Singular Soergel bimodules

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To Markus

Contents

CHAPTER 1

Introduction

Let (W, S) be a Coxeter system and let $T = \bigcup_{w \in W} wSw^{-1}$ denote the reflections in W. A finite dimensional representation V of W is *reflection faithful* if it is faithful and, for all $w \in W$, V^w has codimension 1 in V if and only if w is a reflection. For example, if W is finite then the geometric representation of W is reflection faithful.

Let us fix a reflection faithful representation V of W over a field k of characteristic 0. Let R be the ring of regular functions on V, graded so that V^* has degree 2. The ring R carries a W-action by functoriality.

Let us call a subset $I \subset S$ *finitary* if the associated standard parabolic subgroup $W_I = \langle I \rangle \subset W$ is finite. Given a finitary subset $I \subset S$ denote by R^I the invariants in R under W_I. Furthermore, if $I, J \subset S$ are finitary denote by R^I-Mod-R^J the category of graded (R^I, R^J)-bimodules.

We want to define certain subcategories ${}^I\mathcal{B}^J \subset R^I$-Mod-$R^J$ for all pairs of finitary subsets $I, J \subset S$. Note first that if $I, J, K \subset S$ are finitary and satisfy $I \supset K \subset J$ then both R^I and R^J are graded subrings of R^K and hence we may regard R^K as an object in R^I-Mod-R^J. Roughly, we obtain the categories ${}^I\mathcal{B}^J$ by tensoring all combinations of such bimodules together and taking direct summands.

More precisely, given two finitary subset $I, J \subset S$ we define ${}^I\mathcal{B}^J$ to be the smallest full additive subcategory of R^I-Mod-R^J which contains all objects isomorphic to direct summands of shifts of objects of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

where $I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset \cdots \subset J_{n-1} \supset I_n = J$ are finitary subsets of S. We obtain in this way categories of *singular Soergel bimodules*.

Given any double coset $p \in W_I \setminus W / W_J$ we consider the subvariety

$${}^I\mathrm{Gr}_p^J \subset V/W_I \times V/W_J$$

obtained as the image of the subvariety $\{(x\lambda, \lambda) \mid x \in p, \lambda \in V\}$ of $V \times V$ under the quotient map. The Bruhat order on W descends to a Bruhat order on $W_I \setminus W / W_J$ which we also denote by $\leq$. We write ${}^I\mathrm{Gr}_{\leq p}^J$ (resp. ${}^I\mathrm{Gr}_{<p}^J$) for the union of all ${}^I\mathrm{Gr}_q^J$ with $q \leq p$ (resp. $q < p$). We may regard any $M \in R^I$-Mod-R^J as an $R^I \otimes R^J$-module and hence as a quasi-coherent sheaf on $V/W_I \times V/W_J$, which allows us to speak of support of M or $m \in M$. We denote by $\Gamma_{\leq p} M$ (resp. $\Gamma_{<p} M$) the submodule of sections supported on ${}^I\mathrm{Gr}^J_{\leq p}$ (resp ${}^I\mathrm{Gr}^J_{<p}$).

Our main theorem is the following:

THEOREM 1. *There is a natural bijection:*

$$W_I \backslash W / W_J \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{indecomposable bimodules in } {}^I\mathcal{B}^J \\ \textit{(up to shifts in the grading).} \end{array} \right\}$$

More precisely, for every $p \in W_I \backslash W / W_J$ there exists a unique isomorphism class (up to shifts) of indecomposable bimodules $M \in {}^I\mathcal{B}^J$ whose support is ${}^I\mathrm{Gr}^J_{\leq p}$.

In order to explain why one would be interested in proving such a theorem we need to introduce the Hecke category. Recall that the Hecke algebra is the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_w \mid w \in W\}$ and multiplication:

$$H_w H_s = \begin{cases} H_{ws} & \text{if } ws > w \\ (v^{-1} - v) H_w + H_{ws} & \text{if } ws < w. \end{cases}$$

The Hecke algebra has a *duality involution* which sends H_w to $H_{w^{-1}}^{-1}$ and v to v^{-1} and a self-dual *Kazhdan-Lusztig basis* $\{\underline{H}_w \mid w \in W\}$. For all finitary $I, J \subset S$ we consider the $\mathbb{Z}[v, v^{-1}]$-submodule

$${}^I\mathcal{H}^J = \{h \in \mathcal{H} \mid \underline{H}_s h = h \underline{H}_t = (v + v^{-1}) h \text{ for all } s \in I \text{ and } t \in J\}.$$

The duality involution on $\mathcal{H}$ induces an (anti-linear) endomorphism on each ${}^I\mathcal{H}^J$ and each ${}^I\mathcal{H}^J$ possesses a standard basis $\{ {}^I H^J_p \mid p \in W_I \backslash W / W_J\}$. If $I, J, K \subset S$ are finitary there exists a product

$$\begin{aligned} {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K &\to {}^I\mathcal{H}^K \\ (f, g) &\mapsto f *_J g \end{aligned}$$

which is a certain renormalisation of the product in the Hecke algebra. It is natural to view this structure as a $\mathbb{Z}[v, v^{-1}]$-linear category which we call the *Hecke category*: the objects are finitary subsets $I \subset S$, and the morphisms from I to J consist of the module ${}^I\mathcal{H}^J$.

For any bimodule $M \in {}^I\mathcal{B}^J$ and $p \in W_I \backslash W / W_J$ the subquotient $\Gamma_{\leq p} M / \Gamma_{<p} M$ is isomorphic to a finite direct sum of shifts of certain "standard modules" which may be described explicitly. It is therefore natural to define a character

$$\begin{aligned} \mathrm{ch} : {}^I\mathcal{B}^J &\to {}^I\mathcal{H}^J \\ M &\mapsto \sum h_p \, {}^I H^J_p \end{aligned}$$

where $h_p \in \mathbb{N}[v, v^{-1}]$ counts the graded multiplicity of the standard module in the subquotient $\Gamma_{\leq p} M / \Gamma_{<p} M$.

Our second main theorem is that the collection of categories ${}^I\mathcal{B}^J$ for all finitary subsets $I, J \subset S$ "categorifies" the Hecke category.[1]

Theorem 2. *If $I, J, K \subset S$ are finitary we have a commutative diagram*

$$\begin{tikzcd} {}^I\mathcal{B}^J \times {}^J\mathcal{B}^K \arrow[r, "-\otimes_{R^J}-"] \arrow[d, "\mathrm{ch}\times\mathrm{ch}"] & {}^I\mathcal{B}^K \arrow[d, "\mathrm{ch}"] \\ {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \arrow[r, "-*_J-"] & {}^I\mathcal{H}^K \end{tikzcd}$$

Moreover, one may choose representatives $\{{}^IB_p^J \mid p \in W_I\backslash W/W_J\}$ for each isomorphism class of indecomposable bimodules (up to shifts) such that $\{\mathrm{ch}({}^IB_p^J)\}$ gives a self-dual basis of ${}^I\mathcal{H}^J$ and

$$\mathrm{ch}({}^IB_p^J) = {}^IH_p^J + \sum_{q \le p} g_{q,p}\, {}^IH_q^J \quad \textit{for some } g_{q,p} \in \mathbb{N}[v, v^{-1}].$$

If $I = J = \emptyset$ then ${}^I\mathcal{H}^J$ is the Hecke algebra and we write $\mathcal{B}$ instead of ${}^I\mathcal{B}^J$. In this case Theorem 1 tells us that the isomorphism classes of indecomposable objects in $\mathcal{B}$ are parametrised, up the shifts, by W and Theorem 2 tells us that their characters yield a special basis for the Hecke algebra. This result was obtained by Soergel in [**So6**] (using a slightly different definition of $\mathcal{B}$) and formed the principle motivation for this work. Similar ideas have also been pursued by Dyer in [**Dy1**] and [**Dy2**], and by Fiebig in [**Fie1**], [**Fie2**] and [**Fie3**].

It is natural to ask for a description of the characters of the indecomposable modules in ${}^I\mathcal{B}^J$. We write B_x for a representative of the isomorphism class of indecomposable objects parametrised by $x \in W$, normalised as in Theorem 2. Soergel has proposed the following:

Conjecture 1 ([**So6**], Vermutung 1.13). *For all $x \in W$ we have*

$$\mathrm{ch}(B_x) = \underline{H}_x.$$

For arbitrary finitary subsets $I, J \subset S$ there exists a Kazhdan-Lusztig basis $\{{}^I\underline{H}_p^J\}$ for ${}^I\mathcal{H}^J$. The following relates the objects in the categories ${}^I\mathcal{B}^J$ and $\mathcal{B}$ and shows that Soergel's conjecture implies character formulae for all indecomposable singular bimodules.

Theorem 3. *Let $I, J \subset S$ be finitary, $p \in W_I\backslash W/W_J$ and denote by p_+ the unique element of p of maximal length. Then we have an isomorphism:*

$$R \otimes_{R^I} {}^IB_p^J \otimes_{R^J} R \cong B_{p_+} \quad \textit{in } R\textit{-Mod-}R.$$

[1] The Hecke category is already a category. Thus in order to make this statement more precise we should equip the collection of categories ${}^I\mathcal{B}^J$ together with tensor product ${}^I\mathcal{B}^J \times {}^J\mathcal{B}^K \to {}^I\mathcal{B}^K$ with the structure of a 2-category. We will not need this formalism and will be happy with a rough idea of what "categorification" means in this context.

In particular, if Soergel's conjecture is true then

$$\mathrm{ch}({}^I B_p^J) = {}^I \underline{H}_p^J.$$

Let V be a reflection faithful representation of W over an infinite field of positive characteristic. As shown in [**So6**] it is still possible to define the category $\mathcal{B}$ and one obtains the same classification of indecomposable objects[2] however Conjecture 1 is not expected to be true in general. As pointed out in [**So4**] an understanding of the characters of the indecomposable bimodules (in particular when $\mathrm{ch}(B_x) = \underline{H}_x$) would have important applications in the representation theory of groups of Lie type in positive characteristic.

Our last result is a combinatorial method by which one may verify the characters of the indecomposable bimodules in some cases. We define, based on the W-graph of (W, S), a certain subset $\sigma(W) \subset W$ of *separated elements* and confirm Soergel's conjecture for $x \in \sigma(W)$.

THEOREM 4. *Suppose that $x \in \sigma(W)$. Then* $\mathrm{ch}(B_w) = \underline{H}_w$.

The proof relies on elementary properties of the basis $\{\mathrm{ch}(B_w)\}$ and hence works in arbitrary characteristic.

Of course, in order to apply this theorem it is necessary to know the set $\sigma(W)$. The essential ingredient in the calculation of the set $\sigma(W)$ is the W-graph of the Coxeter system (W, S). Unfortunately, even in simple situations the W-graph can be very complicated and no general description is known. However, using Fokko du Cloux's program *Coxeter* [**dC1**] we use a computer to determine the set $\sigma(W)$ for low rank, finite Coxeter groups.

The simplest situation is when $\sigma(W) = W$. This occurs in type A_n for $n \leq 6$. In other types and type A_n for $n \geq 7$ our techniques are not as effective. In most examples that we have computed $\sigma(W)$ is not the entire Weyl group. However, we are able to confirm the characters for approximately 99% of all indecomposable bimodules in ranks ≤ 6. We also believe that the elements $x \notin \sigma(W)$ for which our methods fail will provide an interesting source of future research.

As a second motivation for studying the categories ${}^I\mathcal{B}^J$ we will describe how, in certain special situations, the categories ${}^I\mathcal{B}^J$ appear as "algebraic models" of certain categories of perverse sheaves. We will not need the following again, however we believe it is important to have in mind.

Let G be a connected reductive algebraic group over $\mathbb{C}$ equipped with the classical topology and $T \subset B \subset G$ a maximal torus and Borel subgroup respectively. Let W be the Weyl group and S its simple

[2]We believe the same should be true for the categories ${}^I\mathcal{B}^J$ of singular Soergel bimodules in positive characteristic, but have not yet pursued this. See Perspective 1) at the end of the introduction.

reflections. For any subset $I \subset S$ of simple reflections we let P_I denote the corresponding standard parabolic subgroup.

Let $V = \mathrm{Lie}\, T$ be the Lie algebra of T, a vector space over $\mathbb{C}$ which has an induced linear action of W. The representation of W on V is reflection faithful and we are therefore in the earlier situation: we write R for the regular functions on V and obtain full subcategories ${}^I\mathcal{B}^J$ of R^I-Mod-R^J for all $I, J \subset S$.

For any pair $I, J \subset S$ we let $P_I \times P_J$ act on G via $(p_1, p_2) \cdot g = p_1 g p_2^{-1}$ and consider the equivariant derived category $D^b_{P_I \times P_J}(G)$ with coefficients in $\mathbb{C}$ as defined in [**BL**]. If $X \subset G$ is stable under $P_I \times P_J$ we write $\underline{X}$ for the equivariant constant sheaf on X, extended by zero to G. As explained in [**So7**], for all triples $I, J, K \subset S$ there exists a convolution functor

$$D^b_{P_I \times P_J}(G) \times D^b_{P_J \times P_K}(G) \to D^b_{P_I \times P_K}(G)$$

which we denote $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} *_{P_J} \mathcal{G}$.

For all pairs $I, J \subset S$ we define ${}^I\mathcal{P}^J$ to be the smallest full additive category of $D^b_{P_I \times P_J}(G)$ which contains all objects isomorphic to direct summands of shifts of objects of the form

$$\underline{P_{J_1}} *_{P_{I_2}} \underline{P_{J_2}} *_{P_{I_3}} \cdots *_{P_{I_{n-1}}} \underline{P_{J_{n-1}}} \tag{1}$$

where $I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset \cdots \subset J_{n-1} \supset I_n = J$ are all subsets of S and for $1 \leq i < n$ we regard $\underline{P_{J_i}}$ as an object in $D^b_{P_{I_i} \times P_{I_{i+1}}}(G)$.

Given any complex algebraic group H, H-variety X and $\mathcal{F} \in D^b_H(X)$ the equivariant hypercohomology $\mathbb{H}^\bullet_H(\mathcal{F})$ is a graded module over $H^\bullet_H(pt)$. As $H^\bullet_{P_I \times P_J}(pt) = R^I \otimes R^J$ (eg. [**Bri**], Proposition 1), given any $\mathcal{F} \in D^b_{P_I \times P_J}(G)$ we may regard $\mathbb{H}^\bullet_{P_I \times P_J}(\mathcal{F})$ as an object in R^I-Mod-R^J.

Using the formalism of Bernstein-Lunts ([**BL**], Theorem 12.7.2) one can show that the equivariant hypercohomology of (1) coincides with the tensor product

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n} \in R^I\text{-Mod-}R^J.$$

It follows that $\mathbb{H}^\bullet_{P_I \times P_J}$ restricts to a functor between ${}^I\mathcal{P}^J$ and ${}^I\mathcal{B}^J$. By combining the arguments in [**So5**] or [**So4**] with Theorem 5.4.1 of this thesis one may show that $\mathbb{H}^\bullet_{P_I \times P_J}$ is fully-faithful. It follows that hypercohomology is essentially surjective and we obtain an equivalence of graded additive categories

$$\mathbb{H}^\bullet_{P_I \times P_J} : {}^I\mathcal{P}^J \xrightarrow{\sim} {}^I\mathcal{B}^J.$$

It is natural to ask what the classification of indecomposable objects in ${}^I\mathcal{B}^J$ means under this equivalence. Using the equivariant version of the decomposition theorem ([**BL**], Theorem 5.3) one may show that every object in ${}^I\mathcal{P}^J$ is isomorphic to a direct sum of shifts of equivariant intersection cohomology complexes. The intersection cohomology complexes are indecomposable and are parametrised by the $P_I \times P_J$ orbits on G, which in turn are parametrised by $W_I \backslash W / W_J$. If we denote by $\mathrm{IC}_p \in D^b_{P_I \times P_J}(G)$ the equivariant intersection cohomology complex corresponding to $p \in W_I \backslash W / W_J$ one has

$$\mathbb{H}^{\bullet}_{P_I \times P_J}(\mathrm{IC}_p) \cong {}^I B^J_p.$$

In some sense this “explains” Theorem 1 in this context. In [**So5**] these techniques are used to prove Conjecture 1 for finite Weyl groups. In [**Hä**], Härterich generalises these techniques to establish Conjecture 1 for certain representations of affine Weyl groups.

Perspectives: We finish this introduction with a list of five areas that we believe deserve further work.

(1) For simplicity we have always assumed that our representation is over a field of characteristic 0. Instead, we probably should require the weaker statement:

$$R \text{ is graded free over } R^I \text{ for all finitary subsets } I \subset S.$$

The only point where we really use characteristic 0 (rather than the above condition) is during the proof of Corollary 3.3.5, and another argument would have to be found in this case.

(2) Given any representation V of W over a field of characteristic zero such that every reflection $t \in T$ fixes a hyperplane, the definition of the categories of singular Soergel bimodules still makes sense. However, at certain points the arguments used to classify the indecomposable objects break down. Somewhat surprisingly, in [**Li2**] Libedinsky has show that, if $I = J = \emptyset$ and there exists an inclusion of representations $V \subset \widetilde{V}$ where the action of W on $\widetilde{V}$ is reflection faithful, then the indecomposable objects defined using V and $\widetilde{V}$ are in bijection and have the same characters. By Proposition 2.1 in [**So6**] it is always possible to find a reflection faithful representation of W containing the geometric representation as a subrepresentation. Hence one may define the category $\mathcal{B}$ by using the geometric representation. It appears straightforward to extend Libedinsky's arguments to the singular situation (that is $I, J \neq \emptyset$), however this should be written down. It should also be noted that in [**Li1**] Libedinsky has recently obtained explicit expressions for all morphisms between “Bott-Samelson” bimodules in $\mathcal{B}$. It is not clear if his techniques generalise to the singular situation.

(3) In [**Kh**] Khovanov has shown how to produce a knot-invariant (related to the HOMFLYPT polynomial) by taking Hochschild homology of a complex of Soergel bimodules known as the

Rouquier complex (see [**Ro**]). In [**WW**] Webster and the author have shown how one may reinterpret some steps in this construction geometrically. It would be interesting to find applications of singular Soergel bimodules in this theory.

(4) The set $\sigma(W) \subset W$ of separated elements (appearing in Theorem 4 above) has the advantage of being simple to define and (relatively) easy to compute. However, the disadvantage is that if $x \notin \sigma(W)$ there is no direct way to obtain more information on the character of B_x, for any given representation. Also, brute force calculation of B_x via computer seems difficult.

In [**Wi**] we hope to explain (extending results of Soergel in [**So4**]) that, after fixing an infinite field k of characteristic not too small (for example $\operatorname{char} k > 5$ is always sufficient) there is an intimate relationship between intersection cohomology complexes of Schubert varieties with coefficients in k, and Soergel bimodules constructed using a representation over the same field. In particular, if the graded dimensions of the stalks of intersection cohomology of a Schubert variety corresponding to $x \in W$ over k are different to those in characteristic 0, then it is not possible to have $\operatorname{ch}(B_x) = \underline{H}_x$.

Hence one may instead search for examples where the stalks of intersection cohomology over $\mathbb{Z}$ have torsion. Recently, in [**Bra**], Braden has made some progress on this problem and discovered examples of 2-torsion in type A_7 and D_4. It would be interesting to combine his techniques with the set $\sigma(W)$. More generally, the importance for representation theory of understanding intersection cohomology in positive characterstic and over $\mathbb{Z}$ is becoming clear (see [**Fie4**], [**Ju**] and [**MV**]).

(5) Let $\Phi \subset \mathfrak{h}$ be a crystallographic root system in a real Euclidean vector space and $W_f \subset W$ the corresponding finite and affine Weyl groups. Let $S \subset W$ be the simple reflections and $I \subset S$ be such that $W_I = W_f$. After choosing a reflection faithful representation of V one may apply the above construction to obtain a tensor category ${}^I\mathcal{B}^I$. In this case ${}^I\mathcal{H}^I$ is an $\mathbb{Z}[v, v^{-1}]$-algebra known as the "adjoint spherical Hecke algebra". It is a fact known as the Satake isomorphism (see [**Lu1**]) that the spherical Hecke algebra is a deformation of the representation ring of the adjoint semi-simple group $G_a^\vee$ with root system $\Phi^\vee$ dual to Φ. Using this fact, one may show that if one normalises the representatives $\{{}^IB_p^I \mid p \in W_I \backslash W / W_I\}$ as in Theorem 2 then any tensor product ${}^IB_p^I \otimes_{R^I} {}^IB_q^I$ is isomorphic to a direct sum of ${}^IB_r^I$ for $r \in W_I \backslash W / W_I$ without shifts. We therefore obtain a tensor subcategory ${}^I\mathcal{B}_0^I$ containing all ${}^IB_p^I$ for $p \in W_I \backslash W / W_I$. It is natural to expect an equivalence of

tensor categories

$$ {}^I\mathcal{B}_0^I \cong \operatorname{Rep} G_a^\vee $$

where $\operatorname{Rep} G_a^\vee$ denotes the tensor category of finite dimensional algebraic representations of $G_a^\vee$. Making this precise should certainly involve [**MV**]. One would hope to be able to enlarge ${}^I\mathcal{B}_0^I$ to a category $\widetilde{{}^I\mathcal{B}_0^I}$ so as to obtain an equivalence

$$ \widetilde{{}^I\mathcal{B}_0^I} \cong \operatorname{Rep} G_{sc}^\vee $$

where $G_{sc}^\vee$ is the simply connected algebraic group with root system $\Phi^\vee$. A further challenge would be make everything work over a field of positive characteristic and possibly find a connection with tilting modules.

