror symmetry

Definition 2.7. Let X be a complex algebraic variety. The E-polynomial (virtual Hodge polynomial) is defined as

$$E(X; u, v) = \sum_{p,q,i} (-1)^i h^{p,q}(Gr^W_{p+q} H^i(X)) u^p v^q$$

where $Gr^W_{p+q} H^i(X)$ is the $p+q$th graded piece of the weight filtration of $H^i(X)$. This has a (pure) Hodge structure, and $h^{p,q}$ is the corresponding Betti number.

Remark 2.8. For pure MHS $h^{p,q} \neq 0$ implies that $p+q=i$. The fibres $\mathcal{M}, \check{\mathcal{M}}, \hat{\mathcal{M}}$ have pure MHS.

Let X be smooth, and assume that a finite group Γ acts on X. Then we can define the *stringy E-polynomial* of the orbifold X/Γ as follows:

$$E_{st}(X/\Gamma, u, v) = \sum_{[\gamma]\in[\Gamma]} E(X_\gamma/C_\gamma; u, v)(uv)^{F(\gamma)}, \tag{2.1}$$

where

- $[\gamma]$ is a conjugacy class of Γ, and X_γ is the fixed point set, C_γ is the centralizer of γ in Γ, acting on X_γ.
- $F(\gamma)$ is the Fermionic shift, defined as $F(\gamma) = \sum w_i$, where γ acts on $TX|_{X_\gamma}$ with eigenvalues $e^{2\pi i w_i}$, $w_i \in [0, 1]$.

Let now B be a Γ-equivariant $U(1)$-gerbe on X. Then more generally, we define

$$E^B_{st}(X/\Gamma), u, v) = \sum_{[\gamma]\in[\Gamma]} E(X_\gamma/C_\gamma, L_{B,\gamma}, u, v)(uv)^{F(\gamma)}, \tag{2.2}$$

where $L_{B,\gamma}$ is the local system on X_γ given by B.

2.7. **Topological mirror test.**

Conjecture 2.9. *(Hausel-Thaddeus, 2003)*
For $(d, n) = (e, n) = 1$

$$E(\check{\mathcal{M}}^d; u, v) = E^{\check{B}^e}_{st}(\hat{\mathcal{M}}^e; u, v),$$

where $\check{B}^e$ *is a* Γ*-equivariant gerbe on* $\check{M}^e$*, defined later.*

This is a cohomological shadow of some equivalence of derived categories of sheaves on the Hitchin fibres.

Theorem 2.10. *(Hausel-Thaddeus,2003)* $n = 2$ *(using Hitchin 1987),* $n = 3$ *(using Gothen 1994).*

Recall the topological mirror symmetry test / conjecture:

$$E(\check{M}^d; u, v) = E_{st}^{\hat{B}}(\hat{M}^e; u, v)$$

In this lecture we will unravel what this means. For simplicity we assume that $d = e$ (and always assume $(d, n) = 1$).

Recall that Γ acts on $\check{M}$ and hence Γ also acts on $H^*(\check{M})$. We get a decomposition

$$H^*(\check{M}) = \bigoplus_{\kappa \in \hat{\Gamma}} H^*_\kappa(\check{M})$$

and because the action of Γ on $\check{M}$ is algebraic, this leads to a decomposition of mixed Hodge structures. Therefore we can write

$$E(\check{M}; u, v) = E(\check{M}; u, v)^\Gamma + E_{var}(\check{M}; u, v)$$

where the first part is the invariants under Γ, and the second denotes the "variant" part. The variant part decomposes further according to non-trivial characters of Γ as follows:

$$E_{var}(\check{M}; u, v) = \sum_{\kappa \in \hat{\Gamma}^*} E_\kappa(\check{M}; u, v).$$

Here $\hat{\Gamma}^*$ denotes the set of non-trivial irreducible linear characters of Γ. A crucial fact is that the action of Γ on $H^*(\check{M})$ is non-trivial, and so E_{var} is non-zero.

