Tilting modules and the anti-spherical module

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(joint work with Simon Riche)

Let G denote a connected reductive algebraic group over a field $\Bbbk = \overline{\Bbbk}$ of positive charateristic and Rep its category of rational representations. Let X^+ denote the dominant weights of G (with respect to a fixed $T \subset B$). To any $\lambda \in X^+$ we have a standard ("Weyl") module Δ_λ, a costandard ("induced") ∇_λ and a simple module L_λ all with highest weight λ and canonical maps

$$\Delta_\lambda \twoheadrightarrow L_\lambda \hookrightarrow \nabla_\lambda$$

identifying L_λ as the head (resp. socle) of Δ_λ (resp. ∇_λ). The set $\{L_\lambda\}_{\lambda \in X^+}$ coincides with the set of isomorphism classes of simple rational G-modules.

Denote by Tilt $\subset$ Rep the full additive subcategory of tilting modules. Recall that a module is tilting if it can be written both as a successive extension of Δ and of ∇ modules. By a theorem of Ringel and Donkin for every $\lambda \in X^+$ there exists an indecomposable tilting module T_λ with highest weight λ and the set $\{T_\lambda\}_{\lambda \in X^+}$ coincides with the set of isomorphism classes of indecomposable tilting modules.

Why are we interested in tilting modules? Here are a few pointers:

(1) From the fundamental vanishing $\mathrm{Ext}^i(\Delta, \nabla) = 0$ for $i > 0$ it follows immediately that $\mathrm{Ext}^i(T, T') = 0$ for $T, T' \in$ Tilt. It is then not difficult to see that one has an equivalence

$$K^b(\mathrm{Tilt}) = D^b(\mathrm{Rep})$$

and one can use the order on X^+ to define a t-structure on $K^b(\mathrm{Tilt})$ which recovers Rep. Thus Tilt provides a "minimal homological skeleton" of Rep.

(2) The usual meaning of tilting object is an E which generates the derived category, and satisfies $\mathrm{Ext}^i(E, E) = 0$. Such an E leads to a derived equivalence with $\mathrm{End}(E)$-modules, with one heart "tilted" with respect to the other. Hence one should really call each T_λ a partial tilting object. Rep contains many other tilting objects in this larger sense.

(3) Steinberg's tensor product theorem leads to a recursive "fractal" structure on the simple characters in Rep [5]. In a similar but more complicated manner, the characters of tilting modules display an (only partially understood) fractal behaviour [6].

(4) An extremely important basic theorem on tilting modules: $T \otimes T' \in$ Tilt if $T, T' \in$ Tilt. This is easily deduced from the statement (due to Humphreys, …, Donkin, Mathieu, Kaneda, …) that any tensor product $\Delta_\lambda \otimes \Delta_\mu$ of Weyl modules has a Δ-filtration. This theorem has the consequence (important later) that translation functors preserve Tilt.

(5) Now suppose that $G = GL_n$. Then the natural module V is a Weyl, induced, simple and tilting module. Applying the previous point we see that any tensor power $V^{\otimes m}$ decomposes (non-canonically) as a direct sum of tilting modules. It is an now an easy consequence of Schur-Weyl duality

that if we knew how to write $V^{\otimes m}$ as direct sum of tilting modules then we would know dimensions of all simple modules for all symmetric groups indexed by partitions with $\leq n$ parts. Of course we could do this if we knew the characters if indecomposable tilting modules, but this seems very difficult. For example we know all tilting modules for GL_2, but already for GL_3 there are infinitely many unknown cases. (Compare this to the fact that the characters of the simple rational $G = GL_3$-modules were already known to Jantzen prior to Lusztig's conjecture.)

In my talk I outlined a conjecture giving a combinatorial model for the category of tilting modules. Let Rep_0 denote the principal block, and Tilt_0 its full subcategory of tilting modules. Let us assume that G is semi-simple, simply connected and that the characteristic p of $\Bbbk$ is larger than the Coxeter number of G.[1]

Let W denote the affine Weyl group associated to G which acts by affine translations on $\mathfrak{h}^*$, the real vector space containing the root system of G. Once we have fixed a fundamental alcove we may identify W with the set the alcoves and we have a bijection:

$$\{\text{dominant alcoves}\} \xrightarrow{\sim} {}^fW$$

where ${}^fW$ denotes minimal coset representatives for $W_f \setminus W$, where $W_f \subset W$ is the finite Weyl group. Given any dominant alcove A we may associate to it in a unique way a highest weight of a module in the principal block, and hence objects $L_A, \Delta_A, \nabla_A, T_A \in \mathrm{Rep}_0$. (These facts are consequences of the linkage principal, which we will not explain here.)