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List of important notation

W, S	a Coxeter group and its simple reflections	*15*
T	the reflections in W	*15*
ℓ	the length function on W	*15*
I, J, K, L	finitary subsets of S, (i.e. $W_I, W_J, \dots$ are finite)	*16*
W_I	the standard parabolic subgroup generated by I	*16*
w_I	the longest element in W_I	*16*
$\pi(I)$, $\widetilde{\pi}(I)$	two Poincaré polynomials of W_I	*16*
$W_I \backslash W / W_J$	the (W_I, W_J)-double cosets in W	*17*
p, q, r	elements of $W_I \backslash W / W_J$	*17*
p_+, p_-	the maximal and minimal elements in p	*17*
$\pi(p)$, $\widetilde{\pi}(p)$	Poincaré polynomials of p	*17*
$w_{I,p,J}$	the longest element in $W_{I \cap p_- J p_-^{-1}}$	*18*
$\pi(I,p,J)$, $\widetilde{\pi}(I,p,J)$	Poincaré polynomials of $W_{I \cap p_- J p_-^{-1}}$	*18*
$\leq$	the Bruhat order on $W_I \backslash W / W_J$	*18*
$\mathcal{H}$	the Hecke algebra	*22*
H_w, $\underline{H}_w$	a standard and Kazhdan-Lusztig basis element	*22*
${}^I\mathcal{H}^J$	a hom space in the Hecke category	*24*
${}^I H_p^J$	a standard basis element in ${}^I\mathcal{H}^J$	*25*
${}^I \underline{H}_p^J$	a Kazhdan-Lusztig basis element in ${}^I\mathcal{H}^J$	*25*
${}^I H^J$	a standard generator of the Hecke category	*25*
$\langle \cdot, \cdot \rangle$	the bilinear form on ${}^I\mathcal{H}^J$	*29*
V	a reflection faithful representation of W	*35*
h_t	an equation for $V^t \subset V$	*35*
R	the graded ring of regular functions on V	*35*
R^I	the W_I-invariants in R	*35*
${}^I R_p^J$	the standard object indexed by $p \in W_I \backslash W / W_J$	*36*
$R(X)$	the enlarged ring of regular functions	*38*
${}^I\mathrm{Gr}_p^J$	the (twisted) graph of p in $V/W_I \times V/W_J$	*46*
${}^I\mathrm{Gr}_C^J$	the union of graphs of all $p \in C \subset W_I \backslash W / W_J$	*46*
$\Gamma_C M$	sections of M with support in ${}^I\mathrm{Gr}_C^J$	*47*
$\Gamma^p M$	the stalk of M at p	*47*
$\Gamma_p^{\leq} M$, $\Gamma_p^{\geq} M$	support subquotients of M	*47*
${}^I\nabla_p^J$, ch_∇	a nabla module and the nabla character	*56*
${}^I\Delta_p^J$, ch_Δ	a delta module and the delta character	*64*
D	the duality functor	*65*
${}^I\mathcal{B}_{BS}^J$	the category of Bott-Samelson bimodules	*67*
${}^I\mathcal{B}^J$	the category of singular Soergel bimodules	*67*
${}^I B_p^J$	an indecomposable singular Soergel bimodule	*76*

CHAPTER 2

Coxeter Groups and Hecke Algebras

1. Coxeter groups

In this section we recall standard facts about Coxeter groups and standard parabolic subgroups, as as well as their Poincaré polynomials and double cosets. We then introduce translation sequences, which are a generalisation of reduced expressions.

1.1. Fundamental properties. In this section we recall standard facts about Coxeter groups. References for this section are [**Hu**] and [**Bo**].

Recall that a *Coxeter system* (W, S) is a group W together with a set of distinguished generators $S \subset W$ subject only to the relations

$$(st)^{m(s,t)} = id \text{ for all } s, t \in S$$

where $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a symmetric function satisfying $m(s, s) = 1$ and $m(s, t) \geq 2$ for all $s \neq t \in S$. The elements of S have order 2 and are called the *simple reflections*. The *reflections* consist of the set

$$T = \bigcup_{w \in W} wSw^{-1} \subset W.$$

A *reduced expression* for $w \in W$ is an expression for w in the elements of S of minimal length. The *length* of w, $\ell(w)$, is the length of a reduced expression for w. If W is finite there is a unique element of longest length, the *longest element* $w_0 \in W$. One has $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for all $w \in W$. The length of an element $w \in W$ may be expressed in terms of reflections:

$$\ell(w) = |\{t \in T \mid wt < w\}| = |\{t \in T \mid tw < w\}|. \tag{1.1.1}$$

The *Bruhat order* $\leq$ on W is the partial order generated by the relation $wt \leq w$ if $t \in T$ and $\ell(wt) \leq \ell(w)$. Alternatively, $x \leq w$ if and only if x may be obtained as a subexpression of a reduced expression for w.

To the Coxeter system (W, S) one may associate a directed, edge-labelled graph $\mathcal{G}_W$, as follows. Its vertices correspond to the elements of W and there is a directed edge from x to tx labelled by t for all $x \in W$ and $t \in T$ such that $x < tx$. The graph $\mathcal{G}_W$ is the *Bruhat graph* of W. If $X \subset W$ is a subset we denote by $\mathcal{G}_X$ the full subgraph with vertices X.

1.2. Parabolic subgroups and double cosets. Given a subset $I \subset S$ we consider the subgroup $W_I \subset W$ generated by I. This is a *standard parabolic subgroup* of W. This is also a Coxeter group with presentation

$$(st)^{m(s,t)} = id \text{ for all } s, t \in I.$$

In this work we will be chiefly concerned with *finite* standard parabolic subgroups.

Definition 1.2.1. *A subset $I \subset S$ is* finitary *if W_I is finite. If $I \subset S$ is finitary we denote by $w_I \in W$ the longest element of W_I. The Poincaré polynomials of W_I are the elements in $\mathbb{N}[v, v^{-1}]$ defined by*

$$\widetilde{\pi}(I) = \sum_{w \in W_I} v^{-2\ell(w)} \quad \textit{and} \quad \pi(I) = v^{\ell(w_I)}\widetilde{\pi}(I).$$

Let $f \mapsto \overline{f}$ be the involution of $\mathbb{Z}[v, v^{-1}]$ which fixes $\mathbb{Z}$ and sends v to v^{-1}. We will call elements $f \in \mathbb{Z}[v, v^{-1}]$ satisfying $f = \overline{f}$ *self-dual.* Because $\ell(w_I x) = \ell(w_I) - \ell(x)$ for all $x \in W_I$ it follows that $\pi(I)$ is self-dual. It will be useful to always define two normalisations of Poincaré polynomials: π will always be self-dual and $\widetilde{\pi}$ will always lie in $\mathbb{N}[v^{-1}]$.

Definition 1.2.2. *Given $I \subset S$ we define*

$$D_I = \{w \in W \mid ws > w \text{ for all } s \in I\} \quad \text{and} \quad {}_I D = (D_I)^{-1}.$$

If $I \subset S$ is finitary we define

$$D^I = \{w \in W \mid ws < w \text{ for all } s \in I\} \quad \text{and} \quad {}^I D = (D^I)^{-1}.$$

The elements of D_I and D^I (resp. ${}_I D$ and ${}^I D$) are called the minimal *and* maximal left *(resp.* right*)* coset representatives.

The terminology is justified by the following proposition.

Proposition 1.2.3. *Let $I \subset S$. Every left coset of W_I contains precisely one element of D_I and this is the unique element of minimal length. Furthermore, $\ell(wu) = \ell(w) + \ell(u)$ for all $w \in D_I$ and $u \in W_I$. If I is finitary, then every left coset contains a unique element of maximal length, this element lies in D^I and $\ell(zu) = \ell(z) - \ell(u)$ for all $z \in D^I$ and $u \in W_I$. Analogous statements hold for right cosets using ${}_I D$ and ${}^I D$.*

Proof. For the case of minimal elements see Proposition 1.10 of [**Hu**] or Proposition 2.2.3 of [**Ca**]. If W_I is finite then the case of maximal elements follows from this. □

Corollary 1.2.4. *If $I \subset J \subset S$ are finitary then*

$$\frac{\widetilde{\pi}(J)}{\widetilde{\pi}(I)} = \sum_{w \in W_J \cap D_I} v^{-2\ell(w)} \in \mathbb{N}[v^{-1}].$$

It is natural to ask if the generalisation of the above proposition is true for double cosets of two standard parabolic subgroups.

DEFINITION 1.2.5. *Given two subsets $I, J \subset S$ we define*

$$ {}_I D_J = {}_I D \cap D_J. $$

If I and J are finitary we define

$$ {}^I D^J = {}^I D \cap D^J. $$

The set ${}_I D_J$ does indeed give minimal distinguished double coset representatives:

PROPOSITION 1.2.6. *Let $I, J \subset S$. Every double coset $p = W_I x W_J$ contains a unique element of ${}_I D_J$ and this is the element of smallest length in p. If I and J are finitary then p also contains a unique element of ${}^I D^J$, and this is the unique element of maximal length.*

PROOF. For the case of minimal elements see [**Ca**], Proposition 2.7.3. The case of maximal elements if W_I and W_J are finite follows by similar arguments. □

DEFINITION 1.2.7. *Let $I, J \subset S$. We denote the set of (W_I, W_J)-double cosets of W by $W_I \backslash W / W_J$. More generally, if $X \subset W$ is a union of (W_I, W_J)-double cosets we write $W_I \backslash X / W_J$ for the (W_I, W_J)-double cosets contained in X. Given $p \in W_I \backslash W / W_J$ we denote by p_- the unique element of minimal length. If I and J are finitary, we denote by p_+ the unique element of maximal length in p. We call p_- and p_+ the* minimal *and* maximal double coset representatives *respectively. The* Poincaré polynomials *of p are the elements in $\mathbb{N}[v, v^{-1}]$ defined by*

$$ \widetilde{\pi}(p) = v^{2\ell(p_-)} \sum_{x \in p} v^{-2\ell(x)} \quad \text{and} \quad \pi(p) = v^{\ell(p_+) - \ell(p_-)} \widetilde{\pi}(p). $$

In will be important in the sequel to be able to describe intersections of (not necessarily standard) parabolic subgroups. This is the subject of Kilmoyer's theorem.

THEOREM 1.2.8 (Kilmoyer). *Let $I, J \subset S$ and $p \in W_I \backslash W / W_J$. Then*

$$ W_I \cap p_- W_J p_-^{-1} = W_{I \cap p_- J p_-^{-1}}. $$

PROOF. See [**Ca**], Theorem 2.7.4. □

The following theorem is a generalisation of Lemma 1.2.3 to double cosets.

THEOREM 1.2.9 (Howlett). *Let $I, J \subset S$ and $p \in W_I \backslash W / W_J$. Setting $K = I \cap p_- J p_-^{-1}$ the map*

$$ \begin{array}{rcl} (D_K \cap W_I) \times W_J & \to & p \\ (u, v) & \mapsto & u p_- v \end{array} $$

is a bijection satisfying $\ell(u p_- v) = \ell(p_-) + \ell(u) + \ell(v)$.

PROOF. See [**Ca**], Theorem 2.7.5. □

The intersection $I \cap p_- J p_-^{-1}$ emerges often enough to warrent special notation.

DEFINITION 1.2.10. *Let $I, J \subset S$ be finitary, $p \in W_I \backslash W / W_J$ and set $K = I \cap p_- J p_-^{-1}$. We define $\widetilde{\pi}(I,p,J) = \widetilde{\pi}(K)$, $\pi(I,p,J) = \pi(K)$ and $w_{I,p,J} = w_K$.*

COROLLARY 1.2.11. *Let $I, J \subset S$ be finitary and $p \in W_I \backslash W / W_J$. We have the identitities:*

$$\ell(p_+) - \ell(p_-) = \ell(w_I) + \ell(w_J) - \ell(w_{I,p,J}) \tag{1.2.1}$$

$$\widetilde{\pi}(p)\widetilde{\pi}(I,p,J) = \widetilde{\pi}(I)\widetilde{\pi}(J) \tag{1.2.2}$$

$$\pi(p)\pi(I,p,J) = \pi(I)\pi(J) \tag{1.2.3}$$

$$\overline{\pi(p)} = \pi(p). \tag{1.2.4}$$

PROOF. The first three statements follow from from Howlett's theorem and Corollary 1.2.4. The last statement follows because $\pi(I)$, $\pi(J)$ and $\pi(I,p,J)$ are all self-dual and therefore so is $\pi(p)$. ☐

PROPOSITION 1.2.12. *Let $I, J \subset S$ and $p \in W_I \backslash W / W_J$. All edges of the Bruhat graph of W restricted to $p \subset W$ are generated by reflections in W_I and W_J. In other words, if x and tx both lie in p then either $t \in W_I$ or $tx = xt'$ for some reflection $t' \in W_J$.*

PROOF. We may assume $tx < x$ and write $x = up_-v$ as in Theorem 1.2.9. After choosing reduced expressions for u, p_- and v we obtain a reduced expression for x by concatenation. By the exchange condition ([**Hu**], Theorem 5.8 or [**Bo**], IV, Proposition 4), we may obtain an expression for tx by omitting a reflection from a reduced expression for x. However, using our reduced expression above, we must omit a reflection from either u or v in order to stay in p, and the proposition follows. ☐

Recall that W becomes a poset when equipped with the Bruhat order.

DEFINITION 1.2.13. *Given finitary $I, J \subset S$ the Bruhat order on $W_I \backslash W / W_J$ (which we also denote by $\leq$) is the partial order defined by setting $p \leq q$ if $p_- \leq q_-$ in W. We say that a subset $C \subset W_I \backslash W / W_J$ is* downwardly *(resp.* upwardly*)* closed *if $p \in C$ and $q \leq p$ (resp. $q \geq p$) implies $q \in C$.*

Given a poset $(X, \leq)$ and $x \in X$ we will often abuse notation and write $\{\leq x\}$ (resp. $\{< x\}$) for the set of elements in X less (resp. strictly less) than x, and similarly for $\{\geq x\}$ and $\{> x\}$.

Let $\phi : X \to Y$ be a map between the underlying sets of two posets $(X, \leq)$ and $(Y, \leq)$. We call ϕ a *morphism of posets* if $x_1 \leq x_2$ implies that $\phi(x_1) \leq \phi(x_2)$. If ϕ is a morphism of posets we call ϕ $\leq$*-strict* if, whenever $\phi(x) \leq y$ for $x \in X$ and $y \in Y$ there exists $x' \in \phi^{-1}(y)$ such that $x \leq x'$. Similarly, ϕ is $\geq$-*strict* if, whenever $y \leq \phi(x)$ there exists $x' \in \phi^{-1}(y)$ such that $x' \leq x$. If ϕ is $\leq$-strict (resp. $\geq$-strict) then $\phi^{-1}(\{\leq y\})$ (resp. $\phi^{-1}(\{\geq y\})$) is equal to $\{x \mid x \leq x' \text{ for some } x' \in \phi^{-1}(y)\}$ (resp. $\{x \mid x \geq x' \text{ for some } x' \in \phi^{-1}(y)\}$. We call ϕ *strict*, if it is both $\leq$-strict and $\geq$-strict.

PROPOSITION 1.2.14. *Let $I \subset K$ and $J \subset L$ be subsets of S. The quotient map*

$$\mathrm{qu} : W_I \backslash W / W_J \to W_K \backslash W / W_L$$

is a strict morphism of posets.

PROOF. In order to show that qu is a morphism of posets we need to show that if $x \leq y$ in W and p and q are the (W_K, W_L)-double cosets containing x and y respectively, then $p_- \leq q_-$. By choosing simple reflections in W_I and W_J which reduce y, and repeatedly applying the fact that $x \leq y$ implies that either $x \leq ys$ or $xs \leq ys$ for $s \in S$ and similarly on the left ([**Hu**], Proposition 5.9) we see that $x' \leq q_-$ for some $x' \in p$ and hence $p_- \leq q_-$. Thus qu is a morphism of posets. Lastly, if $\mathrm{qu}(p) \leq q$ for some $p \in W_I \backslash W / W_J$ and $q \in W_K \backslash W / W_L$ then $\mathrm{qu}(p)_- \leq q_-$. By multiplying by suitable elements $s \in K$ on the left and $t \in L$ on the right and using the previous fact, we see that $p_- \leq w$ for some $w \in q$ and so $p \leq p'$ for some $p' \in \mathrm{qu}^{-1}(q)$. Hence qu is $\leq$-strict. Similar arguments show that qu is $\geq$-strict. $\square$

Let qu be as in the proposition and choose $q \in W_K \backslash W / W_L$. The set $\mathrm{qu}^{-1}(q)$ always has a maximal element p. Because qu is strict it follows that

$$\mathrm{qu}^{-1}(\{\leq q\}) = \{\leq p\} \text{ and } \mathrm{qu}^{-1}(\{\geq q\}) = \{\geq p\}.$$

The following fact will be needed in in the sequel.

LEMMA 1.2.15. *Let $I \subset K$ and $J \subset L$ be finitary subsets of S. If $p \in W_I \backslash W / W_J$ and $q \in W_K \backslash W / W_L$ are such that $p \subset q$ then*

$$\frac{\pi(K, q, L)}{\pi(I, p, J)} \in \mathbb{N}[v, v^{-1}].$$

PROOF. We may assume that either $I = K$ and $J = L$. If $I = K$ then, by imitating the arguments used in the proof of [**Ca**], Lemma 2.7.1 one may show that $I \cap p_- J p_-^{-1} \subset K \cap q_- L q_-^{-1}$ and the lemma follows in this case by Corollary 1.2.4. The case $J = L$ follows by inversion and the fact that two conjugate subsets of S have the same Poincaré polynomials. $\square$

We will need the following proposition when we come to discuss Demazure operators.

PROPOSITION 1.2.16. *Let p be a double coset and $x \in p$. We have*

$$\ell(p_+) - \ell(x) = |\{t \in T \mid x < tx \in p\}|.$$

PROOF. Let $u \in W_I$ and $v \in W_J$ and set $y = uxv \in p$. We claim that for all $t \in T$,

$$x > tx \notin p \Leftrightarrow y > (utu^{-1})y \notin p. \tag{1.2.5}$$

In order to verify this claim it is enough to show that, if $x \in p$

$$x > tx \notin p, s \in W_J \Rightarrow xs > txs$$
$$x > tx \notin p, s \in W_I \Rightarrow sx > (sts)sx.$$

For the first statement note that either $xs > txs$ or $xs < txs$. However, as $x > tx$ the second possibility would imply $x = txs$ by Deodhar's "Property Z" (alternatively this follows from [**Hu**], Proposition 5.9) which contradicts $tx \notin p$. The second statement follows similarly. Thus we have verified (1.2.5). It is also immediate that, for all $t \in T$,

$$tx \in p \Leftrightarrow utu^{-1}y \in p.$$

Now, setting $y = p_+$ and using the above facts together with the maximality of $p_+ \in p$ we follow

$$\begin{aligned} \ell(p_+) - \ell(y_+) &= |\{t \in T \mid p_+ > tp_+\}| - |\{t \in T \mid x > tx\}| \\ &= |\{t \in T \mid p_+ > tp_+ \in p\}| - |\{t \in T \mid x > tx \in p\}| \\ &= |\{t \in T \mid x < tx \in p\}|. \end{aligned}$$ □

1.3. Translation sequences. When studying a Coxeter group W an important role is played by expressions. Their importance becomes particularly obvious when studying the Bruhat order or Hecke algebra. In this work we are interested in the set of double cosets $W_I\backslash W/W_J$ for $I, J \subset S$ and thus would like an analogue of reduced expressions. These are the "translation sequences" of the title.

As a motivation, consider a reduced expression $st\dots u$. Given a simple reflection s, we will write $\langle s\rangle$ for the parabolic subgroup which it generates. It is natural to consider $st\dots u$ as giving a sequence of cosets:

$$\{id\} \subset \langle s\rangle \supset \{s\} \subset s\langle t\rangle \supset \{st\} \subset \cdots \supset \{st\dots u\}.$$

This will be our model for a "translation sequence": roughly speaking it is a sequence of double cosets, in which an inclusion relation is satisfied at every step.

In contrast to the one-sided cosets, an equality between two double cosets W_IxW_J and W_KyW_L does not imply that $W_I = W_K$ or $W_J = W_L$. For our purposes, it will be necessary to keep track of the groups W_I and W_J and not just their double cosets. This leads to some complicated notation.

We begin by defining what a step in a translation sequence may look like:

DEFINITION 1.3.1. *Let $I, J, K, L \subset S$ be finitary, $p \in W_I\backslash W/W_J$ and $q \in W_K\backslash W/W_L$. We call*

(1) (K, q, L) *a* shrinking *of* (I, p, J) *if* $I \supset K$, $J \supset L$ *and* $p \supset q$.

(2) (K, q, L) *a* expansion *of* (I, p, J) *if* $I \subset K$, $J \subset L$ *and* $p \subset q$.