On the other hand, by definition of the stringy E-polynomial (and the fact that Γ is commutative) we have

$$E_{st}^{\hat{B}^d}(\hat{M}; u, v) = E(\check{M}/\Gamma; u, v) + \sum_{\gamma \in \Gamma^*} E(\check{M}_\gamma/\Gamma; L_{B,\gamma}, u, v)(uv)^{F(\gamma)}$$

Of course $E(\check{M}/\Gamma) = E(\check{M})^\Gamma$ and so the topological mirror symmetry test can be rephrased in the form

$$\text{variant part} = \text{stringy contributions}.$$

Note that Γ is canonically isomorphic to $H^1(C, \mathbb{Z}_n)$, where C is our underlying curve. It follows that Poincaré duality gives us a canonical pairing

$$\omega : \Gamma \times \Gamma \to H^2(C, \mathbb{Z}_n) = \mathbb{Z}_n$$

this allows us to identify $\omega : \Gamma \to \Gamma^*$.

This leads to the *refined topological mirror symmetry test*:

$$E_\kappa(\check{M}; u, v) = E(\check{M}_\gamma/\Gamma, L_{\hat{B}^d,\gamma}; u, v)(uv)^{F(\gamma)}$$

where $\gamma = \omega(\kappa)$. This conjecture has roughly the same form as Ngo's main formula.

To understand $\check{M}_\gamma$ we use similar techniques to those of Narasimhan-Ramanan (1975) and obtain the following theorem.

Theorem 2.11 (Hausel-Thaddeus 2003). *Given $\gamma \in \Gamma$ we obtain $L_\gamma \in Pic(C)[n]$ of order $o(\gamma) = m$ dividing n. This yields a covering*

$$\begin{array}{c} C_\gamma \\ \downarrow{\scriptstyle m:1} \\ C \end{array}$$

with Galois group $\mathbb{Z}/m\mathbb{Z}$. If a Higgs bundle $(E, \phi) \in \check{M}$ is fixed by tensoring with L_γ then there exists $(\widetilde{E}, \widetilde{\phi})$ a stable rank n/m Higgs bundle on C_γ such that

$$(E, \phi) = (\pi_* \widetilde{E}, \pi_* \widetilde{\phi})$$

and hence

$$\check{M}_\gamma = \frac{\check{M}(GL_{n/m}, C_\gamma)}{\mathbb{Z}_m = \mathrm{Gal}(C_\gamma/C)} = \begin{array}{c}\textit{“Higgs moduli space for an} \\ \textit{endoscopic group } H_\gamma \textit{ of } SL_n\textit{”.}\end{array}$$

2.8. **The case $n = 2$.** Let us assume $n = 2$ and $d = 1$. Consider circle action of $U(1) \subset \mathbb{C}^*$ on the Higg's moduli space by rescaling the Higgs field (that is, $\lambda \cdot (E, \phi) \mapsto (E, \lambda \cdot \phi)$). We can study the corresponding Morse stratification and obtain a decomposition

$$H^*(\check{M}) = \bigoplus H^{*+\mu_i}(F_i)$$

where the sum is over the connected components of the fixed point set $\check{M}^{\mathbb{C}^*}$, and μ_i denotes the index of F_i with respect to the $U(1)$-action. Note that this decomposition is a decomposition as Γ-modules.

The strata F_i have been described for $n = 2$ by Hitchen in 1987, and by Gothen for $n = 3$ in 1994. The case $n = 4$ seems quite hard.

One obvious fixed point locus if F_0, consisting of those bundles with zero Higg's field. However this doesn't contribute to the variant part as the Γ-action is trivial (recall that this was the crucial result of Harder-Narasimhan).

The other components are labelled by $i = 1, \ldots, g-1$ and are those Higgs bundles of the form

$$F_i = \{(E, \phi) \mid E \cong L_1 \oplus L_2, \phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \varphi \in H^0(L_1^{-1} L_2 K)\}$$

now stability forces $\deg L_2 = 1 - i$ where $i > 0$ (because L_2 is a sub-bundle) and one may check that associating to $(E, \phi) \in F_i$ the divisor of ϕ in $S^{2g-2i-1}(C)$ yields a $2g : 1$ covering

$$F_i \to S^{2g-2i-1}(C)$$

(clearly, if $\Gamma_0 \in \mathrm{Pic}^0(C)$ is 2-torsion then it acts on F_i, but one may moreover check that this action is free).

