Some examples of parity sheaves

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In 2008 I made a bet with Peter Fiebig involving a case of very good wine: I bet that by 2017 there would be "significant progress" on Lusztig's conjecture, and by 2022 a "complete solution". Of course the terms in inverted commas are open to interpretation (there are different versions of Lusztig's conjecture involving different bounds, what about a single counter-example? etc.) However even arrival at a point where we have to debate these questions would indicate significant progress.

In this sense my talk is selfish: I want to convince as many people as possible to start thinking about Lusztig's conjecture, so that I maximise my chances of getting some bottles of good wine. (Not to mention my second selfish motivation: Peter knows a lot more about wine than I do.) The main goal of my talk is to convince you that there are very difficult questions involved, but that things are happening. Recent work shows that Lusztig's conjecture is not the inpenetrable fortress that many think it is.

1. Parity sheaves

Let X denote a complex algebraic variety equipped with a Whitney stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

by locally closed connected smooth subvarieties. We write d_λ for the complex dimension of X_λ and for $i_\lambda : X_\lambda \hookrightarrow X$ the inclusion. Fix a field k (of characteristic $p \geq 0$). Write $\mathrm{Loc}(X_\lambda)$ for the abelian category of local systems of finite dimensional k-vector spaces on X_λ and $D_\Lambda(X)$ for the full subcategory of the derived category of sheaves of k-vector spaces consisting of Λ-constructible complexes.[1]

A *pariversity* is a function $\dagger : \Lambda \to \mathbb{Z}/2\mathbb{Z}$. We will only ever care about two special pariversities: the *constant* pariveristy $\natural(\Lambda) = \overline{0}$; and the *dimension* pariversity $\diamond(\lambda) = \overline{d_\lambda}$. For a fixed pariveristy $\dagger$ we say that a complex $F \in D_\Lambda(X)$ is $\dagger$-*even* if $\mathcal{H}^m(i_\lambda^? F) = 0$ for $\overline{m} \not\cong \dagger(\lambda)$ modulo 2 and all $\lambda \in \Lambda$ and $? \in \{*, !\}$. Furthermore, F is $\dagger$-*parity* if $F \cong F_0 \oplus F_1$ with F_0 and $F_1[1]$ both $\dagger$-even.

Example: $F \in D_\Lambda(X)$ is $\natural$-even if its stalks and costalks vanish in odd degree.

We make the following (strong) assuption on our stratification:

$$H^{\mathrm{odd}}(X_\lambda, \mathcal{L}) = 0 \qquad \text{for all } \mathcal{L} \in \mathrm{Loc}(X_\lambda). \qquad (P)$$

For example, if all the strata are simply connected this is the assumption that $H^{\mathrm{odd}}(X_\lambda) = 0$. Certainly our assumption (P) forces $\mathrm{Loc}(X_\lambda)$ to be semi-simple.

Fix a pariversity λ. A somewhat surprising consequence of the above assumption is the following theorem, which was discovered in joint work with D. Juteau and C. Mautner: Given any indecomposable (=simple) local system $\mathcal{L}$ on X_λ

[1] Also known as the "bounded derived category of Λ-constructible complexes" although this is slightly misleading.

there exists up to isomorphism at most one indecomposable †-parity sheaf $\mathcal{E}^{\dagger}(\lambda, \mathcal{L})$ extending $\mathcal{L}[d_\lambda]$. Moreover, any indecomposable †-parity sheaf is isomorphic to $\mathcal{E}^{\dagger}(\lambda, \mathcal{L})[m]$ for some λ, $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$ and $m \in \mathbb{Z}$. If it exists we call $\mathcal{E}^{\dagger}(\lambda, \mathcal{L})$ a *parity sheaf*.

Some remarks:

i) If they exist, the above theorem shows that parity sheaves are classified in the same was as intersection cohomology complexes. However, parity sheaves need not exist and when they do they need not be perverse.
ii) Condition (P) is very restrictive. Sometimes it is useful to replace it with an equivariant version. For example this allows one to discuss parity sheaves on nilpotent cones (under explicit mild restrictions on p) and toric varieties.
iii) Our work on parity sheaves was inspired by work of Soergel who showed the existence and uniqueness of certain complexes on the flag variety obtained as direct summands of direct images from Bott-Samelson resolutions. He obtained his classification by relating the endomorphism algebras of these complexes to the endomorphism algebras of projective objects in "modular category $\mathcal{O}$". Whilst performing calculations on nilpotent cones with sheaf coefficients of characteristics 2 and 3 we noticed a similar phenomenon to that observed by Soergel. This led us to look for a geometric classification.
iv) The proof of the classification result is formally similar to the classification of tilting objects in highest weight categories by Ringel and Donkin.