A shrinking is reduced *if the maximal elements of* p *and* q *are equal. The expansion is* reduced *if the minimal elements of* p *and* q *are equal and*

$$I \cap p_- J p_-^{-1} = K \cap q_- L q_-^{-1}.$$

We can now define a translation sequence:

Definition 1.3.2. *A* translation sequence *is a triple* $(I_i, p_i, J_i)_{0 \leq i \leq n}$ *where, for all* $0 \leq i \leq n$, $I_i, J_i \subset S$ *are finitary and* $p_i \in W_{I_i} \backslash W / W_{J_i}$. *The sequence is subject to the following conditions:*

(1) $I_0 = J_0$ *and* $id \in p_0$*;*

(2) $(I_{i+1}, p_{i+1}, J_{i+1})$ *is either a shrinking or expansion of* (I_i, p_i, J_i) *for* $0 \leq i < n$, *.*

The translation sequence is reduced *if each shrinking and expansion is reduced. A* left translation sequence *(resp.* right translation sequence*) is a translation sequence* $(I_i, p_i, J_i)_{0 \leq i \leq n}$ *in which* $J_i = J_{i+1}$ *(resp.* $I_i = I_{i+1}$*) for all* $0 \leq i < n$. *The* end-point *of a translation sequence* $(I_i, p_i, J_i)_{0 \leq i \leq n}$ *is the triple* (I_n, p_n, J_n)*.*

Example 1.3.3.

(1) We have described translation sequences as a generalisation of an expression $st \dots u$ *for an element* $w \in W$. *Given such an expression we obtain a right translation sequence (with slight abuse of notation):*

$$(\emptyset, id, \emptyset), (\emptyset, \langle s \rangle, s), (\emptyset, s, \emptyset), (\emptyset, s\langle t \rangle, t), (\emptyset, st, \emptyset), \textit{etc.}$$

This translation sequence has endpoint $(\emptyset, w, \emptyset)$ *and is reduced if and only if* $st \dots u$ *is a reduced expression.*

(2) More generally, a right translation sequence (I, p_i, J_i) *with* $I = \emptyset$ *gives a path in the Coxeter complex of* (W, S) *(see* [**Bro**]*). It would be nice to have a geometric interpretation for translation sequences in general.*

The existence of reduced expressions in a Coxeter group is a triviality. The following proposition (which is important in what follows) show that reduced translation sequences always exist.

Proposition 1.3.4. *Let* $I, J \subset S$ *be finitary and* $p \in W_I \backslash W / W_J$. *Then there exists a reduced right translation sequence* $(I, p_i, J_i)_{0 \leq i \leq n}$ *with end-point* (I, p, J)*.*

Proof. Assume that one of the following is true:

(1) There exists $K \supsetneq J$, necessarily finitary, such that p_+ remains the maximal element in $q = W_I p W_K$;

(2) There exists $K \subsetneq J$ such that, setting $q = W_I p_- W_K$ we have $I \cap p_- J p_-^{-1} = I \cap p_- K p_-^{-1}$.

In case 1), (I, p, J) is a reduced shrinking of (I, q, K) and in case 2), p_- remains the minimal element in q and thus (I, p, J) is a reduced expansion of (I, q, K).

Thus, by induction, we may assume that neither 1) nor 2) is true. We claim that in this case necessarily $I = J$ and $p = W_I W_J = W_I$ which implies the proposition.

As 2) is not true, for all $t \in J$ there exists $s \in I$ with $sp_- = p_-t$. Thus we may write $p_+ = up_-$ for some $u \in W_I$. Now assume for contradiction that $p_-t < p_-$ for some $t \in S$. We cannot have $up_- < up_-t$ as otherwise $p_-t < p_- < up_- < up_-t$ would all be in the same left W_I-coset. Hence $up_- > up_-t$, contradicting the fact that 1) is not possible. We conclude that $p_- = id$. Hence $I \supset J$, $p_+ = w_I$ and $I = J$ by using 1) again. $\square$

2. The Hecke algebra and category

We begin by recalling the Hecke algebra, Kazhdan-Lusztig basis and a certain canonical bilinear form. We then introduce the Hecke category (a certain relative version of the Hecke algebra) and define its standard basis, standard generators, Kazhdan-Lusztig basis and bilinear form.

2.1. The Hecke algebra and Kazhdan-Lusztig basis. Let (W, S) be a Coxeter system. Recall that the *Hecke algebra* $\mathcal{H}$ is the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_w \mid w \in W\}$ and multiplication

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w \\ (v^{-1} - v)H_w + H_{sw} & \text{if } sw < w. \end{cases} \tag{2.1.1}$$

We call $\{H_w\}$ the *standard basis*. Each H_w is invertible and there is an involution on $\mathcal{H}$ which sends H_w to $H_{w^{-1}}^{-1}$ and v to v^{-1}. We will call elements fixed by this involution *self-dual*. One has the following fundamental theorem of Kazhdan and Lusztig ([**KL1**]):

THEOREM 2.1.1. *There exists a unique basis $\{\underline{H}_w \mid w \in W\}$ such that:*

(1) Each $\underline{H}_w$ is self-dual.
(2) One has $\underline{H}_w = \sum_{x \le w} h_{x,w} H_x$ with $h_{w,w} = 1$ and $h_{x,w} \in v\mathbb{Z}[v]$.

PROOF. The original proof is in [**KL1**]. In [**So3**] there is a simpler proof (which uses the above notation). $\square$

We call $\{\underline{H}_w\}$ the *Kazhdan-Lusztig basis* and the coefficients $h_{x,y}$ the *Kazhdan-Lusztig polynomials.*[1]

[1]It should be noted that the $h_{x,y}$ are not exactly the Kazhdan-Lusztig polynomials in the literature. One may write $v^{\ell(x)-\ell(y)} h_{x,y} = P_{x,y}(v^{-2})$ with $P_{x,y} \in \mathbb{Z}[q]$. $P_{x,y}$ is the "real" Kazhdan-Lusztig polynomial.

Clearly $\underline{H}_{id} = H_{id}$ and a calculation shows that $\underline{H}_s = H_s + H_{id}$, which, together with (2.1.1), yields

$$\underline{H}_s H_w = \begin{cases} vH_w + H_{sw} & \text{if } sw > w \\ v^{-1}H_w + H_{sw} & \text{if } sw < w. \end{cases} \tag{2.1.2}$$

If $x < y$, write $\mu(x, y)$ for the coefficient of v in $h_{x,y}$. One has the following multiplication formula ([**So3**], Proposition 2.8):

$$\underline{H}_s \underline{H}_w = \begin{cases} (v + v^{-1})\underline{H}_w & \text{if } sw < w \\ \underline{H}_{sw} + \sum_{x<w;sx<x} \mu(x, w)\underline{H}_x & \text{if } sw > w \end{cases} \tag{2.1.3}$$

If $sw < w$ one may expand $\underline{H}_s\underline{H}_w$ in (2.1.2) and (2.1.3) to conclude that $h_{sx,w} = vh_{x,w}$ if $xs < x$. It follows that if $I \subset S$ if finitary and w_I denotes the longest element in W_I we have

$$\underline{H}_{w_I} = \sum_{x \in W_I} v^{\ell(w_I)-\ell(x)} H_x. \tag{2.1.4}$$

If $s \in W_I$ is a simple reflection one has

$$H_s\underline{H}_{w_I} = \underline{H}_s\underline{H}_{w_I} - vH_{id}\underline{H}_{w_I} = v^{-1}\underline{H}_{w_I}.$$

From which it follows that

$$H_x\underline{H}_{w_I} = v^{-\ell(x)}\underline{H}_{w_I}. \tag{2.1.5}$$

If $K \subset I$ then, combining (2.1.4) and (2.1.5) we obtain

$$\underline{H}_{w_K}\underline{H}_{w_I} = \pi(K)\underline{H}_{w_I}. \tag{2.1.6}$$

There is a $\mathbb{Z}[v, v^{-1}]$-linear anti-involution $i : \mathcal{H} \to \mathcal{H}$ sending H_x to $H_{x^{-1}}$. Following [**Lu2**] we define a bilinear form:

$$\begin{aligned} \mathcal{H} \times \mathcal{H} &\to \mathbb{Z}[v, v^{-1}] \\ (f, g) &\mapsto \langle f, g\rangle = \text{ coefficient of } H_{id} \text{ in } fi(g). \end{aligned}$$

The form has the following alternative description:

Lemma 2.1.2. *We have $\langle H_x, H_y\rangle = \delta_{x,y}$ for all $x, y \in W$.*

Proof. If $x = id$ or lies in S then the formula is immediate from (2.1.1). We may now induct on the length of x. After having chosen $s \in S$ with $xs < x$ we have $\langle H_x, H_y\rangle = \langle H_{xs}, H_yH_s\rangle = \delta_{xs,ys} = \delta_{x,y}$ by induction and (2.1.1) again. □

2.2. The Hecke category. We want to define a certain relative version of the Hecke algebra associated to all pairs of finitary subsets $I, J \subset S$. The most natural way to define this is as an "$\mathbb{Z}[v, v^{-1}]$-linear category". Recall that, given a ring R, an *R-linear category* is a category in which each space of morphisms has the structure of an R-module and composition is R-bilinear.

For all pairs of finitary subsets $I, J \subset S$ define:

$$\begin{aligned} {}^I\mathcal{H} &= \underline{H}_{w_I}\mathcal{H} \\ \mathcal{H}^J &= \mathcal{H}\underline{H}_{w_J} \\ {}^I\mathcal{H}^J &= {}^I\mathcal{H} \cap \mathcal{H}^J \end{aligned}$$

There is no natural multiplication on ${}^I\mathcal{H}^J$. However, given another finitary subset $K \subset S$ we may define a multiplication as follows

$$\begin{aligned} {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K &\to {}^I\mathcal{H}^K \\ (h_1, h_2) &\mapsto h_1 *_J h_2 = \frac{1}{\pi(J)} h_1 h_2. \end{aligned}$$

This well defined by (2.1.6). If $J = \emptyset$ we write $*$ instead of $*_\emptyset$. The existence of this "partial multiplication" is formalised by the following definition.

Definition 2.2.1. *The* Hecke category *is the $\mathbb{Z}[v, v^{-1}]$-linear category defined as follows. The objects are finitary subsets $I \subset S$. The morphisms between two objects I and J consists of the module ${}^I\mathcal{H}^J$. Composition ${}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \to {}^I\mathcal{H}^K$ is given by $*_J$.*

This does indeed define a $\mathbb{Z}[v, v^{-1}]$-linear category. The only point that may not yet be obvious is the existence of the identity endomorphism. However this will be become clear in the discussion below.

Remark 2.2.2. *The Hecke category unifies several different objects:*

1. *The endomorphism ring of $\emptyset \subset S$ is the Hecke algebra.*
2. *For any finitary subset $I \subset S$,* $\mathrm{Hom}(\emptyset, I)$ *is a left module over* $\mathrm{End}(\emptyset) = \mathcal{H}$*. This is an example of a "parabolic Hecke module" introduced by Deodhar in* [**Deo**]*.*
3. *If $\widetilde{W}$ is an affine Weyl group, $W \subset \widetilde{W}$ is the finite Weyl group, and $I \subset S$ corresponds to all simple reflections in W then* $\mathrm{End}(I)$ *is the (adjoint) "spherical Hecke algebra" (see e.g.* [**Lu1**]*).*

Our main goal for the rest of the section is to define a basis for ${}^I\mathcal{H}^J$ for all finitary subsets $I, J \subset S$ and analyse the action of "standard generators" on this basis.

Until Proposition 2.2.4 fix $I, J \subset S$ finitary. We start with a lemma which helps us to decide if an element $h \in \mathcal{H}$ belongs to ${}^I\mathcal{H}^J$.

Lemma 2.2.3. *Let $h = \sum a_y H_y \in \mathcal{H}$. The following are equivalent:*

1. *$h \in {}^I\mathcal{H}$*
2. *$\underline{H}_{w_I} h = \pi(W_I) h$*
3. *$\underline{H}_s h = (v + v^{-1}) h$ for all $s \in I$.*
4. *For all $y \in W$*
$$a_{sy} = v a_y$$
if $s \in W_I$ is a simple reflection and $sy < y$.

The analogous "right handed" statements hold for $\mathcal{H}^J$

PROOF. Straightforward, using (2.1.6) and Proposition 1.2.3. □

In particular, we conclude that $h = \sum a_y H_y$ is in ${}^I\mathcal{H}^J$ if and only if, for all $y \in W$, $a_{sy} = va_y$ and $a_{yt} = va_y$ for all $s \in I$ and $t \in J$ such that $sy < y$ and $yt < y$.

This shows how to find a basis for ${}^I\mathcal{H}^J$ as a $\mathbb{Z}[v, v^{-1}]$-module. Namely, for all $p \in W_I \backslash W / W_J$ define

$$ {}^I H^J_p = \sum_{x \in p} v^{\ell(p_+) - \ell(x)} H_x. $$

It follows that, if $h = \sum a_y H_y$ is in ${}^I\mathcal{H}^J$ then

$$ h = \sum_{p \in W_I \backslash W / W_J} a_{p_+} \, {}^I H^J_p. \tag{2.2.1} $$

The set $\{ {}^I H^J_p \mid p \in W_I \backslash W / W_J \}$ is clearly linearly independent over $\mathbb{Z}[v, v^{-1}]$ and we conclude that they form a basis, which we call the *standard basis* of ${}^I\mathcal{H}^J$.

Using Lemma 2.2.3 and (2.1.3) we see that $\underline{H}_y \in {}^I\mathcal{H}^J$ if and only if y is maximal in its (W_I, W_J)-double coset. In general, if $p \in W_I \backslash W / W_J$ we define

$$ {}^I \underline{H}^J_p = \underline{H}_{p_+}. $$

We have

$$ {}^I \underline{H}^J_p = {}^I H^J_p + \sum_{q < p} h_{q_+, p_+} \, {}^I H^J_q. $$

It follows that $\{ {}^I \underline{H}^J_p \mid p \in W_I \backslash W / W_J \}$ also forms a $\mathbb{Z}[v, v^{-1}]$ basis for ${}^I\mathcal{H}^J$. We will refer to this as the *Kazhdan-Lusztig basis*.

For all finitary subsets $I, J \subset S$ satisfying $I \subset J$ or $J \subset I$ we define

$$ {}^I H^J = {}^I H^J_p \text{ where } p = W_I id W_J. $$

We call call elements of the form ${}^I H^J \in {}^I\mathcal{H}^J$ *standard generators*. The standard generators are the analogues of the elements $\underline{H}_s \in \mathcal{H}$ and we will see below that the set of standard generators generate the Hecke category, which justifies the terminology. The following proposition describes the action of the standard generators on the standard basis.

PROPOSITION 2.2.4. *Let $I, J, K \subset S$ be finitary and assume $J \subset K$ or $J \supset K$. The action of ${}^J H^K$ on the basis $\{ {}^I H^J_p \mid p \in W_I \backslash W / W_J \}$ is as follows:*

(1) If $J \supset K$ then

$$ {}^I H^J_p *_J {}^J H^K = \sum_{q \in W_I \backslash p / W_K} v^{\ell(p_+) - \ell(q_+)} \, {}^I H^J_q. $$

(2) If $J \subset K$ then

$$ {}^I H^J_p *_J {}^J H^K = v^{\ell(q_-)-\ell(p_-)} \frac{\pi(I,q,K)}{\pi(I,p,J)} \, {}^I H^K_q $$

where $q = W_I p W_K$ is the (W_I, W_K)-coset containing p.

Before we prove the proposition we need a lemma.

LEMMA 2.2.5. *Let $I, J \subset S$ be finitary, $x \in W$ and $p = W_I x W_J$. Then*

$$ {}^I H^\emptyset * H_x * {}^\emptyset H^J = v^{\ell(p_-)-\ell(x)} \pi(I,p,J) \, {}^I H^J_p. $$

PROOF. By Howlett's Theorem (1.2.9) we may write $x = up_-v$ with $u \in W_I$, $v \in W_J$ and $\ell(x) = \ell(u) + \ell(p_-) + \ell(v)$. By (2.1.5) we have:

$$ {}^I H^\emptyset * H_x * {}^\emptyset H^J = v^{\ell(p_-)-\ell(x)} \, {}^I H^\emptyset * H_{p_-} * {}^\emptyset H^J $$

Thus we will be finished if we can show that

$$ {}^I H^\emptyset * H_{p_-} * {}^\emptyset H^J = \pi(I,p,J) \, {}^I H^J_p. $$

We write $K = I \cap p_- J p_-^{-1}$ so that $\pi(I,p,J) = \pi(K)$. If $s \in K$ then $sp_- = p_- s'$ for some $p' \in J$ and therefore

$$ \underline{H}_{w_K} H_{p_-} = H_{p_-} \underline{H}_{w_{K'}} \tag{2.2.2} $$

where $K' = p_-^{-1} K p_-$. Because K and K' are conjugate $\pi(K) = \pi(K')$. We define $N \in \mathcal{H}$ by

$$ N = v^{\ell(w_I)-\ell(w_K)} \sum_{u \in D_K \cap W_I} v^{-\ell(u)} H_u $$

and calculate

$$ \begin{aligned} {}^I H^\emptyset * H_{p_-} * {}^\emptyset H^J &= N \underline{H}_{w_K} H_{p_-} \underline{H}_{w_J} && \text{(Proposition 1.2.3)} \\ &= N H_{p_-} \underline{H}_{w_{K'}} \underline{H}_{w_J} && \text{(2.2.2)} \\ &= \pi(K) N H_{p_-} \underline{H}_{w_J} && \text{(2.1.6)} \\ &= \pi(K) v^a \sum_{x \in p} v^{-\ell(x)} H_x && \text{(Howlett's theorem)} \\ &= \pi(K) \, {}^I H^J_p \end{aligned} $$

where the last line follows because

$$ a = \ell(w_I) - \ell(w_K) + \ell(w_J) + \ell(p_-) = \ell(p_+) $$

by Corollary 1.2.11. □

PROOF OF PROPOSITION 2.2.4. Statement (1) follows by (2.1.6) and (2.2.1). We now turn to (2). Let us expand

$$ P = {}^I H^\emptyset * H_{p_-} * {}^\emptyset H^J *_J {}^J H^K $$

in two different ways. As ${}^{\emptyset}H^J * {}^J H^K = {}^{\emptyset}H^K$ by (2.1.6) we obtain, using Lemma 2.2.5:

$$P = {}^I H^{\emptyset} * H_{p_-} * {}^{\emptyset}H^K = v^{\ell(q_-)-\ell(p_-)}\pi(I,q,K)\, {}^I H^K_q.$$

We also have (again using Lemma 2.2.5):

$$P = \pi(I,p,J)\, {}^I H^J_p *_J {}^J H^K.$$

We follow that

$$(2.2.3) \qquad {}^I H^J_p *_J {}^J H^K = v^{\ell(q_-)-\ell(p_-)} \frac{\pi(I,q,K)}{\pi(I,p,J)}\, {}^I H^K_q$$

By Corollary 1.2.15 and the fact that $\mathcal{H}$ is free as a $\mathbb{Z}[v,v^{-1}]$-module. $\square$

Given an element $h \in {}^I\mathcal{H}^J$ we may write $h = \sum \lambda_p\, {}^I H^J_p$. We define the *support* of h to be the finite set

$$\operatorname{supp} h = \{p \in W_I \backslash W / W_J \mid \lambda_p \neq 0\}.$$

A second corollary of the above multiplication formulas is a description of multiplication by a standard generator on the support.

Corollary 2.2.6. *Let $I, J, K \subset S$ be finitary with $J \subset K$ and let*

$$\mathrm{qu} : W_I \backslash W / W_J \to W_I \backslash W / W_K$$

be the quotient map.

(1) *If $h \in {}^I\mathcal{H}^J$ then*

$$\operatorname{supp}(h *_J {}^J H^K) \subset \mathrm{qu}(\operatorname{supp} h).$$

(2) *If $h \in {}^I\mathcal{H}^K$ then*

$$\operatorname{supp}(h *_K {}^K H^J) \subset \mathrm{qu}^{-1}(\operatorname{supp} h).$$

Recall that in Subsection 1.3 we introduced *translation sequences* as a generalisation of reduced expression. We state a proposition, analysing a product associated with a translation sequence.

Proposition 2.2.7. *Let $(I, p_i, J_i)_{0 \le i \le n}$ be a right reduced translation sequence with end-point (I, p, J). Then*

$${}^I H^{J_0} *_{J_0} {}^{J_0} H^{J_1} *_{J_1} \cdots *_{J_{n-1}} {}^{J_{n-1}} H^{J_n} = {}^I H^J_p + \sum_{q<p} \lambda_q\, {}^I H^J_q.$$

Before we prove this proposition, we show how it may be used to show that the standard generators generate the Hecke category (justifying the terminology). We first define what this means.

Let R be a ring and $\mathcal{C}$ be an R-linear category. Suppose we are given a subset $X_{AB} \subset \operatorname{Hom}(A,B)$ for all pairs of objects $A, B \in \mathcal{C}$. We define the *span* of the collection $\{X_{AB}\}$ to be the smallest collection of R-submodules $\{Y_{AB} \subset \operatorname{Hom}(A,B)\}$ such that:

(1) $X_{AB} \subset Y_{AB}$ for all $A, B \in \mathcal{C}$,

(2) The collection $\{Y_{AB}\}$ is closed under composition in $\mathcal{C}$.