The following theorem is due to Hitchin in 1987:

Theorem 2.12. *The Γ action on $H^*(F_i)$ is only non-trivial in the middle degree $2g-2i-1$. We have*

$$\dim H^{2g-2i-1}_{var}(F_i) = (2^g-1)\binom{2g-2}{2g-2i-1}$$

Moreover, if $\kappa \in \hat{\Gamma}*$ then

$$\dim H^{2g-2i-1}_{var}(F_i) = \binom{2g-2}{2g-2i-1}.$$

We now consider the stringy side. $\hat{\mathcal{M}} = \check{\mathcal{M}}/\Gamma$ and $\gamma \in \Gamma^*$. Then Γ leads to a connected covering

$$\begin{tikzcd} C_\gamma \arrow[d, "2:1"] \\ C \end{tikzcd}$$

with Galois group $\mathbb{Z}/2\mathbb{Z}$. Consider the commutative diagram

$$\begin{tikzcd} \check{\mathcal{M}}(GL_1, C_\gamma) = T^*\,\mathrm{Jac}^d(C_\gamma) \arrow[r, "\text{push-forward}"] \arrow[d, "\sim"] & \mathcal{M}^d \arrow[d, "\det"] & \supset \check{\mathcal{M}}^d = \det^{-1}(\Lambda_0) \\ T^*\,\mathrm{Jac}^d(C_\gamma) \arrow[r, "N_m(C_\gamma/C)"] & T^*\,\mathrm{Jac}^d(C) & \ni \Lambda_0 \end{tikzcd}$$

Then the endoscopic Higgs bundles are $\check{\mathcal{M}}(GL_1, C_\gamma) = N_m(C_\gamma/C)^{-1}(\Lambda, 0)$ and we have an isomorphism

$$\check{\mathcal{M}}_\gamma = T^*\,\mathrm{Prym}^d(C_\gamma/C)/\,\mathrm{Gal}(C_\gamma, C).$$

One may then calculate that

$$\dim H^{2g-2i+1}(\check{\mathcal{M}}_\gamma/\Gamma, L_{\hat{B}^d,\gamma}) = \binom{2g-2}{2g-2i-1}.$$

and is zero otherwise. (Note that the presence of the gerbe means that we see odd degrees.)

It follows that in this case one does indeed have equality

$$E_\kappa(\check{\mathcal{M}}; u, v) = E(\check{\mathcal{M}}_\gamma; L_{B,\gamma}, u, v)$$

when $\gamma = \omega(\kappa)$.

One may give a similar proof for $n = 3$. The conjecture remains open for $n > 3$.

3. Topological mirror symmetry for character varieties

We give a quick sketch to see the relation between Higgs bundles and character varieties. Given a Higgs bundle (E,ϕ) one obtains (d_E,ϕ) which solves the Hitchin self-duality equation. It follows that $d_E+\phi+\phi*$ yields a complex flat connection on E, and its monodromy yields a representation

$$\pi_1(C) \to GL_n(\mathbb{C}).$$

Definition 3.1. The *character variety for* GL_n is the space

$$\mathcal{M}_B^d := \{(A_1,B_1,\dots,A_g,B_g) \in GL_n(\mathbb{C}) \mid [A_1,B_1]\dots[A_g,B_g] = \zeta_n^d I_n\}//GL_n$$

The *character variety for* SL_n is the space

$$\mathcal{M}_B^d := \{(A_1,B_1,\dots,A_g,B_g) \in SL_n(\mathbb{C}) \mid [A_1,B_1]\dots[A_g,B_g] = \zeta_n^d I_n\}//PGL_n$$

where the action is always by simultaneous conjugation on all factors, and $\zeta_n = e^{2\pi i/n}$.

Of course in the above quotient the scalar matrices act trivially, however it turns out that the residual action of PGL_n is free.

If d and n are coprime, these two varieties are non-singular.

To get the PGL_n space consider $\Gamma = \mathbb{Z}_n^{2g} \subset (\mathbb{C}^*)^{2g}$. Then Γ acts on $\check{M}_B^d$ and we define the character variety of PGL_n to be

$$\hat{\mathcal{M}}_B^d := \check{\mathcal{M}}_B^d/\Gamma = \mathcal{M}_B^d/(\mathbb{C}^*)^{2g}.$$

4. Lecture 6:

We start off by giving some motivation. Recall that $\mathcal{A}$ denotes the Hitchin base. To each point $a \in \mathcal{A}$ one may associate a spectral curve C_a/C. There are various important subsets of the Hitchin base:

$$\mathcal{A}_{ell} = \text{those points where } C_a \text{ is integral;}$$
$$\mathcal{A}_{hyp} = \text{those points where } C_a \text{ is reduced}$$

we have $\mathcal{A}_{ell} \subset \mathcal{A}_{hyp}$. Furthermore there is the "nilpotent cone" which consists of the fibre over $0 \in \mathcal{A}$.

