Checking Lusztig's conjecture around the Steinberg weight

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Let $(R \subset X, R^\vee \subset X^\vee)$ be a root datum and $G \supset T$ the corresponding split Chevalley group scheme over $\mathbb{Z}$. Fix an algebraically closed field k. A fundamental question in representation theory is to determine the simple rational modules of G_k. Here "determine" means: Can they be parametrised? What are their dimensions? What are their characters? Can one give a uniform construction/description? etc.

Let p denote the characteristic of k. In characteristic $p = 0$ there exists a uniform construction and description. To describe it we choose a system of positive roots $R^+ \subset R$, a basis $\Delta \subset R^+$ and let $B \subset G$ be the Borel subgroup corresponding to the negative roots $R^- = -R^+$. To each $\lambda \in X$ one can associate a line bundle $\mathcal{L}(\lambda) := G \times_B k_\lambda$ over the flag variety $(G/B)_k$. Here k_λ denotes the B-module which is obtained by inflation from the one-dimensional T-module given by the character $\lambda \in X$ of T. If we set

$$\nabla(\lambda) = H^0(G/B, \mathcal{L}(\lambda))$$

then it is known that $\nabla(\lambda)$ is non-zero if and only if λ belongs to the cone of dominant characters:

$$X^+ = \{\lambda \in X \mid \langle \alpha^\vee, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Moreover, in the later case $\nabla(\lambda)$ is a simple G_k-module. One obtains in this way a bijection

$$X^+ \xrightarrow{\sim} \mathrm{Irr} G_k$$

where $\mathrm{Irr} G_k$ denotes the set of isomorphism classes of simple rational G_k-modules. Moreover, the characters are given by Weyl's character formula.

If $p > 0$ the situation is more complicated. Let us assume for simplicity that k is an algebraic closure of $\mathbb{F}_p$, the finite field with p elements. It is a priori obvious that things will be more complicated than (or at the very least different to) the characteristic zero case. The reason is that we have a Frobenius map

$$\mathrm{Fr} : G_k \to G_k$$

obtained by elevating coordinates to the p^{th} power. Hence given any G_k-module V, we can produce another G_k-module $V^{(1)}$ (or in fact infinitely many new modules $V^{(m)}$) by precomposing (m times) with the Frobenius morphism. This operation is called "Frobenius twist". It is easy to see, for example, that it preserves simple modules. It leads to a recursive structure on the category of rational representations of G_k which is only partly understood, and is a big part of the fascination of the subject.

In positive characteristic one may still define the modules $\nabla(\lambda)$ as above, but they are not in general simple. However they contain a unique simple module $L(\lambda)$. It turns out that $L(\lambda)$ still has highest weight λ and hence the above bijection between simple modules and X^+ and $\mathrm{Irr} G_k$ remains true. However the dimensions and characters of $L(\lambda)$ are not known. By Kempf's vanishing theorem one still knows the characters of the module $\nabla(\lambda)$ (as in the case $p = 0$ they are given by Weyl's character formula). Much of the recent work on determining the simple G_k modules focuses on understanding the composition series for the modules $\nabla(\lambda)$.

A cornerstone of the subject are two theorems of Steinberg. Consider the set

$$X_r^+ = \{\lambda \in X^+ \mid \langle \alpha^\vee, \lambda + \rho \rangle < p^r \text{ for all } \alpha \in \Delta\}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. Steinberg's tensor product theorem asserts that if we have weights λ_0, λ_1, ..., λ_m all belonging to X_1^+ then the module

$$L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_m)^{(m)}$$

is simple (and is hence isomorphic to $L(\lambda_0 + p\lambda_1 + \cdots + p^m\lambda_m)$). Hence, in order to understand the simple representations of G_k it is enough to understand $L(\lambda)$ with $\lambda \in X_1^+$.

The second important theorem is Steinberg's restriction theorem. It states that for any $\lambda \in X_r^+$, the restriction of $L(\lambda)$ to the finite group $G(\mathbb{F}_{p^r})$ is simple. Moreover on obtains all simple $kG(\mathbb{F}_{p^r})$-modules in this way. This theorem is probably why we are discussing rational G_k-modules at a conference on finite groups!

To get to Lusztig's conjecture we need to recall two more pieces of structure theory. Consider W_p, the subgroup of all affine transformations of X generated by reflections in the hyperplanes $H_{\alpha,pm} = \langle \alpha^\vee, \lambda + \rho \rangle \in pm$ for all $\alpha \in R^+$ and $m \in \mathbb{Z}$. The linkage principle is the statement:

$$\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda \in W_p \cdot \mu.$$

Denote by $\mathrm{Rep}G_k$ the category of all rational representations of G_k. Given $\pi \subset X^+$ let $\mathrm{Rep}_\pi G_k$ denote the full subcategory of all objects whose composition factors belong to $\{L(\lambda) \mid \lambda \in \pi\}$. The linkage principle implies that we have a decomposition:

$$\mathrm{Rep}G_k = \bigoplus_{\pi \in X/W_p} \mathrm{Rep}_{\pi \cap X^+} G_k.$$

Finally, the translation principle states that, as long as $p > h$ (so that 0, the weight of the trivial module, lies on no hyperplane $H_{\alpha,pm}$) we understand all multiplicities $[\nabla(\lambda) : L(\mu)]$ as long as we understand the multiplicities $[\nabla(x \cdot 0) : L(y \cdot 0)]$ for $x, y \in W_p$ with $x \cdot 0, y \cdot 0 \in X^+$. Steinberg's tensor product theorem even allows us to assume that $x \cdot 0, y \cdot 0 \in X_1^+$.