We say that $\{X_{AB}\}$ *generates* $\mathcal{C}$ if the span of $\{X_{AB}\}$ consists of $\mathrm{Hom}(A, B)$ for all $A, B \in \mathcal{C}$. Less formally, one may refer to the span of any set of morphisms in $\mathcal{C}$ and ask whether they generate the category.

COROLLARY 2.2.8. *The standard generators ${}^{I}H^{J}$ for finitary $I, J \subset S$ with either $I \subset J$ or $I \supset J$ generate the Hecke category.*

PROOF. By Proposition 1.3.4, for every $p \in W_I \backslash W / W_J$ there exists a right reduced translation sequence $(I, p_i, J_i)_{0 \le i \le n}$ with end-point (I, p, J). For each p, choose such a translation sequence and consider the product

$$P_p = {}^{I}H^{J_0} *_{J_0} {}^{J_0}H^{J_1} *_{J_1} \cdots *_{J_{n-1}} {}^{J_{n-1}}H^{J_n}.$$

By the above proposition, after choosing a total ordering on $W_I \backslash W / W_J$ compatible with the Bruhat order, the matrix relating the standard basis $\{ {}^{I}H^{J}_{p}\}$ and the elements $\{P_p\}$ is uni-triangular. In particular, the set $\{P_p\}$ spans ${}^{I}\mathcal{H}^{J}$ as a $\mathbb{Z}[v, v^{-1}]$-module. The corollary then follows. $\square$

REMARK 2.2.9. *It is natural to ask what relations the arrows ${}^{I}H^{J}$ satisfy. We have not looked into this.*

PROOF. We will prove the proposition via induction on n, with the case $n = 0$ being trivial. For $0 \le k \le n$ let us denote by P_k the partial product:

$$P_k = {}^{I}H^{J_0} *_{J_0} {}^{J_0}H^{J_1} *_{J_1} \cdots *_{J_{k-1}} {}^{J_{k-1}}H^{J_k}.$$

By induction we have:

(1) $\mathrm{supp}(P_{n-1}) \subset \{p \mid p \le p_{n-1}\}$;
(2) the coefficient of ${}^{I}H^{J_{n-1}}_{p_{n-1}}$ in P_{n-1} is 1.

There are two cases to consider:

Case 1: $J_{n-1} \supset J_n$: As (I, p_n, J_n) is a reduced shrinking of (I, p_{n-1}, J_{n-1}), p_n is the maximal (W_I, W_{J_n})-double coset in p_{n-1}. It follows from Proposition 2.2.4 that the coefficient of ${}^{I}H^{J_n}_{p_n}$ is 1 and, if

$$\mathrm{qu} : W_I \backslash W / W_{J_n} \to W_I \backslash W / W_{J_{n-1}}$$

denotes the quotient map, then $\mathrm{supp}(P_n) \subset \mathrm{qu}^{-1}(\{\le p_{n-1}\}) = \{\le p_n\}$ by Corollary 2.2.6 and the fact that qu is strict.

Case 2: $J_{n-1} \subset J_n$: Let

$$\mathrm{qu} : W_I \backslash W / W_{J_{n-1}} \to W_I \backslash W / W_{J_n}$$

denote the quotient map. As (I, p_n, J_n) is a reduced expansion of (I, p_{n-1}, J_{n-1}), $\pi(I, p_{n-1}, J_{n-1}) = \pi(I, p_n, J_n)$ and p_{n-1} is minimal in $\mathrm{qu}^{-1}(p_n)$. Hence the coefficient of ${}^{I}H^{J_n}_{p_n}$ is 1 by Proposition 2.2.4 and $\mathrm{supp}(P_n) \subset \mathrm{qu}(\mathrm{supp}\, P_{n-1}) \subset \{\le p_n\}$ as qu is a map of posets. $\square$

In the previous subsection we defined a bilinear form on $\mathcal{H}$. We now generalise this construction and define a bilinear form on each ${}^I\mathcal{H}^J$ for $I, J \subset S$ finitary. Recall that $i : \mathcal{H} \to \mathcal{H}$ denotes the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution sending H_x to $H_{x^{-1}}$. As $\underline{H}_{w_I}$ and $\underline{H}_{w_J}$ are fixed by i it follows that i restricts to an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules

$$i : {}^I\mathcal{H}^J \to {}^J\mathcal{H}^I.$$

We define

$$\begin{aligned} {}^I\mathcal{H}^J \times {}^I\mathcal{H}^J &\to \mathbb{Z}[v, v^{-1}] \\ (f, g) &\mapsto \langle f, g\rangle = \text{ coefficient of } H_{id} \text{ in } f *_J i(g). \end{aligned}$$

We do not include reference to I and J in the notation, and hope that this will not lead to confusion. It follows from the definition that if $I, J, K \subset S$ are finitary and $f \in {}^I\mathcal{H}^J$, $g \in {}^J\mathcal{H}^K$ and $h \in {}^I\mathcal{H}^K$ then

$$\langle f *_J g, h\rangle = \langle f, h *_K i(g)\rangle. \tag{2.2.4}$$

The following lemma describes the bilinear form on the standard basis of ${}^I\mathcal{H}^J$.

Lemma 2.2.10. *Let $I, J \subset S$ be finitary. For all $p, q \in W_I\backslash W/W_J$ we have*

$$\langle\, {}^I H_p^J,\, {}^I H_q^J\rangle = v^{\ell(p_+)-\ell(p_-)} \frac{\pi(p)}{\pi(J)} \delta_{p,q}.$$

Proof. Let $f, g \in {}^I\mathcal{H}^J$ and write $\tilde{f}$, $\tilde{g}$ for the elements f and g regarded as elements of $\mathcal{H}$. It is clear from the definition that

$$\langle f, g\rangle = \frac{1}{\pi(J)}\langle \tilde{f}, \tilde{g}\rangle.$$

where the second expression is the bilinear form calulated in $\mathcal{H}$. We may then calculate using Lemma 2.1. If $p \neq q$ then $\langle\, {}^I H_p^J,\, {}^I H_q^J\rangle = 0$. If $p = q$ we have

$$\langle\, {}^I H_p^J,\, {}^I H_q^J\rangle = \frac{1}{\pi(J)} \sum_{x\in p} v^{2(\ell(p_+)-\ell(x))} = v^{\ell(p_+)-\ell(p_-)} \frac{\pi(p)}{\pi(J)}. \qquad \square$$

CHAPTER 3

Singular Soergel Bimodules

In this chapter we study singular Soergel bimodules. The main goals are Theorems 5.4.2 and 5.5.1 which classify the indecomposable singular Soergel bimodules and show that they provide a categorification of the Hecke category.

In order to describe the contents of this chapter in more detail we briefly recall the definition of the categories ${}^{I}\mathcal{B}^{J}$ of singular Soergel bimodules already given in the introduction. Let R be the graded algebra of regular functions on a reflection faithful representation V of W and for a finitary subset $I \subset S$ let R^I denote the subalgebra of invariants under W_I. Given two finitary subsets $I, J \subset S$ let R^I-Mod-R^J denote the category of graded (R^I, R^J)-bimodules. For any pair $I, J \subset S$ of finitary subsets we define the categories ${}^{I}\mathcal{B}^{J}$ of singular Soergel bimodules to be the smallest full additive subcategory of R^I-Mod-R^J which contains all modules isomorphic to direct summands of shifts of bimodules of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

where $I = I_1 \subset J_1 \supset I_2 \subset \cdots \subset J_{n-1} \supset I_n = J$ are all finitary subsets of S.

Given any module $M \in R^I$-Mod-R^J and an enumeration of the elements in $W_I \backslash W / W_J$ compatible with the Bruhat order one obtains two natural filtrations of M

$$\begin{aligned} \cdots \subset \Gamma_{C(i-1)}M \subset \Gamma_{C(i)}M \subset \Gamma_{C(i+1)}M \subset \cdots \\ \cdots \supset \Gamma_{\check{C}(i-1)}M \supset \Gamma_{\check{C}(i)}M \supset \Gamma_{\check{C}(i+1)}M \supset \cdots \end{aligned}$$

by considering M as a quasi-coherent sheaf on $V/W_I \times V/W_J$. The crucial fact is that, if $M \in {}^{I}\mathcal{B}^{J}$, both filtrations are exhaustive and the subquotients are isomorphic to direct sums of shifted *standard modules*, which are certain bimodules in R^I-Mod-R^J which may be described explicitly.

In order to prove this fact we define *objects with nabla flags* and *objects with delta flags* as those objects for which the subquotients in the first or second filtration respectively are isomorphic to direct sums of shifts of standard modules. We then show that these subcategories are preserved by the functors of restriction and extension of scalars, which we renormalise and rename *translation functors*. Given an object with a nabla or delta flag it is natural to define its *nabla* or *delta* *character* in the Hecke category by counting the graded multiplicities of standard modules in the subquotients of the above filtrations. It turns out that one may describe the effect of translation functors on the character in terms of multiplication with a standard generator in the Hecke category.

By the inductive definition of the objects in ${}^I\mathcal{B}^J$ it follows that they they have both a nabla and a delta flag. This may be exploited to describe $\mathrm{Hom}(M, B)$ and $\mathrm{Hom}(B, M)$ when B is a Soergel bimodule and M has a delta or nabla flag respectively. The classification of the indecomposable objects in ${}^I\mathcal{B}^J$ is then straightforward.

Given the classification of indecomposable objects in ${}^I\mathcal{B}^J$ it is natural to attempt to describe their character in the Hecke category. In case $I = J = \emptyset$, Soergel has conjectured that the character of an (appropriately shifted) indecomposable module parametrised by $w \in W$ is given by the Kazhdan-Lusztig basis element $\underline{H}_w$. We show that if this conjecture is true then the characters of all indecomposable singular Soergel bimodules are given by Kazhdan-Lusztig basis elements.

The structure of this chapter is as follows. In Section 1 we introduce basic notation. In Section 2 we introduce the standard modules and begin to study the effect of restriction and extension of scalars on them. In order to complete the description of extension we need to construct certain filtrations, which we do in Section 3 using Demazure operators. In Section 4 we introduce objects with nabla and delta flags, define their characters and show that these subcategories are preserved by the translation functors. In Section 5 we complete the classification, recall Soergel's conjecture and show that it implies character formulae for all indecomposable singular Soergel bimodules.

1. Bimodules and homomorphisms

Fix a field k of characteristic 0. We consider rings A satisfying

(1.0.5) $A = \oplus_{i \geq 0} A^i$ is a finitely generated, positively graded commutative ring with $A^0 = k$.

We denote by A-Mod and Mod-A the category of graded left and right A-modules. All tensor products are assumed to take place over k, unless otherwise specified. If A_1 and A_2 are two rings satisfying (1.0.5) we write A_1-Mod-A_2 for the category of (A_1, A_2)-bimodules, upon which the left and right action of k agrees. As all rings are assumed commutative we have an equivalence between A_1-Mod-A_2 and $A_1 \otimes A_2$-Mod. We generally prefer to work in A_1-Mod-A_2, but will occasionally switch to $A_1 \otimes A_2$-Mod when convenient.

Given a graded module $M = \oplus M^i$ we define the shifted module $M[n]$ by $(M[n])^i = M^{n+i}$. The endomorphism ring of any finitely generated object in A-Mod, Mod-A or A_1-Mod-A_2 is finite dimensional and hence any finitely generated module satisfies Krull-Schmidt (for example, by adapting the proof in [**Pie**]).

Given a Laurent polynomial with positive coefficients

$$P = \sum a_i v^i \in \mathbb{N}[v, v^{-1}]$$

and an object M in A-Mod, Mod-A or A_1-Mod-A_2, we define

$$P \cdot M = \bigoplus M[i]^{\oplus a_i}.$$

If $P, Q \in \mathbb{N}[v, v^{-1}]$ and M and N are finitely generated modules such that

$$P \cdot M \cong PQ \cdot N$$

we may "cancel P" and conclude (using Krull-Schmidt) that

$$M \cong Q \cdot N.$$

This will prove to be a useful notational convenience.

Given two modules $M, N \in A_1$-Mod-A_2 a morphism $\phi : M \to N$ of (ungraded) (A_1, A_2)-bimodules is of *degree* i if $\phi(M^m) \subset \phi(N^{m+i})$ for all $m \in \mathbb{Z}$. We denote by $\operatorname{Hom}(M, N)^i$ the space of all morphisms of degree i and

$$\operatorname{Hom}(M, N) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}(M, N)^i.$$

We make $\operatorname{Hom}(M, N)$ into an object of A-Mod-B by defining an action of $a \in A$ and $b \in B$ on $f \in \operatorname{Hom}(M, N)$ via

$$(afb)(m) = f(amb) = af(m)b$$

for all $m \in M$. If M and N are objects in A-Mod we similarly define $\operatorname{Hom}_A(M, N) \in A$-Mod. (We will only omit the subscript for morphisms of bimodules but will sometimes write $\operatorname{Hom}_{A_1 - A_2}(M, N)$ if the context is not clear. We *never* use $\operatorname{Hom}(M, N)$ to denote external (i.e. degree 0) homomorphisms.)

One may check that, if $P, Q \in \mathbb{N}[v, v^{-1}]$, then

$$\operatorname{Hom}(P \cdot M, Q \cdot N) \cong \overline{P} Q \cdot \operatorname{Hom}(M, N).$$

where $P \mapsto \overline{P}$ denotes the involution on $\mathbb{N}[v, v^{-1}]$ sending v to v^{-1}.

In the sequel we will need various natural isomorphisms between homomorphism spaces, which we recall here. Let A_1, A_2 and A_3 be three rings satisfying (1.0.5). Let $M_{ij} \in A_i$-Mod-A_j for $i, j \in \{1, 2, 3\}$. In A_1-Mod-A_3 one has isomorphisms

$$\operatorname{Hom}_{A_1 - A_3}(M_{12} \otimes_{A_2} M_{23}, M_{13})$$

$$\cong \operatorname{Hom}_{A_1 - A_2}(M_{12}, \operatorname{Hom}_{A_3}(M_{23}, M_{13})) \tag{1.0.6}$$

$$\cong \operatorname{Hom}_{A_2 - A_3}(M_{23}, \operatorname{Hom}_{A_1}(M_{12}, M_{13})) \tag{1.0.7}$$

because all three modules describe the same subset of maps $M_{12} \times M_{23} \to M_{13}$. For similar reasons, if $N \in A_1$-Mod one has an isomorphism in A_1-Mod,

$$\mathrm{Hom}_{A_1}(M_{12} \otimes_{A_2} M_{23}, N) \cong \mathrm{Hom}_{A_2}(M_{23}, \mathrm{Hom}_{A_1}(M_{12}, N)). \tag{1.0.8}$$

Furthermore, this is an isomorphism in $A_1 \otimes A_3$-Mod if both sides are made into $A_1 \otimes A_3$-modules in the only natural way possible.

If M_{32} is graded free of finite rank as a right A_2-module one has an isomorphism

$$\mathrm{Hom}_{A_2}(M_{32}, M_{12}) \cong M_{12} \otimes_{A_2} \mathrm{Hom}_{A_2}(M_{32}, A_2) \tag{1.0.9}$$

in A_1-Mod-A_3. Indeed, there is a natural map from the right hand to the left hand side, which is an isomorphism under the above assumptions. (The structure of both sides as an object in A^1-Mod-A^3 is again the natural one).

2. Invariants, graphs and standard modules

In this section we introduce standard modules, which are the building blocks of Soergel bimodules. Due to the inductive definition of Soergel bimodules, it will be necessary to be able to precisely describe the effect of extension and restriction of scalars on standard modules. Restriction turns out to be straightforward (Lemma 2.2.3). Extension of scalars is more complicated, and we first need to define certain auxillary (R, R)-bimodules $R(p)$.

The structure is as follows. In Section 2.1 we define what it means for a representation to be reflection faithful and recall some facts about invariant subrings. In the Section 2.2 we define standard objects and analyse the effect of restriction of scalars on them. In Section 2.3 we define the bimodules $R(p)$ and in Section 2.4 we use them to describe extension of scalars. In Section 2.5 we introduce the notion of support, which will be essential in what follows.

2.1. Reflection faithful representations and invariants. Let (W, S) be a finite Coxeter system and recall that we denote by

$$T = \bigcup_{w \in W} wSw^{-1}$$

the reflections in W. A *reflection faithful* representation of W is a finite dimensional representation V of W such that:

(1) The representation is faithful,
(2) We have $\operatorname{codim} V^w = 1$ if and only if w is a reflection.

If W is finite it is straightforward to see that the geometric representation ([**Hu**], Proposition 5.3) is reflection faithful because it preserves a positive definite bilinear form. If W is infinite, this is not the case in general. However, one has:

PROPOSITION 2.1.1 ([**So6**], Proposition 2.1). *Given any Coxeter system (W, S) there exists a reflection faithful representation of W on a finite dimensional real vector space V.*

For the rest of this chapter let us a fix a reflection faithful representation V of W over a field of characteristic zero.

Because of our assumptions all reflections $t \in T$ act via

$$t(\lambda) = \lambda - 2h_t(\lambda)v_t \tag{2.1.1}$$

for some linear form $h_t \in V^*$ and vector $v_t \in V$. The pair (h_t, v_t) is only determined up to a some choice of scalars. However, one may choose $h_t \in V^*$ such that

$$xh_s = h_t \text{ if } xsx^{-1} = t \tag{2.1.2}$$

where we regard V^* as a W-module via the contragredient action. The elements $h_t \in V^*$ (which we will think of as equations for the hyperplane V^t) will be important in the sequel. For this reason we make a fixed choice of the set $\{h_t \mid t \in T\}$ with the only restriction being that (2.1.2) should hold.

LEMMA 2.1.2. *The elements of $\{h_t \mid t \in T\} \subset V^*$ are pairwise linearly independent.*

PROOF. Let us suppose that $V^s = V^t$ for some reflections $s, t \in T$. For parity reasons st is not a reflection. However V^{st} is of codimension at most 1 and hence must be all of V, as our representation is reflection faithful. In other words st is the identity and so $s = t$. This implies the lemma. $\square$

Let R be the graded ring of regular functions on V, with V^* sitting in degree 2. Because k is an infinite field we may identify R with the symmetric algebra on V^*. As W acts on V it also acts on R on the left via

$$(wf)(\lambda) = f(w^{-1}\lambda) \quad \text{for all } \lambda \in V.$$

If $w \in W$ we denote by R^w the invariants under w. If $I \subset S$ we denote by R^I the invariants under W_I. Recall that $\widetilde{\pi}(I)$ denotes a Poincaré polynomial of W_I (see Section 1.2). One has the fundamental theorem:

THEOREM 2.1.3. *The ring R is a graded free module over R^I. One has an isomorphism of graded R^I-modules:*

$$R \cong \widetilde{\pi}(I) \cdot R^I.$$

We will return to this theorem in Section 3 where we give a sketch of a proof. The following corollary (which may be seen as a relative version of the above statement) will also be important.

COROLLARY 2.1.4. *Let $I \subset J$ be subsets of S. Then R^I is a graded free module over R^J and one has*

$$R^I \cong \frac{\widetilde{\pi}(J)}{\widetilde{\pi}(I)} \cdot R^J.$$

in R^J-Mod.

PROOF. Using the above theorem and the transitivity of restriction we conclude that there exists an isomorphism

$$\widetilde{\pi}(I) \cdot R^I \cong \widetilde{\pi}(J) \cdot R^J$$

in R^J-Mod. However $\widetilde{\pi}(J)/\widetilde{\pi}(I) \in \mathbb{N}[v, v^{-1}]$ by Corollary 1.2.4, and so we may divide both sides by $\widetilde{\pi}(I)$ to obtain the result. □

2.2. Singular standard modules. In this section we define "standard modules". These are graded (R^I, R^J)-bimodules indexed by triples (I, p, J) where $I, J \subset S$ are finitary and and $p \in W_I \backslash W / W_J$ is a double coset.

DEFINITION 2.2.1. *Let $I, J \subset S$ be finitary, $p \in W_I \backslash W / W_J$ and define $K = I \cap p_- J p_-^{-1}$. The* standard module indexed by (I, p, J), *denoted ${}^I R_p^J$, is the ring R^K of W_K-invariant functions in R. We make ${}^I R_p^J$ into an object in R^I-Mod-R^J by defining left and right actions as follows:*

$$r \cdot m = rm \qquad \text{for } r \in R^I \text{ and } m \in {}^I R_p^J$$
$$m \cdot r = m(p_- r) \qquad \text{for } m \in {}^I R_p^J \text{ and } r \in R^J$$

(where rm and $(p_- r)m$ denotes multiplication in R^K). If $I = J = \emptyset$ we write R_w instead of ${}^I R_w^J$.

This action is well-defined because if $r \in R^I$ (resp. $r \in R^J$) then r (resp. $p_- r$) lies in R^K. In the future we will supress the dot in the notation for the left and right action. If p contains $id \in W$ we sometimes omit p and write simply ${}^I R^J$

EXAMPLE 2.2.2. *Some examples of standard objects.*

1. *If either I or J is empty then $I \cap p_- J p_-^{-1} = \emptyset$ for all $p \in W_I \backslash W / W_J$ and hence*

$$ {}^I R_p^J \cong R$$

   with the (R^I, R^J)-bimodule structure as in the definition.
2. *If I and J are both non-empty, the graded dimension of ${}^I R_p^J$ usually varies across double cosets. For example, if $W = S_3$ is the symmetric group on three letters and $I = J$ consists of one simple reflection then W contains two (W_I, W_J)-double cosets*

which we will call p_1 and p_2. Assume that $id \in p_1$. Then, as graded vector spaces:

$$\begin{aligned} {}^I R^J_{p_1} &\cong R^I, \\ {}^I R^J_{p_2} &\cong R. \end{aligned}$$

The following lemma describes the effect of restriction of scalars on standard objects.