Ngŏ understands the decomposition theorem over $\mathcal{A}_{ell}$ and this was extended by Chaudouard and Laumon to the hyperbolic locus. (Note that the complement of $\mathcal{A}_{hyp}$ to $\mathcal{A}_{ell}$ in the Hitchin base has quite small codimension.)

We define

$$\check{M}_{ell} := \chi_{SL_n}^{-1}(\mathcal{A}_{ell})$$
$$\hat{M}_{hyp} := \chi_{PGL_n}^{-1}(\mathcal{A}_{ell})$$

where χ_{SL_n} (resp. χ_{PGL_n}) denotes the SL_n (resp. PGL_n) Hitchin map.

At this stage it appears that Ngŏ's results imply the topological mirror symmetry conjecture for $\check{M}_{ell}$ and $\hat{M}_{hyp}$. It is not clear how to "spread this out" to get the result in general.

Let us briefly point out that, if one works over appropriate versions defined over finite fields, and $a \in \mathcal{A}_a$ denotes a point in the Hitchin base then the number of points in the fibre over a can be expressed as an orbital integral for $SL_n(\mathbb{F}_q((t)))$. Hence these fibres may be thought of as encoding "harmonic analysis for $SL_n(\mathbb{F}_q((t)))$".

If one passes to the Betti picture (which replaces the total space of the Hitchin fibration with the (diffeomorphic) character variety) then counting points leads to "harmonic analysis for $SL_n(\mathbb{F}_q))$," as we will explain in this lecture! (Note that $SL_n(\mathbb{F}_q)$ is now a finite group, so one expects things to be much simpler.)

4.1. **Character varieties and the topological mirror symmetry conjecture.** Recall the definitions of the various character varieties from last lecture.

$$\mathcal{M}_B^d := \{(A_1, B_1, \ldots, A_g, B_g) \in GL_n(\mathbb{C}) \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\}//PGL_n$$
$$\check{\mathcal{M}}_B^d := \{(A_1, B_1, \ldots, A_g, B_g) \in SL_n(\mathbb{C}) \mid [A_1, B_1] \ldots [A_g, B_g] = \zeta_n^d I_n\}//PGL_n$$
$$\hat{\mathcal{M}}_B^d := \check{\mathcal{M}}_B^d/\Gamma = \mathcal{M}_B^d/(\mathbb{C}^*)^{2g}.$$

where $\Gamma = (\mathbb{Z}_n)^{2g} \subset (\mathbb{C}^*)^{2g}$.

Recall that the *non-abelian Hodge theorem* asserts that we have canonical diffeomorphisms

$$\mathcal{M}_{Dol}^d \cong \mathcal{M}_{DR}^d \cong \mathcal{M}_B^d$$

and similarly for the SL_n and PGL_n versions. Moreover, the later "Riemann-Hilbert" map $\mathcal{M}_{DR}^d \cong \mathcal{M}_B^d$ is an analytic isomorphism.

It follows that SYZ's mirror symmetry proposal should be true for the Betti version also (that is, for the character variety).

This motivates the Betti-Version of the topological mirror symmetry conjecture. (Hausel-Villegas, 2004):

Suppose $(d, n) = (e, n) = 1$. Then we have equality of mixed Hodge polynomials

$$E(\check{M}_B^d; u, v) = E_{st}^{\hat{B}^d}(\mathcal{M}_B^e; u, v)$$

note that, because the complex structures are different, this really is a different conjecture from that for the total space of the Hitchin fibration.

4.2. **An arithmetic technique to calculate E-polynomials.** Recall the definition of the E-polynomial of a complex variety X:

$$E(X;u,v)=\sum_{i,p,q}(-1)^i h^{p,q}(Gr_k^W H_c^i(X))u^p v^q$$

where $W_0\subseteq W_1\subseteq\ldots\subseteq W_i\subseteq\ldots\subseteq W_{2k}=H^k(X)$ is the weight filtration.

