The Hecke algebra and Soergel bimodules

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(joint work with Ben Elias)

Let (W, S) be a Coxeter system and $\mathcal{H} = \bigoplus_{x \in W} \mathbb{Z}[q^{\pm 1/2}]T_x$ its Hecke algebra. In 1970 Kazhdan and Lusztig [7] defined a remarkable basis $C'_x = q^{-\ell(x)/2}(\sum_{y \le x} P_{y,x} T_y)$ for $\mathcal{H}$ and conjectured:

(1) $P_{y,x} \in \mathbb{N}[q]$;
(2) if $C'_x C'_y = \sum \mu^z_{xy} C'_z$ then $\mu^z_{xy} \in \mathbb{N}[q^{\pm 1/2}]$.

This conjecture became known as the *Kazhdan-Lusztig positivity conjecture*. It was solved by Kazhdan and Lusztig [8] for finite and affine Weyl groups by interpreting $P_{y,x}$ as Poincaré polynomials of local intersection cohomology of Schubert varieties. Their proof was generalized using Kac-Moody flag varieties to crystallographic Coxeter groups (groups in which all orders of products of two simple reflections belong to $\{2, 3, 4, 6, \infty\}$) by a number of authors. Kazhdan-Lusztig polynomials have gone on to occupy a central place in Lie theory and representation theory, starting with the famous Kazhdan-Lusztig conjectures on the characters of simple highest weight modules over complex semi-simple Lie algebras. The Kazhdan-Lusztig conjecture was solved in 1981 by Beilinson and Bernstein [1] and Brylinski and Kashiwara [3].

In 1990 Soergel [10] defined certain modules over the coinvariant algebra and used them to give another proof of the Kazhdan-Lusztig conjecture. Like the previous proof of Beilinson-Bernstein and Brylinski-Kashiwara, Soergel's proof relies on results from geometry. However Soergel reduced the reliance to one result, the famous "decomposition theorem" of Beilinson, Bernstein, Deligne and Gabber [2]. Moreover the decomposition theorem has a simple algebraic translation: one hopes that a certain module over the coinvariant algebra decomposes as a direct sum in a way dictated by the Hecke algebra. Though easily stated, this algebraic statement has resisted an algebraic proof for the last twenty years.

In a sequel [11] Soergel introduced equivariant analogues of his modules which have come to be known as Soergel bimodules. More recently [12] Soergel has shown that his bimodules always provided a categorification of the Hecke algebra, and conjectured that a basis given by indecomposable Soergel bimodules categorifies the Kazhdan-Lusztig basis. He explained why his conjecture implies the Kazhdan-Lusztig positivity conjectures.

Since its proof by Beilinson, Bernstein, Deligne and Gabber the decomposition theorem has had two other proofs. The first, by Saito, uses his technology of mixed Hodge modules. He replaces the reliance of the earlier proof on Frobenius weights with weights in the sense of mixed Hodge theory. More recently, de Cataldo and Migliorini [4, 5] have given a proof of the decomposition theorem using classical Hodge theory. Using several beautiful arguments they translate the geometric statement of the decomposition theorem to questions about forms and filtrations on the cohomology of smooth projective varieties, and use Hodge theory (more precisely the weak Lefschetz theorem and the Hodge-Riemann bilinear relations) to prove that these statements hold. In the process they uncover several remarkable signature properties of forms involved in the decomposition theorem, leading to what they dub the "decomposition theorem with signs".

In [6] Ben Elias and the author establish these Hodge theoretic properties for Soergel bimodules in a purely algebraic/combinatorial manner. By adapting the arguments of de Cataldo and Migliorini we are able to deduce Soergel's conjecture. Hence the Kazhdan-Lusztig positivity conjecture holds. The evidence is mounting that Soergel bimodules provide the right setting in which to study Kazhdan-Lusztig theory for arbitrary Coxeter groups.

References

[1] A. Beilinson, J. Bernstein, *Localisation de g-modules.* C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.

[2] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers.* Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.

[3] J.-L. Brylinski, M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems.* Invent. Math. **64** (1981), no. 3, 387–410.

[4] M. A. de Cataldo, L. Migliorini, *The hard Lefschetz theorem and the topology of semismall maps.* Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 759–772.

[5] M. A. de Cataldo, L. Migliorini, *The Hodge theory of algebraic maps.* Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693–750.

[6] B. Elias, G. Williamson, *The Hodge theory of Soergel bimodules.* preprint, arxiv:1212.0791.

[7] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras.* Invent. Math. **53** (1979), no. 2, 165–184.

[8] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality.* Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 185203.

[9] M. Saito, *Introduction to mixed Hodge modules.* Actes du Colloque de Théorie de Hodge (Luminy, 1987).

[10] W. Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe.* J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.

[11] W. Soergel, *The combinatorics of Harish-Chandra bimodules.* J. Reine Angew. Math. **429** (1992), 49–74.

[12] W. Soergel, *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln ber Polynomringen.* J. Inst. Math. Jussieu **6** (2007), no. 3, 501–525.