LEMMA 2.2.3. *Let $w \in W$, $I, J \subset S$ be finitary and $p = W_I w W_J$ be the (W_I, W_J)-double coset containing w. Then in R^I-Mod-R^J we have an isomorphism:*

$${}_{R^I}(R_w)_{R^J} \cong \widetilde{\pi}(I,p,J) \cdot {}^I R^J_p.$$

Furthermore, if $I \subset K$, $J \subset L$ are finitary and $q = W_K p W_L$ then

$${}_{R^K}({}^I R^J_p)_{R^L} \cong \frac{\widetilde{\pi}(K,q,L)}{\widetilde{\pi}(I,p,J)} \cdot {}^K R^L_q$$

in R^K-Mod-R^L.

PROOF. If $v \in W_J$ then R_w and R_{wv} become isomorphic when we view them as objects in R-Mod-R^J. Similarly, if $u \in W_I$ then the map $r \mapsto ur$ gives an isomorphism between R_w and R_{uw} when regarded as objects in R^I-Mod-R. Thus we may assume without loss of generality that $w = p_-$. Define $K = I \cap p_- J p_-^{-1}$ so that $\widetilde{\pi}(I,p,J) = \widetilde{\pi}(K)$. The first isomorphism follows from the definition of ${}^I R^J_p$ and the decomposition

$$R \cong \widetilde{\pi}(K) \cdot R^K$$

of Theorem 2.1.3.

For the second statement note that, by the transitivity of restriction and the above isomorphism we have

$$\widetilde{\pi}(I,p,J) \cdot {}_{R^K}({}^I R^J_p)_{R^L} = \widetilde{\pi}(K,q,L) \cdot {}^K R^L_q \quad \text{in } R^K\text{-Mod-}R^L.$$

As $\widetilde{\pi}(K,q,L)/\widetilde{\pi}(I,p,J) \in \mathbb{N}[v,v^{-1}]$ by Lemma 1.2.15 we may divide by $\widetilde{\pi}(I,p,J)$ and the claimed isomorphism follows. □

2.3. Enlarging the regular functions. Our ultimate aim for the rest of this section is to understand the effect of extending scalars on standard modules. However, in order to do this we need to introduce certain auxillary modules $R(X) \in R$-Mod-R corresponding to finite subsets $X \subset W$ which we think of as an enlargement of a certain ring of regular functions.

Given $w \in W$ we define its (twisted) graph

$$\mathrm{Gr}_w = \{(w\lambda, \lambda) \mid \lambda \in V\}$$

which we view as a closed subvariety of $V \times V$. Given a finite subset $X \subset W$ we denote by Gr_X the subvariety

$$\mathrm{Gr}_X = \bigcup_{w \in X} \mathrm{Gr}_w .$$

We will denote by $\mathcal{O}(\mathrm{Gr}_X)$ the regular functions on Gr_X which has the structure of an R-bimodule via the inclusion $\mathrm{Gr}_X \hookrightarrow V \times V$.

For all $x \in W$ consider the inclusion

$$\begin{aligned} i_x : V &\hookrightarrow V \times V \\ \lambda &\mapsto (\lambda, x^{-1}\lambda). \end{aligned}$$

This provides an isomorphism of V with Gr_x and an explicit identification of R_x and $\mathcal{O}(\mathrm{Gr}_x)$ as R-bimodules.

The following lemma will be important in the next section.

LEMMA 2.3.1. *Let $I \subset S$ be finitary. We have an isomorphism of graded k-algebras*

$$R \otimes_{R^I} R \cong \mathcal{O}(\mathrm{Gr}_{W_I}).$$

PROOF. (See [**So2**], Lemma 2.2.2) Clearly the surjection $R \otimes R \twoheadrightarrow \mathcal{O}(\mathrm{Gr}_{W_I})$ factorisises to yield a map

$$R \otimes_{R^I} R \twoheadrightarrow \mathcal{O}(\mathrm{Gr}_{W_I}).$$

We claim that this map is the required isomorphism.

As a left R-module, $R \otimes_{R^I} R$ is isomorphic to $\widetilde{\pi}(I) \cdot R$ by Theorem 2.1.3. The subvariety Gr_{W_I} is a union of $|W|$ hyperplanes, each of which is isomorphic to V under the first projection $V \times V \to V$. Hence $\mathrm{Quot}\, R \otimes_R \mathcal{O}(\mathrm{Gr}_{W_I})$ has dimension $|W|$ over $\mathrm{Quot}\, R$. Let K be the kernel of the above surjection. Because both modules have the same dimension over $\mathrm{Quot}\, R$ after applying $\mathrm{Quot}\, R \otimes_R -$ we see that $\mathrm{Quot}\, R \otimes_R K = 0$. However, $R \otimes_{R^W} R$ is torsion free as a left R-module and hence so is K. We conclude that K is zero, establishing the claim. ☐

Recall that, for all $t \in T$, we have chosen an equation $h_t \in V^*$ for the hyperplane fixed by t. We will denote by $(h_t) \subset R$ the ideal generated by h_t. We now come to the definition of the R-bimodules $R(X)$.

DEFINITION/PROPOSITION 2.3.2. *Let $X \subset W$ be a finite subset. Consider the subspace*

$$R(X) = \left\{ f = (f_x) \in \bigoplus_{x \in X} R \;\middle|\; \begin{array}{c} f_x - f_{tx} \in (h_t) \\ \textit{for all } t \in T \textit{ and } x, tx \in X \end{array} \right\} \subset \bigoplus_{x \in X} R.$$

Then $R(X)$ is a graded k-algebra under componentwise multiplication and becomes an object of R-Mod-R if we define left and right actions of $r \in R$ via

$$\begin{aligned} (rf)_x &= rf_x \\ (fr)_x &= f_x(xr) \end{aligned}$$

for $f = (f_x) \in R(X)$*. If a pair of subgroups* $W_1, W_2 \subset W$ *satisfy* $W_1X = X = XW_2$ *then* $R(X)$ *carries commuting left* W_1*- and right* W_2*-actions if we define*

$$(uf)_x = uf_{u^{-1}x} \qquad \text{for } u \in W_1,$$
$$(fv)_x = f_{xv^{-1}} \qquad \text{for } v \in W_2.$$

If $X = \{x\}$ *is a singleton then* $R(X) \cong R_x$*. If* $X = \{x, y\}$ *consists of two elements we write* $R_{x,y}$ *instead of* $R(X)$*.*

PROOF. It is straightforward to check that $R(X)$ is a graded subring containing k. In order to see that the left and right R-operations preserve $R(X)$ it is therefore enough to check that $(r)_{x\in X}$ and $(xr)_{x\in X}$ are elements of $R(X)$ for all $r \in R$. This is clear for $(r)_{x\in X}$ and for $(xr)_{r\in X}$ it follows from the formula $tg = g - g(v_t)h_t$ for $g \in V^*$. The right W_2-operation clearly preserves $R(p)$. For the left W_1-operation if $x, tx \in X$ one has, using (2.1.2),

$$(wf)_x - (wf)_{tx} = w(f_{w^{-1}x} - f_{w^{-1}tx}) \in (w(h_{w^{-1}tw})) = (h_t).$$

The operations clearly commute and the fact that $R(X) \cong R_x$ if $X = \{x\}$ is immediate from the definitions. □

REMARK 2.3.3.

(1) *We have defined* $R(X)$ *for general finite subsets* $X \subset W$ *but will only ever need two cases:*
   (a) $X = p$ *is a* (W_I, W_J)*-double coset for finitary* $I, J \subset S$*.*
   (b) $X = \{x, tx\}$ *for some* $x \in W$ *and reflection* $t \in T$*.*
(2) *The graded ring* $R(X)$ *has a natural description in terms of the Bruhat graph of* W *introduced in Section 1.1. Let* $\mathcal{G}_X$ *be the full subgraph of the Bruhat graph of* W *with vertices* X*. Then an element of* $R(X)$ *can be thought of as a choice of* $f_x \in R$ *for every vertex* $x \in \mathcal{G}_p$*, subject to the conditions that* $f_x - f_y$ *lies in* (h_t) *whenever* x *and* y *are connected by an edge labelled* t*. Under this description the left action of* R *is just the diagonal action, and the right action is the diagonal action "twisted" by the label of each vertex. The left* W_1*- and right* W_2*-actions are induced (with a twist for the action of* W_1*) by the left and right multiplication action of* W_1 *ad* W_2 *on* X*.*

The following proposition gives a useful alternative description of $R(X)$.

PROPOSITION 2.3.4. *Let* $X \subset W$ *be a finite set. There exists an exact sequence in* R*-Mod-*R

$$0 \to R(X) \to \bigoplus_{x\in X} R_x \to \bigoplus_{\substack{x<tx\in X\\ t\in t}} R_x/(h_t)$$

where the maps are as described in the proof.

PROOF. The first map is the inclusion of $R(X)$ into $\bigoplus_{x\in X} R_x$ which is clearly a morphism of R-bimodules. We describe the second map by describing its components

$$R_x \to R_y/(h_t).$$

This map is zero if $x \notin \{y, ty\}$. Otherwise it is given by

$$f \mapsto \epsilon_{x,tx} f + (h_t)$$

where $\epsilon_{x,tx}$ is defined by

$$\epsilon_{x,tx} = \begin{cases} 1 & \text{if } x < tx \\ -1 & \text{if } x > tx \end{cases}.$$

This is a morphism in R-Mod-R because this is true of the quotient map $R_x \to R_y/(h_t)$ whenever $x = y$ or $x = ty$. Lastly a tuple $(f_x) \in \oplus R_x$ is mapped to zero if $f_x = f_{tx}$ in $R_x/(h_t)$ for all $x, tx \in X$ and $t \in T$, which is exactly the condition for (f_x) to belong to $R(X)$. $\square$

The following lemma explains the title of this subsection.

LEMMA 2.3.5. *Let $X \subset W$ be finite. The map*

$$\rho : \mathcal{O}(\mathrm{Gr}_X) \to R(X)$$
$$f \mapsto (i_x^* f)_{x\in p}$$

is well-defined, injective and a morphism in R-Mod-R.

PROOF. Any regular function $f \in \mathcal{O}(\mathrm{Gr}_X)$ is determined by its restriction to all Gr_x for $x \in X$, which is just the tuple

$$(i_x^* f)_{x\in p} \in \bigoplus_{x\in p} R.$$

We claim that this tuple lies in $R(X)$. Indeed, we just need to check that $i_x^* f$ and $i_{tx}^* f$ agree on V^t if $x, tx \in p$ for some $t \in T$ and this is straightforward. It follows that the map is an injection of graded k-algebras, in particular an injection in R-Mod-R. $\square$

In general the map

$$\rho : \mathcal{O}(\mathrm{Gr}_X) \hookrightarrow R(X)$$

is not surjective. The question as to when it is seems quite subtle, as the following examples show.

EXAMPLE 2.3.6.

(1) *We begin with an example which does not fit into the above framework, but is nevertheless instructive. Let Z_1, Z_2 and Z_3 be three distinct one dimensional linear subspaces of k^2 and let*

$$Z = Z_1 \cup Z_2 \cup Z_3.$$

The ring of regular functions on Z is a proper subspace of the space of triples (f_1, f_2, f_3) with $f_i \in \mathcal{O}(Z_i)$, such that

$f_1(0) = f_2(0) = f_3(0)$ (which is the analogue of $R(X)$ above). Indeed, both rings are graded and the dimensions of the graded components are $(1, 2, 3, 3, \dots)$ and $(1, 3, 3, 3, \dots)$ respectively.

(2) *It is a straightforward consequence of Lemma 2.3.1 that if p is a left of right coset of a finite standard parabolic subgroup then ρ is always an isomorphism.*

(3) *We now give an example where ρ is not surjective. This example was pointed out by Matthew Dyer. Consider $W = S_4$ acting via permutations of coordinates on $V = k^4$. We may identify $R = k[X_1, X_2, X_3, X_4]$ with W acting via $w(X_i) = X_{w(i)}$. Denote the simple reflections by r, s, t indexed so that $rt = tr$. Let $I = J = \{r, t\}$ and consider the double coset $p = W_I s W_J$. We claim that*

$$\mathcal{O}(\mathrm{Gr}_p) \subsetneq R(p).$$

Indeed $\mathcal{O}(\mathrm{Gr}_p)$ is cyclic as an $R \otimes R$-module and we will see in Theorem 2.4.1 of the next section that

$$R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \cong R(p).$$

If $\rho : \mathcal{O}(\mathrm{Gr}_p) \to R(p)$ were sujective, then $R(p)$ would also be cyclic and hence ${}^I R_p^J$ would be cyclic as an $R^I \otimes R^J$-module.

We claim however, that ${}^I R_p^J$ is not *cyclic as an $R^I \otimes R^J$-module. This is seen already in degree 2. The image of $R^I \otimes R^J$ acting on 1 in ${}^I R_p^J \cong R$ in degree 2 consists of*

$$\langle X_1 + X_2, X_3 + X_4, X_2 + X_4 \rangle \subsetneq R^2.$$

(4) *However it might be still true that, if $I, J \subset S$ are finitary then $R^{I \cap J}$ is generated by the subrings R^I and R^J. This is true for the symmetric group S_n acting as above on $k[X_1, X_2, \dots, X_n]$ if k is of characteristic 0, as one may see by considering Newton power sums. This trick was pointed out by Olivier Mathieu.*

Because $R(X)$ has the structure of a graded k-algebra we have an injection

$$R(X) \hookrightarrow \mathrm{Hom}(R(X), R(X)).$$

However, as $R(X)$ is generally not cyclic as an R-bimodule, $R(X)$ may have more endomorphisms. The following shows that this is not the case.

Proposition 2.3.7. *For all finite subsets $X \subset W$ we have*

$$\mathrm{Hom}(R(X), R(X)) = R(X).$$

Proof. For the course of the proof it will be more convienient to regard $R(X)$ as graded left $R \otimes R$-module. Let $\varphi : R(X) \to R(X)$ be a morphism in $R \otimes R$-Mod and denote by $f = (f_x)_{x \in X}$ the image of 1. Choose $m = (m_x) \in R(X)$. We will be finished if we can show that

$\varphi(m)_z = m_z f_z$ for all $z \in X$. Let us choose $z \in X$ and let $g \in R \otimes R$ be a function that vanishes on Gr_y for $z \neq y$ but not on Gr_z, and let (g_x) denote its image in $R(X)$ (the result of acting with g on $1 \in R(X)$). Note that

$$(gm)_x = \delta_{x,z} g_x m_x$$

and so gm is in the image of $R \otimes R$. Hence

$$g_z \varphi(m)_z = (g\varphi(m))_z = \varphi(gm)_z = f_z g_z m_z$$

and hence $\varphi(m)_z = m_z f_z$ as g_z is non-zero. $\square$

2.4. Standard modules and extension of scalars. The aim of this subsection is to study the effect of extension of scalars on standard modules. That is, we want to understand the bimodules

$$R^K \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R^L \in R^K\text{-Mod-}R^L$$

where $K \subset I$ and $L \subset J$ are finitary. The key is provided by the bimodules $R(X)$ introduced in the previous section.

For the rest of this subsection fix finitary subsets $I, J \subset S$ and a double coset $p \in W_I \backslash W / W_J$. Recall that the bimodules $R(p)$ have commuting left W_I- and right W_J-actions. Of course we can make this into a left $W_I \times W_J$ action by defining $(u,v)m = umv^{-1}$ for all $m \in R(p)$.

Theorem 2.4.1. *Let $I \supset K$ and $J \supset L$. There exists an isomorphism*

$$R^K \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R^L \xrightarrow{\sim} R(p)^{W_K \times W_L}$$

in R^K-Mod-R^L.

The theorem will take quite a lot of effort to prove. In Lemmas 2.4.2 and 2.4.3 below we construct a morphism

$$R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \to R(p).$$

commuting with natural actions of $W_K \times W_L$ on both sides. By considering invariants one may reduce the theorem to showing that this map is an isomorphism.

Let us first describe the $W_I \times W_J$ actions. By Proposition 2.3.2 and the discussion at the beginning of this section there is an $W_I \times W_J$-action on $R(p)$. We define a $W_I \times W_J$-action on $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ via

$$(u,v) f \otimes g \otimes h = uf \otimes g \otimes vh.$$

It is easy to see that this action is well-defined.

The following lemma tells us how to find the standard module ${}^I R_p^J$ as a submodule of $R(p)$.

Lemma 2.4.2. *In R^I-Mod-R^J we have an isomorphism*

$$R(p)^{W_I \times W_J} \cong {}^I R_p^J.$$

Proof. Let $K = I \cap p_- J p_-^{-1}$ and choose $f \in R(p)^{W_I \times W_J}$. If $u \in W_K$ then $up_- = p_- v$ for some $v \in W_J$ and $f_{p_-} = ((u, v)f)_{p_-} = uf_{p_-}$. In other words $f_{p_-} \in R^K$. Hence we obtain a map

$$\begin{aligned} R(p)^{W_I \times W_J} &\to {}^I R_p^J \\ (f_x) &\mapsto f_{p_-} \end{aligned}$$

which is obviously injective and a morphism in R^I-Mod-R^J.

It remains to show surjectivity. To this end choose $m \in {}^I R_p^J$ and consider the tuple $f = (f_x) \in \oplus_{x \in p} R$ where, for each $x \in p$ we choose $u \in W_I, v \in W_J$ with $x = up_- v$ and define $f_x = um$. This is well defined because if $up_- v = u' p_- v'$ with $u, u' \in W_I$ and $v, v' \in W_J$ then $u^{-1} u' \in W_I \cap p_- W_J p_-^{-1} = W_K$ by Kilmoyer's Theorem (1.2.8), and hence $um = u'm$ as m is invariant under W_K. The tuple (f_x) also lies in $R(p)$ as if x and tx both lie in p then by Proposition 1.2.12 either $t \in W_I$ (in which case $f_{tx} = tf_x$) or $tx = xt'$ for some reflection t' in W_J (in which case $f_x = f_{tx}$). Lastly, it it easy to check that f is $W_I \times W_J$ invariant. As f gets mapped to m under the above map, we see that the map is indeed surjective. □

Having identified ${}^I R_p^J$ as a (R^I, R^J)-submodule of $R(p)$ we obtain by adjunction a morphism

$$\mu : R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \to R(p).$$

We will see below that this is an isomorphism. However first we need:

Lemma 2.4.3. *The morphism μ commutes with the $W_I \times W_J$-actions on both modules.*

Proof. This is a technical but straightforward calculation. Let $a = r_1 \otimes m \otimes r_2 \in R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ and $(u, v) \in W_I \times W_J$. We want to show that $\mu((u, v)a) = (u, v)\mu(a)$.

Under μ, a gets mapped to $f = (f_z) \in R(p)$ where

$$f_z = r_1 (xm)(zr_2)$$

if $z = xp_- y$ with $x \in W_I$ and $y \in W_J$. Similarly $(u, v)a = ur_1 \otimes m \otimes vr_2$ gets mapped to $\tilde{f} = (\tilde{f}_z) \in R(p)$ where

$$\tilde{f}_z = ur_1 (xm)(zvr_2).$$

We need to show that $(u, v)f = \tilde{f}$. This follows from

$$((u, v)f)_z = uf_{u^{-1}zv} = u(r_1 (u^{-1} xm)(u^{-1} zvr_2)) = ur_1 (xm)(zvr_2) = \tilde{f}_z.$$

□

Proof of Theorem 2.4.1. By considering $W_K \times W_L$ invariants it is enough to show that the morphism μ construted above is an isomorphism. This will follow from two facts which we verify below:

(1) Both $R(p)$ and $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ are isomorphic to $\widetilde{\pi}(p) \cdot R$ as graded left R-modules;

(2) The morphism μ is injective.

Indeed (1) says that each graded component of $R(p)$ and $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ is of the same (finite) dimension over k. Using (2) we then see that φ is an isomorphism on each graded component and hence is an isomorphism.

We start by establishing (1) for $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$. Choose $w \in p$. By Theorem 2.1.3 we have an isomorphism of left R-modules:

$$R \otimes_{R^I} R_w \otimes_{R^J} R \cong \widetilde{\pi}(I)\widetilde{\pi}(J) \cdot R.$$

Hence, by Lemma 2.2.3 we have (again as left R-modules):

$$\widetilde{\pi}(I, p, J) \cdot R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \cong \widetilde{\pi}(I)\widetilde{\pi}(J) \cdot R$$

Dividing by $\widetilde{\pi}(I, p, J)$ and using Lemma 1.2.15 we conclude that

$$R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \cong \widetilde{\pi}(p) \cdot R \qquad \text{in } R\text{-Mod} \tag{2.4.1}$$

as claimed.

It seems much harder to establish (1) for $R(p)$. This is Corollary 3.3.4 of the next section, which we prove using Demazure operators.