$\mathcal{M}_B$ has a Hodge-Tate type MHS, that is, $h^{p,q}\neq 0$ unless $p=q$ in its MHS. In this case the E-polynomial is a polynomial of uv, i.e

$$E(X;u,v)=E(X,uv):=\sum_{i,k}(-1)^i\dim(Gr_k^W H_c^i(X))(uv)^k, \tag{4.1}$$

but the MHS is not pure, i.e $k\neq i$ when $h^{(k/2,k/2)}\neq 0$.

We say that a variety $X/\mathbb{Z}$ is polynomial count, if

$$E(q)=|X(\mathbb{F}_q)|$$

is polynomial in q.

Theorem 4.1 (Katz, 2006)**.** *For a polynomial count variety $X/\mathbb{Z}$*

$$E(X/\mathbb{C},q)=|X(\mathbb{F}_q)|.$$

Example 4.2. Define $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ over $\mathbb{Z}$ as the subscheme $\{xy=1\}$ of $\mathbb{A}^2$. Then

$$E(\mathbb{C}^*;q)=|(\mathbb{F}_q^*)=q-1|.$$

Since $H_c^2(\mathbb{C}^*)$ has weight q and $H_c^1(\mathbb{C}^*)$ has weight 1, substitution to (4.1) gives the same result.

4.3. **Arithmetic harmonic analysis for $\mathcal{M}_B$.** By Fourier transform on a finite group G one gets the following Frobenius-type formula:

$$|\left\{a_1,b_1,\ldots,a_g,b_g\in G|\prod[a_i,b_i]=z\right\}|=\sum_{\chi\in Irr(G)}\frac{|G|^{2g-1}}{\chi(1)^{2g-1}}\chi(z).$$

Therefore assuming that $\zeta_n\in\mathbb{F}_q^*$, i.e $n|q-1$, we get

$$|\mathcal{M}_B^d(\mathbb{F}_q)|=(g-1)\sum_{\chi\in Irr(GL_n(\mathbb{F}_q))}\frac{|GL_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}}\cdot\frac{\chi(\zeta_n^d\cdot I)}{\chi(1)}. \tag{4.2}$$

$\mathrm{Irr}(GL_n(\mathbb{F}_q))$ has a combinatorial description by Green from 1955, and $|\mathcal{M}_B^d(\mathbb{F}_q)|$ can be calculated explicitely, which turns out to be a polynomial, so Katz's theorem applies and (4.2) gives the E-polynomial.

The same Frobenius-type formula is valid in the $SL(n)$-case:

$$|\check{\mathcal{M}}_B^d(\mathbb{F}_q)|=(g-1)\sum_{\chi\in Irr(GL_n(\mathbb{F}_q))}\frac{|SL_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}}\cdot\frac{\chi(\zeta_n^d\cdot I)}{\chi(1)}, \tag{4.3}$$

Here the character table of $SL_n(\mathbb{F}_q)$ is much trickier! After much work of Lusztig, the character table of $\mathrm{Irr}(SL_n(\mathbb{F}_q))$ has only been completed by Bonnafé and Shaji in 2006. For $\chi \in \mathrm{Irr}(GL_n(\mathbb{F}_q))$ the splitting

$$\chi|_{SL_n(\mathbb{F}_q)} = \sum \chi_i$$

is evenly spread out on $\zeta_n^d \cdot I$, and using this Mereb calculated the formula in 2009, showing that $|\check{\mathcal{M}}_B^d(\mathbb{F}_q)|$ is polynomial, and by Katz's theorem this gives $E(\check{\mathcal{M}}_B^d(\mathbb{F}_q); q)$.

Recall that the topologiacal mirror test has the following form:

$$E(\check{\mathcal{M}}_B^d; q) = E_{inv}(\check{\mathcal{M}}_B^d; q) + \sum_{\gamma \in \Gamma^*} E(\hat{\mathcal{M}}_{B,\gamma}^d), L_{B,\gamma}, q) \cdot q^{F(\gamma)}.$$

As we saw before the invariant part is $E_{inv}(\check{\mathcal{M}}_B^d; q) = E(\hat{\mathcal{M}}_B^d; q)$, and it remains to calculate the variant part, i.e the terms $E(\hat{\mathcal{M}}_{B,\gamma}^d), L_{B,\gamma}, q)$. An ongoing work of Hausel, Mereb and Villegas evaluates these by similar (twisted) arithmetic techniques using Deligne's twisted character formula. This seems to match with Mereb's result, which would give the Betti version of TMS.

5. Shadows of Kapuskin-Witten's S-duality

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