2. Parity sheaves on flag varieties

For the rest of this talk we will restrict ourselves to the case of $X = G/B$ for a Kac-Moody group G and Borel subgroup B. The reader can certainly think about a finite flag variety of a connected complex reductive algebraic group G, for example $G = GL_n(\mathbb{C})$.

We let W denote the Weyl group of G and consider the stratification with $\Lambda = W$ given by the Bruhat decomposition:

$$X = G/B = \bigsqcup_{\lambda \in \mathrm{L}} X_\lambda = \bigsqcup_{x \in W} BxB/B.$$

Each X_λ is isomorphic to an affine space, and hence our parity assumption (P) is satisfied. One can also show that in this case (using an inductive "Deligne" construction) that parity sheaves exist and are unique for any pariversity †.

The following examples hopefully convince the reader of the usefulness of the notion of parity sheaves:

i) if k is a field of characteristic 0 (or of any sufficiently large characteristic) then $\mathcal{E}^{\natural}(w) \cong \mathrm{IC}(\overline{X_w})$, the intersection cohomology complex of the Schubert variety $\overline{X_w}$.
ii) in any characteristic one has $\mathcal{E}^{\diamond}(w) = T(w)$, the indecomposable tilting sheaf with support $\overline{X_w}$.

From now on we focus on the case of $\dagger = \natural$. Under this understanding we omit $\dagger$ from all notation below.

3. The p-canonical basis

As above, we assume that $X = G/B$ with its Bruhat stratification. Recall that W is the Weyl group and let S denote the subset of simple reflections. We order W using the Bruhat order.

We consider $\mathcal{H}$ the Hecke algebra of (W,S). This is an associative $\mathbb{Z}[v^{\pm 1}]$-algebra which is free as an $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_w \mid w \in W\}$. The multiplication is determined by

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w, \\ (v^{-1} - v)H_w + H_{sw} & \text{if } sw < w. \end{cases}$$

Recall that the Hecke algebra $\mathcal{H}$ possesses a remarkable Kazhdan-Lusztig basis $\{\underline{H}_w \mid w \in W\}$. For example $\underline{H}_s = H_s + vH_{id}$. It has the following positivity properties:

i) $\underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x$ where $h_{x,w} \in v\mathbb{N}[v]$;
ii) $\underline{H}_x \underline{H}_y = \sum \mu_{xy}^z \underline{H}_z$ with $\mu_{xy}^z \in \mathbb{N}[v^{\pm 1}]$.

We now recall the geometric meaning of this basis. Given a finite dimensional vector space $V = \bigoplus V^i$ let $\mathrm{ch}V = \sum \dim V^{-i} v^i$ denote its Poincaré polynomial. Given $F \in D^b_W(X)$ define

$$\mathrm{ch}F = \sum_{x \in W} \mathrm{ch}H^*(F_x) v^{-\ell(x)} H_x.$$

Then a fundamental theorem of Kazhdan-Lusztig states that $\mathrm{chIC}(x,\mathbb{Q}) = \underline{H}_x$ ("Kazhdan-Lusztig polynomials encode local rational intersection cohomology of Schubert varieties"). This result is the key to understanding the positivity properties stated above.

Because $\mathcal{E}(x,\mathbb{Q}) \cong \mathrm{IC}(x,\mathbb{Q})$ we are tempted to try to understand $\mathrm{ch}\mathcal{E}(x,k)$ in a similar fashion for arbitrary k. Let us set

$${}^p\underline{H}_x := \mathrm{ch}\mathcal{E}(x,k).$$

(Recall that p denotes the characteristic of k.)

Then we have:

i) ${}^p\underline{H} = H_w + \sum_{x<w} {}^p h_{x,w} H_x$ with ${}^p h_{x,w} \in \mathbb{N}[v^{\pm 1}]$. Hence $\{\underline{H}_w \mid w \in W\}$ is a basis, which we call the *p-canonical basis* of $\mathcal{H}$,
ii) ${}^p\underline{H}_w = \sum {}^p m_{x,y} \underline{H}_x$ with ${}^p m_{x,w} \in \mathbb{N}[v^{\pm 1}]$,
iii) ${}^p\underline{H}_x {}^p\underline{H}_y = \sum {}^p\mu_{xy}^z {}^p\underline{H}_z$ with ${}^p\mu_{x,y}^z \in \mathbb{N}[v^{\pm 1}]$.