The rest of the proof will be concerned with (2). Choose again $w \in p$. Using Lemma 2.3.1 we may identify $R \otimes_{R^I} R_w \otimes_{R^J} R$ with the regular functions on the variety

$$Z = \left\{ (\lambda, \mu, \nu) \,\middle|\, \begin{array}{c} \lambda = u\mu \text{ for some } u \in W_I \\ \mu = wv\nu \text{ for some } v \in W_J \end{array} \right\} \subset V \times V \times V.$$

We have an obvious projection map $Z \to \mathrm{Gr}_p$ sending (λ, μ, ν) to (λ, ν) and hence we have a morphism in R-Mod-R (in fact of k-algebras)

$$\mathcal{O}(\mathrm{Gr}_p) \to R \otimes_{R^I} R_w \otimes_{R^J} R.$$

Taking $w = p_-$ this map lands in $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ regarded as a submodule of $R \otimes_{R^I} R_w \otimes_{R^J} R$. We conclude the existence of a commutative diagram

$$\begin{tikzcd} & \mathcal{O}(\mathrm{Gr}_p) \arrow[dl] \arrow[dr, "\rho"] & \\ R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R \arrow[rr, "\varphi"] & & R(p) \end{tikzcd}$$

where ρ is as in Lemma 2.3.5.

We now argue that all arrows become isomorphisms after tensoring with $\operatorname{Quot} R$. As ρ is injective and $\operatorname{Quot} R$ is flat over R it is enough to show that all modules have dimension $|p|$ over $\operatorname{Quot} R$ after applying $\operatorname{Quot} R \otimes_R -$. This is indeed the case:

(1) $\mathcal{O}(\mathrm{Gr}_p)$: For the same reasons as in the proof of Lemma 2.3.1.
(2) $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$: This follows from (2.4.1)
(3) $R(p)$: By applying $\operatorname{Quot} R \otimes_R -$ to the exact sequence in Proposition 2.3.4.

We conclude that all maps (in particular μ) become isomorphisms after applying $\operatorname{Quot} R \otimes_R -$.

To conclude the proof, note that by the above arguments $R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R$ is torsion free as a left R-module. Hence μ is injective if and only if this is true after applying $\operatorname{Quot} R \otimes_R -$. Thus μ is injective as claimed. $\square$

We may use this theorem to determine the morphisms between standard modules. Recall that ${}^I R_p^J$ was defined as a subring of R, and therefore has the structure of a k-algebra compatible with its (R^I, R^J)-bimodule structure. Therefore we certainly have an injection

$$ {}^I R_p^J \hookrightarrow \operatorname{Hom}({}^I R_p^J, {}^I R_p^J). $$

The following proposition makes this more precise.

Corollary 2.4.4. *For $p, q \in W_I \backslash W / W_J$ we have*

$$ \operatorname{Hom}({}^I R_p^J, {}^I R_q^J) = \begin{cases} {}^I R_p^J & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} $$

Proof. Extension of scalars give us an injection

$$ \operatorname{Hom}({}^I R_p^J, {}^I R_q^J) \to \operatorname{Hom}(R \otimes_{R^I} {}^I R_p^J \otimes_{R^J} R, R \otimes_{R^I} {}^I R_q^J \otimes_{R^J} R) $$

because we may again restrict to R^I-Mod-R^J. By the above theorem the latter module is isomorphic to $\operatorname{Hom}(R(p), R(q))$. This is 0 if $p \neq q$ because $\operatorname{Hom}(R_x, R_y) = 0$ if $x \neq y$. Otherwise $\operatorname{Hom}(R(p), R(p)) = R(p)$ by Proposition 2.3.7, and so $\operatorname{Hom}({}^I R_p^J, {}^I R_p^J)$ consists of those $\alpha \in \operatorname{Hom}(R(p), R(p))$ for which $\alpha(1) \in {}^I R_p^J$. Hence $\operatorname{Hom}({}^I R_p^J, {}^I R_p^J) = {}^I R_p^J$ as claimed. $\square$

2.5. Support. Let X be an affine variety over k and A its k-algebra of regular functions. We will make use of the equivalence between (finitely-generated) A-modules and (quasi)-coherent sheaves on X (see [**Har**], Chapter II, Corollary 5.5). If M is an A-module, and $\mathcal{M}$ is the corresponding quasi-coherent sheaf on X, then the *support* of $\mathcal{M}$, which we will denote $\operatorname{supp} M$ by abuse of notation, consists of those points $x \in X$ for which $\mathcal{M}_x \neq 0$. The support of a section $m \in M$, denoted $\operatorname{supp} m$, is the support of the submodule generated by m. It follows from the definition that if $M' \hookrightarrow M \twoheadrightarrow M''$ is an exact sequence of A-modules then

$$ \operatorname{supp} M = \operatorname{supp} M' \cup \operatorname{supp} M''. \tag{2.5.1} $$

If M is finitely generated then the support of M is the closed subvariety of X determined by the annihilator of M ([**Har**], II, Exercise 5.6(b)).

Let $f : X \to Y$ be a map of affine varieties and $A \to B$ be the corresponding map of regular functions. If M and N are A- and B-modules respectively, then

$$f(\operatorname{supp} N) \subset \operatorname{supp}({}_A N) \subset \overline{f(\operatorname{supp} N)}, \tag{2.5.2}$$

$$\operatorname{supp}(B \otimes_A M) = f^{-1}(\operatorname{supp} M). \tag{2.5.3}$$

The first is an exercise, and the second is Exercise 19(viii), Chapter 3 of [**AM**] for finitely generated M, but seems to be true in general (in any case we only need it for finitely generated M). It follows that if f is finite (hence closed) and N is finitely generated, then

$$f(\operatorname{supp} N) = \operatorname{supp}({}_A N). \tag{2.5.4}$$

The rest of this section will be concerned with applying notions of support to objects in R^I-Mod-R^J, where $I, J \subset S$ are finitary. This is possible as we may regard any such object as an $R^I \otimes R^J$-module. We identify $R^I \otimes R^J$ with the regular functions on the quotient $V/W_I \times V/W_J$. Thus, given any $M \in R^I$-Mod-R^J, $\operatorname{supp} M \subset V/W_I \times V/W_I$.

In Section 2.3, we defined the twisted graph $\mathrm{Gr}_x \subset V \times V$ as well as Gr_C for finite subsets $C \subset W$. For a double coset $p \in W_I \backslash W / W_J$ denote by ${}^I\mathrm{Gr}^J_p$ the image of Gr_p under the quotient map $V \times V \to V/W_I \times V/W_J$. The subvariety ${}^I\mathrm{Gr}^J_p$ is equal to the image of Gr_x for any $x \in p$ and thus is irreducible. Given any set $C \subset W_I \backslash W / W_J$, we define

$$ {}^I\mathrm{Gr}^J_C = \bigcup_{p \in C} {}^I\mathrm{Gr}^J_p $$

which we understand as a subvariety if C is finite, and as a set if C is infinite.

We will be interested in $M \in R^I$-Mod-R^J whose support is contained in ${}^I\mathrm{Gr}^J_C$ for some finite set $C \subset W_I \backslash W / W_J$. Given finitary $I \subset K$ and $J \subset L$ we have functors of restriction and extension of scalars between R^I-Mod-R^J and R^K-Mod-R^L. Because the inclusion $R^K \otimes R^L \to R^I \otimes R^J$ corresponds to the finite map

$$ V/W_I \times V/W_J \to V/W_K \times V/W_L $$

we may translate (2.5.3) and (2.5.4) as follows:

LEMMA 2.5.1. *Let $I \subset K$ and $J \subset K$ be finitary subsets of S and let*

$$ \mathrm{qu} : W_I \backslash W / W_J \to W_K \backslash W / W_L $$

denote the quotient map.

(1) *If $M \in R^I$-Mod-R^J and* $\operatorname{supp} M = {}^I\mathrm{Gr}^J_C$ *for some finite subset $C \subset W_I \backslash W / W_J$ then* $\operatorname{supp}({}_{R^K} M_{R^L}) = {}^K\mathrm{Gr}^L_{\mathrm{qu}(C)}$.

(2) *If $N \in R^K$-Mod-R^L and* $\operatorname{supp} M = {}^I\mathrm{Gr}^J_{C'}$ *for some finite subset $C' \subset W_K \backslash W / W_L$ then* $\operatorname{supp}(R^I \otimes_{R^K} M \otimes_{R^L} R^J) = {}^I\mathrm{Gr}^J_{\mathrm{qu}^{-1}(C')}$.

The same is true with "=" replaced with "⊂" throughout.

Given a set $C \subset W_I \backslash W/W_J$ and $M \in R^I$-Mod-R^J we denote by $\Gamma_C M$ the submodule of sections with support in ${}^I\mathrm{Gr}_C^J$. That is

$$\Gamma_C M = \{m \in M \mid \operatorname{supp} m \subset {}^I\mathrm{Gr}_C^J\}.$$

Recall from Proposition 1.2.12 that the Bruhat order on W descends to a partial order on $W_I \backslash W/W_J$ and that, given $p \in W_I \backslash W/W_J$, we write $\{\leq p\}$ for the set of elements in $W_I \backslash W/W_J$ which are smaller than p (and similarly for $\{< p\}$, $\{\geq p\}$ and $\{> p\}$). We also abbreviate

$${}^I\mathrm{Gr}_{\leq p}^J = {}^I\mathrm{Gr}_{\{\leq p\}}^J \text{ and } \Gamma_{\leq p} M = \Gamma_{\{\leq p\}} M$$

and analogously for ${}^I\mathrm{Gr}_{<p}^J$, $\Gamma_{<p} M$, ${}^I\mathrm{Gr}_{\geq p}^J$ etc. The following additional notation will be useful:

$$\begin{aligned}
\Gamma^p M &= M/\Gamma_{\neq p} M \\
\Gamma_p^{\leq} M &= \Gamma_{\leq p} M/\Gamma_{<p} M \\
\Gamma_p^{\geq} M &= \Gamma_{\geq p} M/\Gamma_{>p} M.
\end{aligned}$$

Recall that in Subsection 2.3 we defined $R(X) \in R$-Mod-R for any finite subset $X \subset W$.

Lemma 2.5.2. *The support of $f = (f_x) \in R(X)$ is Gr_C, where*

$$C = \{x \in X \mid f_x \neq 0\}.$$

Proof. Because we may identify R_x as an $R \otimes R$-module with the regular functions on the irreducible Gr_x it follows that every $0 \neq m \in R_x$ has support equal to Gr_x. The lemma than follows by considering the embedding of $R(X)$ in $\bigoplus_{x \in X} R_x$. □

Lemma 2.5.3. *Let $I, J \subset S$ be finitary and $p \in W_I \backslash W/W_J$. The support of any non-zero $m \in {}^I R_p^J$ is ${}^I\mathrm{Gr}_p^J$.*

Proof. This follows from (2.5.4), Lemma 2.5.2 above and the fact that we may view ${}^I R_p^J$ as an (R^I, R^J)-submodule of $R(p)$ (Lemma 2.4.2). □

3. Equivariant Schubert calculus

In this section we introduce Demazure operators. We use them for two purposes:

1. To establish the self-duality (up to shifts) of certain rings R^J, viewed as modules over invariant subrings $R^I \subset R^J$ (Section 3.2).
2. To construct filtrations on the modules $R(p)$, where $p \subset W_I \backslash W/W_J$ is a finite double coset (Section 3.3).

3.1. Demazure operators. In this section we recall the definition and basic properties of "classical" Demazure operators essentially following [**Dem**]. Recall that (W, S) is a Coxeter system, V is a reflection faithful representation of W and R denotes the ring of regular functions on V.

If $t \in W$ is a reflection and $f \in R$ then $f - tf$ vanishes on the hyperplane fixed by t and hence is divisible by h_t. We define the *Demazure operator*

$$\partial_t : R[2] \to R$$

by

$$\partial_t(f) = \frac{f - tf}{2h_t}.$$

It is a morphism of R^t-modules.

Let $f, g \in R$ and $t \in W$ be a reflection. The following properties of ∂_t are immediate:

$$\partial_t f = 0 \quad \text{if and only if} \quad tf = f \tag{3.1.1}$$

$$\partial_t((\partial_t f)g) = \partial_t(f(\partial_t g)) \tag{3.1.2}$$

Let $I \subset S$ be a finitary subset and $W_I \subset W$ the corresponding finite parabolic subgroup. An element $f \in R$ is W_I*-anti-invariant* if $wf = (-1)^{\ell(w)} f$ for all $w \in W_I$. This is equivalent to requiring that $tf = -f$ for all reflections $t \in W_I$. Denote by d_I the product

$$d_I = \prod_{t \in W_I \cap T} h_t \in R^{2\ell(w_I)}$$

which may be seen to be W_I-anti-invariant. Let $J_I : R \to R$ denote the "projection onto W_I-anti-invariants" operator:

$$J_I = \frac{1}{|W_I|} \sum_{w \in W_I} (-1)^{\ell(w)} w$$

We recall the following theorem, due to Demazure [**Dem**] and Bernstein, Gelfand and Gelfand [**BGG**].

THEOREM 3.1.1. *Let $I \subset S$ be a finitary subset.*

(1) The W_I-anti-invariant elements of R build a cyclic R^I-submodule of R generated by d_I.

(2) If $st \dots u$ is a reduced expression for the longest element $w_I \in W_I$ then

$$\partial_s \partial_t \dots \partial_u = J_I / d_I$$

considered as endomorphisms of R.

(3) Let $st \dots u$ be an expression in the elements of I of length n. Then the operator

$$\partial_s \partial_t \dots \partial_u : R[2n] \to R$$

is zero unless $st\dots u$ is a reduced expression, in which case it only depends on $w = st\dots u \in W_I$ and not on the choice of reduced expression. Thus one obtains well defined operators

$$\partial_w : R[2\ell(w)] \to R$$

for all $w \in W_I$.

(4) *The elements $\{\partial_w(d_I) \mid w \in W_I\}$ give a free graded basis for R as an R^I-module. In particular,*

$$R \cong \widetilde{\pi}(W_I) \cdot R^I$$

as an R^I-module.

Comment on proof: (1) is straightforward. For an elegant proof of (2) see Proposition 1 in [**Dem**]. For (3) note that, if $s, t \in S$ are simple reflections such that st has finite order then, taking W_I to be the parabolic subgroup generated by s and t, (2) implies that that ∂_s and ∂_t satisfy the braid relations. (3) is then a consequence of (2) and Tit's theorem (see [**Bo**], p. 16, Prop. 5) that one obtains all reduced expressions for an element by applying braid relations to a fixed reduced expression. It is straightforward to see that the $\{\partial_w(d_I) \mid w \in W_I\}$ span R as an R^I module. Then, assuming the existence of a relation of minimal degree, one may always apply a Demazure operator to obtain a non-trivial relation of smaller degree, which provides the contradiction establishing (4). □

3.2. Duality. As a first application of Demazure operators we will establish the self-duality (up to shifts) of certain invariant subrings of R. Given a finite parabolic subgroup $W_I \subset W$ the the Demazure operator ∂_{w_I} corresponding to the longest element of W_I allows us to define an R^I-bilinear form on R via

$$\begin{array}{rcl} R \times R & \to & R^I[-2\ell(w_I)] \\ (f, g) & \mapsto & (f, g)_I = \partial_{w_I}(fg). \end{array}$$

This form is R^I-bilinear because ∂_{w_I} is a morphism of R^I-modules and is well-defined because the image of $f \in R$ under ∂_{w_I} is W_I-invariant by (3.1.1).

Lemma 3.2.1. *Let $x, y \in W_I$. One has*

$$(\partial_x d_I, \partial_y d_I)_I \in \delta_{xw_I,y} + (R^I)^+$$

where $(R^I)^+ \subset R$ denotes the elements of positive degree.

Proof. Notice that

$$\partial_{w_I}((\partial_x d)(\partial_y d)) = \partial_{w_I}(d(\partial_{x^{-1}}\partial_y d))$$

by repeated application of (3.1.2). Now if $xw_I = y$ then

$$\partial_{x^{-1}}\partial_y d = \partial_{w_0}(d) = 1$$

and the lemma follows in this case. If $xw_0 \neq y$ there are two possibilities:

(1) $\ell(x^{-1}) + \ell(y) > \ell(x^{-1}y)$, or
(2) $\ell(x^{-1}) + \ell(y) = \ell(x^{-1}y) < \ell(w_0)$.

In case 1) $\partial_{x^{-1}}\partial_y = 0$. In case 2) $\partial_{x^{-1}}\partial_y(d)$ is of degree strictly greater than 0, and thus so is $\partial_{w_0}(d(\partial_{x^{-1}}\partial_y d))$. □

We will now use this form to establish the self-duality of $R[\ell(w_I)]$ as an R^I-module.

PROPOSITION 3.2.2. *The map*

$$\varphi : R[\ell(w_I)] \to \mathrm{Hom}_{R^I}(R[\ell(w_I)], R^I)$$

given by

$$\varphi(f) = (f, -)_I$$

is an ismorphism in R-Mod.

PROOF. That the map is a morphism of R-modules follows from the fact that, for all $f, g, h \in R$ we have $(fg, h)_I = (f, gh)_I$. The injectivity of φ is equivalent to the non-degeneracy of $(\cdot,\cdot)_I$ which follows from Theorem 3.1.1 and the above lemma. As R^I-modules we have

$$\mathrm{Hom}_{R^W}(R[\ell(w_I)], R^W) \cong \mathrm{Hom}_{R^W}(\pi(I)\cdot R^W, R^W) \cong \pi(I)\cdot R^W \cong R[\ell(w_I)]$$

and hence φ must be an isomorphism. □

COROLLARY 3.2.3. *Let $I \subset J$ be subsets of S. Then*

$$\mathrm{Hom}_{R^J}(R^I[\ell(w_J) - \ell(w_I)], R^J) \cong R^I[\ell(w_J) - \ell(w_I)].$$

PROOF. We have the following isomorphisms in R^J-Mod:

$$\begin{aligned}
\pi(I) \cdot \mathrm{Hom}_{R^J}(R^I[\ell(w_J) - \ell(w_I)], R^J) &\cong \\
&\cong \mathrm{Hom}_{R^J}(\pi(I) \cdot R[\ell(w_J) - \ell(w_I)], R^J) \\
&\cong \mathrm{Hom}_{R^J}(R[\ell(w_J)], R^J) \\
&\cong R[\ell(w_J)] \qquad \text{(Proposition 3.2.2)} \\
&\cong \pi(I) \cdot R^I[\ell(w_J) - \ell(w_I)]
\end{aligned}$$

The corollary then follows by dividing both sides by $\pi(I)$. □

3.3. Demazure operators on $R(X)$. The aim of this subsection is to define Demazure operators on $R(X)$ and use them to construct filtrations on $R(p)$ for finite double cosets $p \subset W$, as well as invariant subrings thereof. This discussion was influenced by [**KT**], where a similar situation is discussed.

Recall that in Section 2.3 we defined, for all finite sets $X \subset W$ a bimodule $R(X) \in R$-Mod-R. Moreover, given subgroups $W_1, W_2 \subset W$ such that $W_1X = X = XW_2$, the bimodule $R(X)$ carries commuting left W_1- and right W_2-actions.

Definition/Proposition 3.3.1. *Let $X, W_1, W_2 \subset W$ be as above.*

(1) *For all reflections $t \in W_1$ there exists an operator $f \mapsto \partial_t f$ on $R(X)$, the* left Demazure operator to t, *uniquely determined by*
$$f - tf = 2h_t(\partial_t f) \quad \text{for all } f \in R(p).$$
This is a morphism in R^t-Mod-R.

(2) *For all reflections $t \in W_2$ there exists an operator $f \mapsto f\partial_t$ on $R(X)$, the* right Demazure operator to t, *uniquely determined by*
$$f - ft = (f\partial_t)2h_t \quad \text{for all } f \in R(p).$$
This is a morphism in R-Mod-R^t.

Proof. We first treat the case of the left Demazure operator. Uniqueness is clear as $R(X)$ is torsion free as a left R-module. Rewriting the condition at $x \in p$ we see that, if $f \in R(X)$, $\partial_t f$ must be given by
$$(\partial_t f)_x = \frac{f_x - tf_{tx}}{2h_t}.$$
A priori this defines an element of $\operatorname{Quot} R$. However, by definition of $R(X)$, $f_x - f_{tx}$ and hence $f_x - tf_{tx}$ lies in (h_t). Thus $(\partial_t f)_x \in R$ for all $x \in X$.

It remains to see that $\partial_t f \in R(X)$. Because $f - tf \in R(X)$ it is clear that
$$(\partial_t f)_x - (\partial_t f)_{t'x} \in (h_{t'})$$
whenever $t' \neq t$ and $x, t'x \in X$. Writing out the definitions, on also sees that
$$(\partial_t f)_x - (\partial_t f)_{tx}$$
it t-anti-invariant, and hence $(\partial_t f)_x - (\partial_t f)_{tx} \in (h_t)$. It follows that $\partial_t f \in R(X)$ and hence the left Demazure operator to t exists.

It is clear that the left Demazure operator for $t \in W_I$ commutes with multiplication on the left with a t-invariant function. For the right action of $r \in R$ on $f \in R(X)$ one has
$$(\partial_t(fr))_x = \frac{(fr)_x - t(fr)_{tx}}{2h_t} = \frac{f_x - f_{tx}}{2h_t} xr = ((\partial_t f)r)_x.$$
In particular, $f \mapsto \partial_t f$ is a morphism in R^t-Mod-R as claimed.