For every simple reflection $s \in W$ we have an exact biajoint functor $\theta_s : \mathrm{Rep}_0 \to \mathrm{Rep}_0$ ("translation through the s-wall"). These functors preserve Tilt_0. It is an easy exercise in the combinatorics of these functors to see that we have an isomorphism of Grothendieck groups

$$\mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W \xrightarrow{\sim} [\mathrm{Tilt}_0] = [\mathrm{Rep}_0]$$

such that the (right) action of a translation functor $[\Theta_s]$ on the right hand side corresponds to multiplication by $(1 + s)$ on the left hand side. The left hand module is sometimes called the "anti-spherical module", which explains the title of this talk.

Now let $\mathcal{D}$ denote the diagrammatic category of Soergel bimodules determined by the affine Cartan matrix corresponding to G over $\Bbbk$, as defined in [2]. This is a graded monoidal category with hom spaces enriched in graded $\Bbbk[\alpha_0, \dots, \alpha_m]$-modules, where $\alpha_0, \dots, \alpha_m$ denote the simple (affine) roots and $\deg \alpha_i = 2$. In [2] it is proved that the isomorphism classes of indecomposable objects are classified up to shift by W, and that the split Grothendieck group of $\mathcal{D}$ is isomorphic to the Hecke algebra H of W. (This theorem is a natural generalization of a theorem of Soergel.) We write B_w for the self-dual indecomposable object indexed by w.

A natural categorification of the anti-spherical module is given by

$$\mathcal{AS} := \Bbbk \otimes_{\mathcal{D}_f} \mathcal{D}$$

[1]These assumptions are for simplicity only, there should be a variants for small p.

where the tensor product means that quotient by the ideal of $\mathcal{D}$ generated by morphisms which factor through objects B_w for $w \notin {}^fW$ as well as morphisms of the form $R^{>0} \cdot f$ (polynomials of positive degree). The category $\mathcal{AS}$ is a right $\mathcal{D}$-module category enriched in graded finite dimensional vector spaces. Let $\mathcal{AS}_{\mathrm{deg}}$ denote the category obtained from $\mathcal{AS}$ by forgetting the grading on hom spaces (the "degrading").

Main Conjecture: We can equip Tilt_0 with the structure of a right $\mathcal{D}$-module category with $B_s := \theta_s$. Moreover, we have an equivalence of right $\mathcal{D}$-module categories:

$$\mathcal{AS}_{\mathrm{deg}} \xrightarrow{\sim} \mathrm{Tilt}_0$$

Notes and consequences of the conjecture:

(1) Basically it says that two natural ways of categorifying the anti-spherical module are equivalent. Hence it can be seen as a "uniqueness of categorification" statement. Rather surprisingly (for me) the first statement in the conjecture implies the second.
(2) The conjecture implies that Tilt_0 admits a grading (defined by $\mathcal{AS}$) and that this grading is defined over $\mathbb{Z}$ (because $\mathcal{D}$ is defined over $\mathbb{Z}$). Over fields of characteristic zero we understand $\mathcal{D}$ quite well (Kazhdan-Lusztig conjectures . . .). It follows that the conjecture immediately implies Lusztig's conjecture for large p.
(3) There is an analogue of this conjecture over $\mathbb{C}$ where Tilt_0 is replaced by the principal block of representations of the quantum group at an ℓ^{th} root of unity. This conjecture has recently been checked for $U_q(\mathfrak{sl}_2)$ by Andersen and Tubbenhauer [1].
(4) This conjecture should be Koszul dual to a conjecture of Finkelberg and Mirkovic [4]. Perhaps it is more tractable because translation functors are "built in" to the equivalence.
(5) A version of the conjecture for singular weights should imply a conjecture of Rickard [7] and the version for the quantum group should imply a conjecture of Chuang-Miyachi [3].
(6) There is also a part of the conjecture explaining the tilting tensor product theorem in terms of the action of Gaitsgory's central sheaves, but we're running out of space...

References

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