Lusztig's conjecture then expresses

$$[\nabla(x \cdot 0) : L(y \cdot 0)] \text{ for } x \cdot 0, y \cdot 0 \in X_1^+$$

in terms of an affine Kazhdan-Lusztig polynomial. It is important to note that the assumption that $x \cdot 0, y \cdot 0 \in X_1^+$ is essential: even though the statement makes sense for any $x \cdot 0, y \cdot 0 \in X^+$ it is certainly false in full generality (even though it will remain true within the "Jantzen region").

Let us pause to note that even if Lusztig's conjecture is true this is a slightly unsatisfactory state of affairs. For weights outside of the X_1^+ there is no conceptual understanding of the situation: in order to calculate a character many iterations of Steinberg's tensor product theorem and Lusztig's conjecture may be necessary.

It is a result due to Andersen, Jantzen and Soergel that Lusztig's conjecture is true for large p, and an effective (but enormous) bound has recently been provided by Fiebig. On the other hand, there is very little experimental evidence for the validity of Lusztig's conjecture. It is known in rank 2 (using Jantzen's sum formula), for a few rank 3 cases, for a few primes in higher rank ...

In the late 1990's Soergel suggested that a useful toy-model for Lusztig's conjecture would be provided by looking "around the Steinberg weight". To be precise, assume from now on that $p > h$, let W denote the Weyl group of our root system acting on X in the standard way and let $\mathrm{st} = (p-1)\rho$ denote the Steinberg weight (the extremal vertex of the fundamental box X_1). Consider the sets

$$\Omega = \{\mathrm{st} + x\rho \mid x \in W\}, \ \leq \Omega = \{\lambda \mid \lambda \leq p\rho\} \text{ and } < \Omega = \leq \Omega \setminus \Omega.$$

Now consider the quotient category

$$\mathcal{O}_p = \mathrm{Rep}_{\leq \Omega} G_k / \mathrm{Rep}_{< \Omega} G_k.$$

We let

$$L(x) := \overline{L(\mathrm{st} + x\rho)}, \quad \nabla(x) := \overline{\nabla(\mathrm{st} + x\rho)}$$

denote the images in $\mathcal{O}_p$. Then $\mathcal{O}_p$ is a finite length, abelian, highest weight category and Lusztig's conjecture predicts

$$[\nabla(x) : L(y)] = h_{w_0 y, w_0 x}(1)$$

where w_0 denotes the longest element of W and $h_{w_0 y, w_0 x} \in \mathbb{Z}[v]$ is a Kazhdan-Lusztig polynomial (this time for the finite Weyl group). Let us emphasise that (as far as we know) the above statement is weaker than the original statement. As we mentioned above, the above should be thought of as a toy model or "sanity check" for Lusztig's original conjecture.

Recently (building on the work of Soergel and Elias-Khovanov) Elias and the author proved the existence of a $\mathbb{Z}$-algebra A which is a free and finitely generated over $\mathbb{Z}$ such that

i) $A_{\mathbb{C}} - \mathrm{mod} \cong \mathcal{O}_0$ "principal block of category $\mathcal{O}$" for $\mathfrak{g} = \mathrm{Lie}\, G_{\mathbb{C}}$.
ii) $A_k - \mathrm{mod} \cong \mathcal{O}_k$ "modular category $\mathcal{O}$".

Moreover, A may be described by generators and relations. Let us make the following remakes:

a) One can think of A as interpolating between characteristic zero representation theory ($\mathcal{O}_0$ is where Kazhdan-Lusztig polynomials made their first appearance "in nature") and modular representation theory. Hence one can think of the above result as "freeing p".
b) A admits a grading and hence $\mathcal{O}_p$ admits a grading $\widetilde{\mathcal{O}_p}$. With Riche and Soergel we have recently proved a "modular Koszul duality":

$$D^b(\widetilde{\mathcal{O}_p}) \cong D^b_{(B^\vee_{\mathbb{C}})}\widetilde{(G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}, k)}$$

c) A_k is Morita equivalent to an ext algebra of parity sheaves on $G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}$.

d) By results of Fiebig there is also a version of A which controls the full Lusztig conjecture (related to the affine Weyl group, rather than the finite Weyl group). It is unclear to the author what this category has to do with the whole principal block of Rep G_k.

Following the "freeing p" reasoning, one can also study the representation theory of A in characteristics below the Coxeter number (where $\mathcal{O}_p$ stops behaving well). Consider the statement

$$(*)_p : \qquad \text{the decomposition matrix of } A \text{ is trivial.}$$

Because of ii) above, Lusztig's conjecture would imply that $(*)_p$ is true for $p > h$.

Using the explicit description of A, we can do computer calculations to check $(*)_p$ in low rank. Here is a summary of the cases where $(*)_p$ holds:

A_n	B_n	D_n	F_4	G_2	E_6 (partial)
all p for $n < 6$ $p \neq 2$ for $n = 7$	$p \neq 2$ for $n < 6$	$p \neq 2$ for $n < 6$	$p \neq 2, 3$	$p \neq 2, 3$	$p \neq 2, 3$

The entry $p \neq 2$ in A_7 is due to Braden (2002). The exclusions $p \neq 2, 3$ in E_6 are due to Polo and Riche. The entries $p \neq 3$ for F_4 and E_6 give a counterexample to Fiebig's "GKM-conjecture". Thanks are also due to Jean Michel for help speeding up my programs significantly.

Recently Polo has found an example to show that $(*)_p$ fails in A_{4p-1}. So the situation is more complicated than one might have thought ...

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