We now treat the case of the right Demazure operator for a reflection $t \in W_2$. The operator is clearly unique if it exists and $f\partial_t$ for $f \in R(X)$ must be given by
$$(f\partial_t)_x = \frac{f_x - f_{xt}}{2xh_t}.$$
Similarly to above one checks that $(f\partial_t)_x \in R$ for all $x \in X$ and then that $f\partial_t \in R(X)$, using the definition of $R(X)$ and (2.1.2). It is then straighforward to see that $f \mapsto f\partial_t$ is a morphism in R-Mod-R^t. $\square$

Recall from Section 2.5 that the support of an element $f \in R(X)$ is easy to calculate: it is the set Gr_A where $A = \{x \in X \mid f_x \neq 0\}$. The following lemma is then an immediate consequence of the definition of the Demazure operators.

LEMMA 3.3.2. *Let $f \in R(X)$ such that* $\operatorname{supp} f \subset \mathrm{Gr}_A$ *for some $A \subset X$.*

(1) If $t \in W_I$ is a reflection then $\operatorname{supp} \partial_t f \subset \mathrm{Gr}_{A \cup tA}$.
(2) If $t \in W_J$ is a reflection then $\operatorname{supp} f \partial_t \subset \mathrm{Gr}_{A \cup At}$.

For the rest of this section fix two finitary subsets $I, J \subset S$ as well as a double coset $p \in W_I \backslash W / W_J$. We now come to the main theorem of this section, which purports the existence of certain special elements in $R(p)$.

THEOREM 3.3.3. *There exists $\phi_x \in R(p)$ for all $x \in p$, unique up to a scalar, such that*

(1) $\deg \phi_x = 2(\ell(p_+) - \ell(x))$,
(2) $\operatorname{supp} \phi_x \subset \mathrm{Gr}_{\leq x}$ *and* $(\phi_x)_x \neq 0$.

The set $\{\phi_w \mid w \in p\}$ builds a basis for $R(p)$ as a left or right R-module.

PROOF. Let us first assume that there exists $\phi_x \in R(p)$ for all $x \in p$ satisfying the conditions of the theorem. We will argue that they are then uique and form a basis for $R(p)$ as a left or right R-module.

Suppose that $f \in R(p)$ has support contained in Gr_A for some downwardly closed subset $A \subset p$ and choose $x \in A$ maximal. As $f_{tx} = 0$ for all $t \in T$ with $x < tx \in p$, from the definition of $R(p)$ we see that f_x is divisible by

$$\alpha_x = \prod_{\substack{t \in T \\ x < tx \in p}} h_t.$$

As $\deg \alpha_x = 2|\{t \in T \mid x < tx \in p\}| = 2(\ell(p_+) - \ell(x))$ by Proposition 1.2.16 we see that $(\phi_x)_x$ is a non-zero scalar multiple of α_x. Hence, we may find $r \in R$ such that

$$\operatorname{supp}(f - r\phi_x) \subset \mathrm{Gr}_{A \setminus \{x\}}.$$

It follows by induction that the $\{\phi_x\}$ span $R(p)$ as a left R-module. They are clearly linearly independent when we consider $R(p)$ as a left R-module by the support conditions. Hence they form a basis for $R(p)$ as a left R-module. Identical arguments show that they are also a basis for $R(p)$ as a right R-module.

We can also use the above facts to see that ϕ_x for $x \in p$ is unique up to a scalar. Indeed, if ϕ_x and ϕ'_x are two candiates we may find $\lambda \in k$ such that $\phi_x - \lambda\phi'_x$ is supported on $\mathrm{Gr}_{\leq x \setminus \{x\}}$. By the above arguments $\phi_x - \lambda\phi'_x$ has degree strictly greater than $2(\ell(p_+) - \ell(x))$ and hence is zero.

It remains to show existence. To get started consider $\vartheta = (\vartheta_x) \in \oplus_{x \in p} R$ defined by

$$\vartheta_x = \begin{cases} \alpha_{p_-} & \text{if } x = p_- \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\vartheta \in R(p)$ and $\deg \vartheta = 2(\ell(p_+ - \ell(p_-))$ (again by Proposition 1.2.16). Hence we may set $\phi_{p_-} = \vartheta$.

Now assume by induction that we have found ϕ_x for all $x \in p$ with $\ell(x) < m$ and choose $y \in p$ of length m. By Howlett's theorem (1.2.9) there exists a simple reflection $s \in W_I$ or $t \in W_J$ such that either $y > sy \in p$ or $y > yt \in p$. In the first case consider $\vartheta = \partial_s \phi_{sy} \in R(p)$. We have

(1) $\deg \vartheta = \deg \phi_{sy} - 2 = 2(\ell(p_+) - \ell(y))$,
(2) $\operatorname{supp} \vartheta \subset \mathrm{Gr}_{\leq y}$ (by Lemma 3.3.2) and $\vartheta_y \neq 0$ because $(\phi_{sy})_{sy} \neq 0$.

Hence we may set $\phi_y = \vartheta$. Similarly in the second case we may take $\phi_y = \phi_{yt} \partial_t$. It follows by induction that the elements $\{\phi_w \mid w \in p\}$ exist. $\square$

The first corollary of this theorem is a description of $R(p)$ as a left R-module, needed during the proof of Theorem 2.4.1.

Corollary 3.3.4. *As left graded R-modules we have an isomorphism*

$$R(p) \cong \widetilde{\pi}(p) \cdot R.$$

Proof. If $P = \sum_{x \in p} v^{2(\ell(x) - \ell(p_+))}$ it follows from the theorem that

$$R(p) \cong P \cdot R \text{ in } R\text{-Mod}.$$

However

$$P = \overline{v^{2\ell(p_+)} \sum_{x \in p} v^{-2\ell(x)}} = \overline{v^{\ell(p_+) - \ell(p_-)} \pi(p)} = v^{\ell(p_-) - \ell(p_+)} \pi(p) = \widetilde{\pi}(p)$$

using the self-duality of $\pi(p)$ (1.2.4) for the third step. $\square$

Corollary 3.3.5. *Let $K \subset I$, $L \subset J$ and $C \subset W_K \backslash W / W_L$ be downwardly closed. For all $q \in C$ maximal such that $q \subset p$ we have an isomorphism in R^K-Mod-R^L:*

$$\Gamma_C R(p)^{W_K \times W_L} / \Gamma_{C \setminus \{q\}} R(p)^{W_K \times W_L} \cong {}^K R_q^L [2(\ell(q_+) - \ell(p_+))].$$

Proof. For the course of the proof let us write ϕ_w^p (resp. ϕ_y^q) for the functions in $R(p)$ (resp. $R(q)$) given to us by Theorem 3.3.3. These are well defined up to a scalar and we make a fixed but arbitrary choice. Also denote by $\mathrm{qu} : W \to W_K \backslash W / W_L$ the quotient map.

The map $(f_x)_{x \in p} \mapsto (f_x)_{x \in q}$ from $R(p)$ to $R(q)$, in which we forget f_x for $x \notin q$, allows us to identify $\Gamma_{\mathrm{qu}^{-1}(C)} R(p) / \Gamma_{\mathrm{qu}^{-1}(C \setminus \{q\})} R(p)$ with an ideal in $R(q)$. Keeping this in mind we obtain a map

$$R(q)[2(\ell(q_+)-\ell(p_+))] \to \Gamma_{\mathrm{qu}^{-1}(C)}R(p)/\Gamma_{\mathrm{qu}^{-1}(C\setminus\{q\})}R(p)$$
$$1 \mapsto \phi^p_{q_+}.$$

As $\partial_s\phi^p_{q_+} = \phi^p_{q_+}\partial_t = 0$ for all $s \in K$ and $t \in L$, $\phi^p_{q_+}$ is $W_K \times W_L$-invariant. Thus $(\phi^p_{q_+})_x \neq 0$ for all $x \in q$, and the above map is injective.

Let us consider the image of $\phi^q_x \in R(p)$ for $x \in q$ in the right hand side. It has degree

$$\deg \phi^q_x + \deg \phi^p_{q_+} = 2(\ell(p_+) - \ell(x))$$

and has support contained in $\mathrm{Gr}_{\leq x}$. Hence, by the uniqueness statement in Theorem 3.3.3, it is a non-zero scalar multiple of (the image of) ϕ^p_x. It is a consequence of Theorem 3.3.3 that ϕ^p_x for $x \in q$ build a basis for the right hand side as a left R-module, and we conclude that the map is an isomorphism.

The $W_K \times W_L$ action on $R(p)$ preserves both $\Gamma_{\mathrm{qu}^{-1}(C)}R(p)$ and $\Gamma_{\mathrm{qu}^{-1}(C\setminus\{q\})}R(p)$ and hence we have a $W_K\times W_L$-action on both modules. As $W_K \times W_L$ acts through k-algebra automorphisms the above map commutes with the $W_K \times W_L$-action on both modules. Taking $W_K \times W_L$-invariants (which is exact as we are in characteristic 0) and using Lemma 2.4.2, we follow that

$${}^KR^L_p[2(\ell(q_+)-\ell(p_+))] \cong (\Gamma_{\pi^{-1}(C)}R(p))^{W_K\times W_L}/(\Gamma_{\pi^{-1}(C\setminus\{q\})}R(p))^{W_K\times W_L}.$$

However, by (2.5.4), $(\Gamma_{\mathrm{qu}^{-1}(C)}R(p))^{W_K\times W_L} = \Gamma_C(R(p)^{W_K\times W_L})$ and similarly for $\Gamma_{\mathrm{qu}^{-1}(C\setminus\{q\})}R(p)$. The claimed isomorphism then follows. $\square$

In the sequel it will be useful to have the above corollary in a slightly different form.

Corollary 3.3.6. *Let $J \supset K$ and $C \subset W_I\backslash W/W_K$ be downwardly closed. If $q \in C$ is maximal and $p \supset q$ then we have an isomorphism in R^I-Mod-R^K*

$$\Gamma_C({}^IR^J_p \otimes_{R^J} R^K)/\Gamma_{C\setminus\{q\}}({}^IR^J_p \otimes_{R^J} R^K) \cong {}^IR^K_q[2(\ell(q_+)-\ell(p_+))].$$

Proof. By Theorem 2.4.1 we have an isomorphism

$${}^IR^J_p \otimes_{R^J} R^K \cong R(p)^{W_I\times W_K}$$

in R^I-Mod-R^K. The claim is then an immediate consequence of Corollary 3.3.5. $\square$

4. Flags, characters and translation

In this section we define and study the categories of objects with nabla and delta flags. These categories provide the first step in the categorication of the Hecke category.

Recall from the introduction to this chapter than to any $M \in R^I$-Mod-R^J one may associate two filtrations, and that M has a nabla (resp. delta) flag if these filtrations are exhaustive and the successive quotients in the first (resp. second) filtration are isomorphic to a finite direct sum of shifts of standard modules. Given an object with a nabla or delta flag it is natural to consider its "character" in ${}^I\mathcal{H}^J$, which counts the graded multiplicity of standard modules these subquotients.

The key results of this section are Theorems 4.1.5 and 4.3.3, which tell us that if $J \subset K$ then the functors of restriction and extension of scalars between R^I-Mod-R^J and R^I-Mod-R^K restrict to functors between the corresponding categories of objects with nabla or delta flags. Moreover, after normalisation, one may describe the effect of these functors on the characters in terms of multiplication in the Hecke category.

The structure of this section is as follows. In Section 4.1 we define the subcategory of modules with nabla flags and the nabla character, and begin the proof of Theorem 4.1.5. The proof involves certain technical splitting and vanishing statements, which we postpone to Section 4.2. In Section 4.3 we define the subcategory of modules with delta flags and the delta character, as well as a duality which is used to relate the categories of object with delta and nabla flags and prove Theorem 4.3.3.

4.1. Objects with nabla flags and translation. For the duration of this section fix finitary subsets $I, J \subset S$. Denote by ${}^I\mathcal{R}^J$ the full subcategory of modules $M \in R^I$-Mod-R^J such that:

(1) M is finitely generated, both as a left R^I-module, and as a right R^J-module;
(2) there exists a finite subset $C \subset W_I\backslash W/W_J$ such that $\operatorname{supp} M \subset {}^I\mathrm{Gr}^J_C$.

Recall that we call a subset $C \subset W_I\backslash W/W_J$ *downwardly closed* if

$$C = \{p \in W_I\backslash W/W_J \mid p \leq q \text{ for some } q \in C\}.$$

We now come to the definition of objects with nabla flags.

Definition 4.1.1. *The category of* objects with nabla flags*, denoted ${}^I\mathcal{F}^J_\nabla$, is the full subcategory of modules $M \in {}^I\mathcal{R}^J$ such that, for all downwardly closed subsets $C \subset W_I\backslash W/W_J$ and maximal elements $p \in C$, the subquotient*

$$\Gamma_C M/\Gamma_{C\backslash\{p\}}M$$

is isomorphic to a direct sum of shifts of modules of the form ${}^IR^J_p$ (which is necessarily finite because $M \in {}^I\mathcal{R}^J$).

We begin with a lemma that simplifies the task of checking whether a module $M \in {}^I\mathcal{R}^J$ belongs to ${}^I\mathcal{F}^J_\nabla$. We call an enumeration $p_1, p_2, \ldots$ of the elements of $W_I\backslash W/W_J$ a *refinement of the Bruhat order* if $p_i \leq p_j$ implies that $i \leq j$. If we let $C(m) = \{p_1, p_2, \ldots, p_m\}$ then all the sets $C(m)$ are downwardly closed, and $p_m \in C(m)$ is maximal. Hence, if

$M \in {}^I\mathcal{F}^J_\nabla$ then $\Gamma_{C(m)}M/\Gamma_{C(m-1)}M$ is isomorphic to a direct sum of shifts of ${}^IR^J_{p_m}$. In fact, the converse is true:

Lemma 4.1.2 (Soergel's "hin-und-her" lemma). *Let $p_1, p_2, \dots$ and $C(m)$ be as above. Suppose $M \in {}^I\mathcal{R}^J$ is such that, for all m, the subquotient*

$$\Gamma_{C(m)}M/\Gamma_{C(m-1)}M$$

is isomorphic to a direct sum of shifts of ${}^IR^J_{p_m}$. Then $M \in {}^I\mathcal{F}^J_\nabla$.

Moreover, if $p = p_m$ then the natural map

$$\Gamma_{\le p}M/\Gamma_{<p}M \to \Gamma_{C(m)}M/\Gamma_{C(m-1)}M$$

is an isomorphism.

Proof. Let $C \subset W_I\backslash W/W_J$ be a downwardly closed subset and $p \in C$ be maximal. We need to show that

$$\Gamma_C M/\Gamma_{C\setminus\{p\}}M$$

is isomorphic to a direct sum of shifts of modules of the form ${}^IR^J_p$.

Let $p, p' \in W_I\backslash W/W_J$ be incomparable in the Bruhat order. We will see in the next section (Lemma 4.2.2) that $\mathrm{Ext}^1_{R^I\otimes R^J}({}^IR^J_p, {}^IR^J_{p'}) = 0$. In particular, if p_i and p_{i+1} are incomparable in the Bruhat order then $\Gamma_{C(i+1)}M/\Gamma_{C(i-1)}M$ is isomorphic to a direct sum of shifts of modules ${}^IR^J_{p_i}$ and ${}^IR^J_{p_{i+1}}$. Hence, if we let C' be associated to the sequence obtained by swapping two elements q_i and q_{i+1} we see that the natural maps

$$\begin{array}{rcl} \Gamma_{C(i)}M/\Gamma_{C(i-1)}M & \to & \Gamma_{C'(i+1)}M/\Gamma_{C'(i)}M \\ \Gamma_{C'(i)}M/\Gamma_{C'(i-1)}M & \to & \Gamma_{C(i+1)}M/\Gamma_{C(i)}M \end{array}$$

are isomorphisms.

Now let $C \subset W_I\backslash W/W_J$ be downwardly closed and $p \in C$ maximal. After swapping finitely many many elements of our sequence we may assume $C(m) = C$ and $p_m = p$ and the first statement follows. The second statement follows by taking $C = \{\le p\}$. $\square$

We now want to define the "character" of an object $M \in {}^I\mathcal{F}^J_\nabla$. It is natural to renormalise ${}^IR^J_p$ and define

$${}^I\nabla^J_p = {}^IR^J_p[\ell(p_+)].$$

If p contains the identity, we sometimes omit p and write ${}^I\nabla^J$.

By assumption, if $M \in {}^I\mathcal{F}^J_\nabla$ we may find polynomials $g_p(M) \in \mathbb{N}[v, v^{-1}]$ such that, for all $p \in W_I\backslash W/W_J$ we have

$$\Gamma_{\le p}M/\Gamma_{<p}M \cong g_p(M)\cdot {}^I\nabla^J_p.$$

We now define the *nabla character* by

$$\begin{array}{rcl} \mathrm{ch}_\nabla : {}^I\mathcal{F}^J_\nabla & \to & {}^I\mathcal{H}^J \\ M & \mapsto & \displaystyle\sum_{p\in W_I\backslash W/W_J} \overline{g_p(M)}\, {}^IH^J_p. \end{array}$$

We now come to the definition of translation functors, which (up to a shift) are the functors of extension and restriction of scalars. Note that, if $K \subset L$ are finitary subsets of S we have an inclusion $R^L \subset R^K$.

Definition 4.1.3. *Let $K \subset S$ be finitary.*

(1) If $J \subset K$ the functor of "translating onto the wall" is:

$$\begin{array}{rcl} - \cdot {}^J\vartheta^K : R^I\text{-}Mod\text{-}R^J & \to & R^I\text{-}Mod\text{-}R^K \\ M & \mapsto & M_{R^K}[\ell(w_K) - \ell(w_J)]. \end{array}$$

(2) If $J \supset K$ the functor of "translating out of the wall" is:

$$\begin{array}{rcl} - \cdot {}^J\vartheta^K : R^I\text{-}Mod\text{-}R^J & \to & R^I\text{-}Mod\text{-}R^K \\ M & \mapsto & M \otimes_{R^J} R^K. \end{array}$$

Remark 4.1.4. *Of course it is also possible to define translation functors "on the left". We have chosen to only define and work with translation functors acting on one side because it simplifies the exposition considerably.*

The following theorem is fundamental to all that follows. It shows that translation functors preserve the categories of objects with nabla flags and that we may describe the effect of translation functors on characters.

Theorem 4.1.5. *Let $K \subset S$ be finitary with $J \subset K$ or $K \subset J$.*

(1) If $M \in {}^I\mathcal{F}^J_\nabla$ then $M \cdot {}^J\vartheta^K \in {}^I\mathcal{F}^K_\nabla$.

(2) The following diagrams commute:

$$\begin{tikzcd} {}^I\mathcal{F}^J_\nabla \arrow[r, "- \cdot {}^J\vartheta^K"] \arrow[d, "\mathrm{ch}_\nabla"] & {}^I\mathcal{F}^K_\nabla \arrow[d, "\mathrm{ch}_\nabla"] \\ {}^I\mathcal{H}^J \arrow[r, "- *_J {}^JH^K"] & {}^I\mathcal{H}^K \end{tikzcd} \qquad \begin{tikzcd} {}^I\mathcal{F}^J_\nabla \arrow[r, "{[1]}"] \arrow[d, "\mathrm{ch}_\nabla"] & {}^I\mathcal{F}^J_\nabla \arrow[d, "\mathrm{ch}_\nabla"] \\ {}^I\mathcal{H}^J \arrow[r, "\cdot v^{-1}"] & {}^I\mathcal{H}^J \end{tikzcd}$$

Before we can prove this we will need a preparatory result.

Proposition 4.1.6. *Let $J \subset K$ be finitary and*

$$\mathrm{qu} : W_I \backslash W / W_J \to W_I \backslash W / W_K$$

be the quotient map. Let $C \subset W_I \backslash W / W_K$ be downwardly closed.

(1) If $M \in {}^I\mathcal{F}^J_\nabla$ then

$$(\Gamma_{\mathrm{qu}^{-1}(C)} M)_{R^K} = \Gamma_C(M_{R^K}).$$

(2) If $M \in {}^I\mathcal{F}^K_\nabla$ then

$$(\Gamma_C M) \otimes_{R^K} R^J = \Gamma_{\mathrm{qu}^{-1}(C)}(M \otimes_{R^K} R^J).$$

Proof. (1) is a direct consequence of (2.5.4). For (2) consider the exact sequence

$$\Gamma_C M \hookrightarrow M \twoheadrightarrow M / \Gamma_C M.$$

Because $M \in {}^I\mathcal{F}^J_\nabla$ the left (resp. right) module has a filtration with subquotients isomorphic to a direct sum of shifts of ${}^IR^J_p$ with $p \in C$ (resp. $p \notin C$). Applying the exact functor $- \otimes_{R^K} R^J$ we obtain an exact sequence

$$\Gamma_C M \otimes_{R^K} R^J \hookrightarrow M \otimes_{R^K} R^J \twoheadrightarrow M/\Gamma_C M \otimes_{R^K} R^J.$$

By exactness, the left (resp. right) modules have a filtration with subquotients a direct sum of shifts of ${}^IR^J_p \otimes_{R^K} R^J$ with $p \in C$ (resp. $p \notin C$). By Corollary 3.3.6, ${}^IR^J_p \otimes_{R^K} R^J$ has a filtration with subquotients isomorphic to (a shift of) ${}^IR^J_q$ with $q \in \mathrm{qu}^{-1}(p)$. Moreover the support of any non-zero element in ${}^IR^J_q$ is precisely ${}^I\mathrm{Gr}^J_q$ (Lemma 2.5.3). Thus the above exact sequence is equal to

$$\Gamma_{\mathrm{qu}^{-1}(C)}(M \otimes_{R^K} R^J) \hookrightarrow M \otimes_{R^K} R^J \twoheadrightarrow M/\Gamma_{\mathrm{qu}^{-1}(C)}(M \otimes_{R^K} R^J)$$

which implies the proposition. □

We can now prove the Theorem 4.1.5.