A note on the proofs of these positivity properties: i) follows easily from the definition of ch and the facts that $i_x \mathcal{E}(x,k) = \underline{k}_{X_x}[\ell(x)]$ and $\mathrm{supp}\mathcal{E}(x,k) \subset \overline{X_x}$. One may show that the characters of indecomposable parity sheaves only depend on the characteristic of k. ii) then follows from the fact that $\mathcal{E}(x,\mathbb{F}_p)$ admit "integral forms" $\mathcal{E}(x,\mathbb{Z}_p)$ and $\mathcal{E}(x,\mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to a direct sum of intersection cohomology complexes. Lastly parity sheaves admit lifts to the equivariant derived category $D_B(X,k)$ where there is a convolution formalism categorifying the multiplication in the Hecke algebra. iii) then follows because the convolution of two parity sheaves is isomorphic to a direct sum of shifts of parity sheaves.

One knows essentially nothing about the basis ${}^p\underline{H}_x$ in general except that, for fixed x, ${}^p\underline{H}_x = \underline{H}_x$ for big enough p, but we can't say how big is enough. To get a feeling for this basis we ask three questions:

- Q1) For which p and x is ${}^p\underline{H}_x = \underline{H}_x$?
- Q2) For fixed p understand the equivalence classes generated by $x \sim y$ if ${}^p m_{x,y} \neq 0$.
- Q3) Describe ${}^p m_{x,y}$ in general.

In the hope of giving some understanding of what these questions involve we consider parity sheaves on the affine Grassmannian which are equivariant with respect to $G[[t]]$-orbits. Equivalently, we consider the elements ${}^p\underline{H}_x$ where x is an element of the affine Weyl group which is maximal in its double coset for the finite Weyl group. Because parity sheaves correspond to tilting modules (if $p > h+1$), we can translate the above questions as follows:

- i) Q1: when is $T(\lambda) = \Delta(\lambda)$ ($\Leftrightarrow \Delta(\lambda) = L(\lambda)$)? This is known (but complicated). For example, it holds if λ belongs to the fundamental alcove.
- ii) Q2: which standard modules may occur in composition factors of tilting modules? For regular weights this is the linkage principle.
- iii) Q3: determine the multiplicities of standard modules in tilting modules. This is unknown (and is presumably very difficult).

Examples related to Lusztig's conjecture:

- i) Soergel has shown that ${}^p\underline{H}_x = \underline{H}_x$ for $p > h$ on a finite flag variety is equivalent to Lusztig's conjecture "around the Steinberg weight".
- ii) Fiebig has shown that ${}^p\underline{H}_x = \underline{H}_x$ for certain elements of the affine Weyl group (those indexing weights in the intersection of the principal block and fundamental box) for $p > h$ implies Lusztig's conjecture.

Given i) above it seems sensible to do experiments. Experiments have been made possible by the following result (building on work of Libedinsky and Elias-Khovanov) of Elias and the author: The monoidal category of Soergel bimodules can be described by generators and relations.

We present a summary of our findings: We have $\underline{H}_x = {}^p\underline{H}_x$ for all p in the following table

A_n	B_n	D_n	F_4	G_2	E_6 (partial)
all p for $n < 6$ $p \neq 2$ for $n = 7$	$p \neq 2$ for $n < 6$	$p \neq 2$ for $n < 6$	$p \neq 2,3$	$p \neq 2,3$	$p \neq 2,3$

The entry $p \neq 2$ in A_7 is due to Braden (2002). With his help I have recently been able to determine the full p-canonical basis in A_7 (38 of the 40320 elements of A_7 satisfy ${}^p\underline{H}_x \neq \underline{H}_x$). The exclusions $p \neq 2,3$ in E_6 are due to Polo and

Riche. The entries $p \neq 3$ for F_4 and E_6 give a counterexample to Fiebig's "GKM-conjecture". Thanks are also due to Jean Michel for helping me speed up my programs significantly.

Recently Polo has shown that for all primes p there is an x in a Weyl group of type A_{4p-1} with ${}^p\underline{H}_x \neq \underline{H}_x$!! So to summarise: the next few years will be interesting ones as far as Lusztig's conjecture is concerned!

References

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[3] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*. J. Pure Appl. Algebra, 2000.

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