Proof of Theorem 4.1.5. It is easy to see that $M \cdot {}^J\vartheta^K \in {}^I\mathcal{R}^K$ using Lemma 2.5.1 and the fact that R^J is finite over R^K in the case that $J \supset K$. We split the proof into two cases.

Case 1: Translating out of the wall ($J \supset K$): We first prove part (1) of the theorem. Let

$$\mathrm{qu} : W_I\backslash W/W_K \to W_I\backslash W/W_J$$

be the quotient map. Because qu is a surjective morphism of posets we may choose an enumeration $p_1, p_2, \dots$ of the elements of $W_I\backslash W/W_K$ refining the Bruhat order such that, after deleting repetitions, $\mathrm{qu}(p_1)$, $\mathrm{qu}(p_2)$, $\dots$ is a listing of the elements of $W_I\backslash W/W_J$ refining the Bruhat order. Fix $q \in W_I\backslash W/W_J$ and $p = p_m \in \mathrm{qu}^{-1}(q)$ and define

$$C(n) = \{p_1, p_2, \dots, p_n\}.$$

By the hin-und-her lemma (4.1.2) it is enough to show that

$$\Gamma_{C(m)}(M \otimes_{R^J} R^K)/\Gamma_{C(m-1)}(M \otimes_{R^J} R^K)$$

is isomorphic to a direct sum of shifts of ${}^IR^K_p$.

The set $F = \mathrm{qu}(C(m))$ is downwardly closed and contains q as a maximal element. As $M \in {}^I\mathcal{F}^J_\nabla$ there exists an exact sequence

$$\Gamma_{F\setminus\{q\}} M \hookrightarrow \Gamma_F M \twoheadrightarrow P \cdot {}^IR^J_q$$

for some $P \in \mathbb{N}[v, v^{-1}]$. Applying $- \otimes_{R^J} R^K$ and using Proposition 4.1.6 we conclude an exact sequence

$$\Gamma_{\mathrm{qu}^{-1}(F\setminus\{q\})}(M \otimes_{R^J} R^K) \hookrightarrow \Gamma_{\mathrm{qu}^{-1}(F)}(M \otimes_{R^J} R^K) \twoheadrightarrow P \cdot {}^IR^J_q \otimes_{R^J} R^K$$

As $\Gamma_{\mathrm{qu}^{-1}(F\setminus\{q\})}(M \otimes_{R^J} R^K)$ is contained in both $\Gamma_{C(m)}(M \otimes_{R^J} R^K)$ and $\Gamma_{C(m-1)}(M \otimes_{R^J} R^K)$ by the third isomorphism theorem we will be finished if we can show that

$$\Gamma_{C(m)}({}^I R_q^J \otimes_{R^J} R^K)/\Gamma_{C(m-1)}({}^I R_q^J \otimes_{R^J} R^K)$$

is isomorphic to a direct sum of shifts of ${}^I R_p^J$. But this is precisely the statement of Corollary 3.3.6. Hence $M \cdot {}^J\vartheta^K \in {}^I\mathcal{F}_\nabla^K$.

We now prove (2). The commutativity of the right hand diagram is clear. As $- \cdot {}^J\vartheta^K$ is exact and every element in ${}^I\mathcal{F}_\nabla^J$ is an extension of the nabla modules we only have to check the commutativity of the left hand diagram for a nabla module. That is, we have to verify that

$$\mathrm{ch}_\nabla({}^I\nabla_q^J) *_J {}^J H^K = \mathrm{ch}_\nabla({}^I\nabla_q^J \cdot {}^J\vartheta^K).$$

By Proposition 2.2.4 the left hand side is equal to

$${}^I H_q^J *_J {}^J H^K = \sum_{p \in W_I \backslash q / W_J} v^{\ell(q_+) - \ell(p_+)}\, {}^I H_p^K.$$

For the right hand side note that:

$$\begin{aligned}
\Gamma_{\leq p}({}^I\nabla_q^J \otimes_{R^J} R^K)/\Gamma_{<p}({}^I\nabla_q^J \otimes_{R^J} R^K) &\cong \\
&\cong \Gamma_{\leq p}({}^I R_q^J \otimes_{R^J} R^K)/\Gamma_{<p}({}^I R_q^J \otimes_{R^J} R^K)[\ell(q_+)] \\
&\cong {}^I R_p^K[2\ell(p_+) - \ell(q_+)] && \text{(Corollary 3.3.6)} \\
&\cong v^{\ell(p_+) - \ell(q_+)} \cdot {}^I\nabla_p^K
\end{aligned}$$

Therefore, by definition of ch_∇,

$$\mathrm{ch}_\nabla({}^I\nabla_q^J \cdot {}^J\vartheta^K) = \sum_{p \in W_I \backslash q / W_J} v^{\ell(q_+) - \ell(p_+)}\, {}^I H_p^K.$$

This completes the proof in case $J \supset K$.

Case 2: Translating onto the wall ($J \subset K$): Denote (as usual) by qu the quotient map

$$\mathrm{qu} : W_I \backslash W / W_J \to W_I \backslash W / W_K.$$

Let $C \subset W_I \backslash W / W_K$ be downwardly closed and choose $q \in C$ maximal. Consider the exact sequence

$$\Gamma_{\mathrm{qu}^{-1}(C\setminus\{q\})} M \hookrightarrow \Gamma_{\mathrm{qu}^{-1}(C)} M \twoheadrightarrow \Gamma_{\mathrm{qu}^{-1}(C)} M / \Gamma_{\mathrm{qu}^{-1}(C\setminus\{q\})} M.$$

As $M \in {}^I\mathcal{F}_\nabla^J$ the right-hand module has a filtration with subquotients isomorphic to direct sums of shifts ${}^I R_p^J$ with $p \in \mathrm{qu}^{-1}(q)$. In Proposition 4.2.5 in the next subsection we will see that any such module splits as a direct sum of shifts of ${}^I R_q^K$ upon restriction to R^K. This implies that $M_{R^K} \in {}^I\mathcal{F}_\nabla^K$ because, by Proposition 4.1.6, the restriction to R^I-Mod-R^K of the above exact sequence is identical to

$$\Gamma_{C\setminus\{q\}}(M_{R^K}) \hookrightarrow \Gamma_C(M_{R^K}) \twoheadrightarrow \Gamma_C(M_{R^K})/\Gamma_{C\setminus\{q\}}(M_{R^K}).$$

We now turn our attention to (2). As above, it is enough to check the commutativity of the left-hand diagram for a nabla module. Let $p \in W_I \backslash W / W_J$ and $q = \mathrm{qu}(p)$. We need to check that

$$\mathrm{ch}_\nabla({}^I\nabla_p^J) *_J {}^JH^K = v^{\ell(q_-)-\ell(p_-)} \frac{\pi(I,q,K)}{\pi(I,p,J)} \, {}^IH_q^K = \mathrm{ch}_\nabla({}^I\nabla_p^J \cdot {}^J\vartheta^K)$$

where the first equality follows from Proposition 2.2.4. By definition of ch_∇ this follows from the isomorphism

$${}^I\nabla_p^J \cdot {}^J\vartheta^K \cong v^{\ell(p_-)-\ell(q_-)} \frac{\pi(I,q,K)}{\pi(I,p,J)} \cdot {}^I\nabla_q^J$$

which we prove in Lemma 4.1.7 below. □

LEMMA 4.1.7. *Let $J \subset K$, $p \in W_I \backslash W / W_J$ and $q = W_I p W_K$. We have an isomorphism*

$${}^I\nabla_p^J \cdot {}^J\vartheta^K \cong v^{\ell(p_-)-\ell(q_-)} \frac{\pi(I,q,K)}{\pi(I,p,J)} \cdot {}^I\nabla_q^K.$$

PROOF. By Lemma 2.2.3 we have

$$\begin{aligned}
({}^I\nabla_p^J) \cdot {}^J\vartheta^K &\cong ({}^IR_p^J)_{R^K}[\ell(p_+) + \ell(w_K) - \ell(w_J)] \\
&\cong \frac{\widetilde{\pi}(I,q,K)}{\widetilde{\pi}(I,p,J)} \cdot {}^IR_q^J[\ell(p_+) + \ell(w_K) - \ell(w_J)] \\
&\cong v^a \frac{\pi(I,q,K)}{\pi(I,p,J)} \cdot {}^I\nabla_q^J
\end{aligned}$$

where

$$\begin{aligned}
a &= \ell(w_{I,p,J}) - \ell(w_{I,q,K}) + \ell(p_+) - \ell(q_+) + \ell(w_K) - \ell(w_J) \\
&= (\ell(p_+) - \ell(w_I) - \ell(w_J) + \ell(w_{I,p,J})) - \\
&\qquad (\ell(q_+) - \ell(w_I) - \ell(w_K) + \ell(w_{I,q,K})) \\
&= \ell(p_-) - \ell(q_-)
\end{aligned}$$

by Corollary 1.2.11. □

4.2. Vanishing and splitting. This is a technical section in which we prove two vanishing statements which were postponed in the last section.

Let us begin with some generalities. Let A be a ring. An extension between two A-modules

$$M \to E \to N$$

gives an element of $\mathrm{Ext}^1_A(N, M)$ by considering the long exact sequence associated to $\mathrm{Hom}(-, M)$ and looking at the image of id_M in $\mathrm{Ext}^1(N, M)$; the sequence splits if and only if this class is zero.

Now let $A' \to A$ be a homomorphism of rings. If M and N are A-modules one has maps

$$r_m : \mathrm{Ext}^m_A(N, M) \to \mathrm{Ext}^m_{A'}(N, M).$$

We will need the following facts:

(1) An extension between M and N splits upon restriction to A' if and only its class lies in the kernel of the map

$$r_1 : \mathrm{Ext}^1_A(N, M) \to \mathrm{Ext}^1_{A'}(N, M).$$

(2) A short exact sequence $M' \hookrightarrow M \twoheadrightarrow M''$ yields a commutative diagram of long exact sequences:

$$\begin{tikzcd} \arrow[r] & \mathrm{Ext}^1_A(M'', N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(M, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(M', N) \arrow[r] \arrow[d] & {} \\ \arrow[r] & \mathrm{Ext}^1_{A'}(M'', N) \arrow[r] & \mathrm{Ext}^1_{A'}(M, N) \arrow[r] & \mathrm{Ext}^1_{A'}(M', N) \arrow[r] & {} \end{tikzcd} \tag{4.2.1}$$

(3) Similarly, if $N' \hookrightarrow N \twoheadrightarrow N''$ is a short exact sequence, we obtain a commutative diagram of long exact sequences:

$$\begin{tikzcd} \arrow[r] & \mathrm{Ext}^1_A(M, N') \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(M, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(M, N'') \arrow[r] \arrow[d] & {} \\ \arrow[r] & \mathrm{Ext}^1_{A'}(M, N') \arrow[r] & \mathrm{Ext}^1_{A'}(M, N) \arrow[r] & \mathrm{Ext}^1_{A'}(M, N'') \arrow[r] & {} \end{tikzcd} \tag{4.2.2}$$

These facts become transparent when reinterpreted in the derived category (see e.g. [**Wie**]).

Given a vector space W, denote by $\mathcal{O}(W)$ its graded ring of regular functions.

LEMMA 4.2.1. *(Lemma 5.8 in* [**So6**]*) Let W be a finite dimensional vector space and $U, V \subset W$ two linear subspaces. Then*

$$\mathrm{Ext}^1_{\mathcal{O}(W)}(\mathcal{O}(U), \mathcal{O}(V))$$

is only non-trivial if $V \cap U$ is V or a hyperplane in V. In the later case it is generated by the class of any short exact sequence of the form

$$\mathcal{O}(V)[-2] \overset{\alpha\cdot}{\hookrightarrow} \mathcal{O}(V \cup U) \twoheadrightarrow \mathcal{O}(U)$$

with $\alpha \in W^*$ a linear form satisfying $\alpha|_U = 0$ and $\alpha|_V \neq 0$.

We now turn to our situation, with the goal of analysing extensions between standard modules. Notationally it proves more convenient to work with left modules, which we may do using the equivalences A_1-Mod-$A_2 \cong A_1 \otimes A_2$-Mod as all our rings are assumed commutative. We will do this for the rest of ths subsection without further comment.

Using the identification of R_x with $\mathcal{O}(\mathrm{Gr}_x)$ and Lemma 4.2.1 we see that $\mathrm{Ext}^1_{R\otimes R}(R_x, R_y)$ is non-zero only when Gr_x and Gr_y intersect in codimension 1. As

$$\mathrm{Gr}_x \cap \mathrm{Gr}_y \cong V^{x^{-1}y}$$

and the representation of W on V is reflection faithful, this occurs only when $y = xt$ for some reflection $t \in T$. We conclude that there are no extensions between R_x and R_y unless $x \neq yt$ for some reflection $t \in W$.

Now let $p, p' \in W_I \backslash W / W_J$ and suppose we have an extension of the form

$$ {}^I R^J_p \hookrightarrow E \twoheadrightarrow {}^I R^J_{p'}. $$

we may extend scalars to obtain an exact sequence

$$ R \otimes_{R^I} {}^I R^J_p \otimes_{R^J} R \hookrightarrow \tilde{E} \twoheadrightarrow R \otimes_{R^I} {}^I R^J_{p'} \otimes_{R^J} R. $$

If we again restrict to $R^I \otimes R^J$ we obtain a number of copies of our original extension. By Theorem 2.4.1 we have an isomorphism

$$ R \otimes_{R^I} {}^I R^J_p \otimes_{R^J} R \cong R(p). $$

Therefore our extension takes the form

$$ R(p) \hookrightarrow \tilde{E} \twoheadrightarrow R(p'). $$

LEMMA 4.2.2. *Suppose that $p, p' \in W_I \backslash W / W_J$ are not comparable in the Bruhat order. Then*

$$ \mathrm{Ext}^1_{R^I \otimes R^J}({}^I R^J_p, {}^I R^J_{p'}) = 0. $$

PROOF. By the above discussion it is enough to show that there are no extensions between $R(p)$ and $R(p')$. As p and p' are incomparable, there are no pairs $x \in p$ and $x' \in p'$ with $x' = xt$ for some $t \in T$. Thus (again by the above discussion), $\mathrm{Ext}^1_{R \otimes R}(R_x, R_{x'})$ for all $x \in p$, $x' \in p'$. By Corollary 3.3.5, $R(p)$ (resp. $R(p')$) has a filtration with successive subquotients R_x for $x \in p$ (resp. $x \in p'$). By induction and the long exact sequence of Ext it follows first that $\mathrm{Ext}^1_{R \otimes R}(R(p), R_{x'}) = 0$ for all $x' \in p'$, and then that $\mathrm{Ext}^1_{R \otimes R}(R(p), R(p')) = 0$. $\square$

Our goal for the rest of this section is to prove Proposition 4.2.5 below. We start with two preparatory lemmas.

LEMMA 4.2.3. *If $x \in W$ and $t \in T$ then the map*

$$ r_1 : \mathrm{Ext}^1_{R \otimes R}(R_x, R_{xt}) \to \mathrm{Ext}^1_{R \otimes R^t}(R_x, R_{xt}) $$

induced by the inclusion $R \otimes R^t \hookrightarrow R \otimes R$ is zero.

PROOF. Given $c \in R \otimes R$ of degree 2, vanishing on Gr_x but not on Gr_{xt} we obtain an extension

$$ R_{xt}[-2] \overset{\cdot c}{\hookrightarrow} R_{x,xt} \twoheadrightarrow R_x. \tag{4.2.3} $$

By Lemma 4.2.1, it is enough to show that (4.2.3) splits upon restriction to $R \otimes R^t$. Consider the map $R_{x,xt} \to R_{xt}[-2]$ sending f to the image of $f\partial_t$, where ∂_t is the (right) Demazure operator introduced in Section 3.3. This is a morphism of $R \otimes R^t$-modules. As c vanishes on Gr_x but not on Gr_{xt}, $c\partial_t$ is non-zero, hence is a non-zero scalar for degree reasons. Thus a suitable scalar multiple of this map provides a splitting of (4.2.3) over $R \otimes R^t$. $\square$

LEMMA 4.2.4. *Let $I, J \subset K$ be finitary subsets of S and $p, p' \in W_I \backslash W / W_J$ be such that $p \neq p'$ but $W_I p W_K = W_I p' W_K$. Then every extension between ${}^I R^J_p$ and ${}^I R^J_{p'}$ splits upon restriction restriction to $R^I \otimes R^K$.*

PROOF. Note that by the above discussion it is enough to show that every extension between $R(p)$ and $R(p')$ splits upon restriction to $R^I \otimes R^K$. First note that if $x \in p'$ and $y \in p$ with $x = yt$ for some reflection $t \in W$, then either $t \in W_K$ or $x = t'y$ for some $t' \in W_I$ by Proposition 1.2.12. The second possibility is impossible however, as $p \neq p'$. We conclude, using the previous lemma, that if $x \in p'$ and $y \in p$ then either $\mathrm{Ext}_{R \otimes R}(R_x, R_y) = 0$ or the map $\mathrm{Ext}_{R \otimes R}(R_x, R_y) \to \mathrm{Ext}_{R^I \otimes R^K}(R_x, R_y)$ is zero.

We now proceed similarly to as in the proof of Lemma 4.2.2. Inducting over a filtration on $R(p)$ and using (4.2.2) we conclude that the map

$$\mathrm{Ext}^1_{R \otimes R}(R_x, R(p)) \to \mathrm{Ext}^1_{R^I \otimes R^K}(R_x, R(p))$$

induced by the inclusion $R^I \otimes R^K \hookrightarrow R \otimes R$ is zero for all $x \in p'$. Inducting again using (4.2.1) we see that the map $\mathrm{Ext}^1_{R \otimes R}(R(p'), R(p)) \to \mathrm{Ext}^1_{R^I \otimes R^K}(R(p'), R(p))$ is zero, which establishes the lemma. $\square$

PROPOSITION 4.2.5. *Let $I, J \subset K$ be finitary subsets of S and let $q \in W_I \backslash W / W_K$. Let $B \in {}^I\mathcal{F}^J_\nabla$ and suppose that $\operatorname{supp} B \subset {}^I\mathrm{Gr}^J_C$ for some $C \subset W_I \backslash q / W_J$. Then the restriction $B_{R^K} \in R^I$-Mod-R^K is isomorphic to a direct sum of shifts of standard modules ${}^I R^K_q$.*

PROOF. Choose $p \in C$ maximal in the Bruhat order. As $B \in {}^I\mathcal{F}^J_\nabla$ we have an exact sequence

$$\Gamma_{C \setminus \{p\}} B \hookrightarrow B \twoheadrightarrow P \cdot {}^I R^J_p \tag{4.2.4}$$

for some $P \in \mathbb{N}[v, v^{-1}]$. As $\Gamma_{C \setminus \{p\}} B \in {}^I\mathcal{F}^J_\nabla$ we may induct over a suitable filtration of $\Gamma_{C \setminus \{p\}} B$ and conclude, with the help of Lemma 4.2.4, that (4.2.4) splits upon restriction to $R^I \otimes R^K$.

Now let us choose a listing $p_1, p_2, \ldots p_n$ of the elements of C refining the Bruhat order and let $C(m) = \{p_1, p_2, \ldots, p_m\}$ denote the first m elements as usual. Using downward induction and the above argument it follows that, in R^I-Mod-R^K, we have an isomorphism

$$B_{R^K} \cong \bigoplus (\Gamma_{C(m)} B / \Gamma_{C(m-1)} B)_{R^K}.$$

The proposition then follows as $({}^I R^J_p)_{R^K}$ is isomorphic to a direct sum of shifts of ${}^I R^K_q$ where $q = pW_K$ by Corollary 2.2.3. $\square$

4.3. Delta flags and duality. In this section we define a category of objects with delta flags, ${}^I\mathcal{F}^J_\Delta$, which is "dual" to ${}^I\mathcal{F}^J_\nabla$. Just as in the case of objects with nabla flags the translation functors preserve ${}^I\mathcal{F}^J_\Delta$ and their effect on a "delta character"

$$\mathrm{ch}_\Delta : {}^I\mathcal{F}^J_\Delta \to {}^I\mathcal{H}^J$$

can be described in terms of the Hecke category.

Of course it would be possible to repeat the same arguments as those used for objects with nabla flags. However, one may define a duality

$$D : {}^I\mathcal{F}^J_\nabla \xrightarrow{\sim} {}^I\mathcal{F}^{J\,opp}_\Delta$$

commuting with the translation functors. This allows us to use what we already know about objects with nabla flags to follow similar statements for objects with delta flags.

For the rest of this section fix a pair $I, J \subset S$ of finitary subsets. Recall that we call a subset $U \subset W_I\backslash W/W_J$ *upwardly closed* if

$$U = \{p \in W_I\backslash W/W_J \mid p \geq q \text{ for some } q \in C\}.$$

Definition 4.3.1. *The category of* objects with Δ-flags*, denoted ${}^I\mathcal{F}^J_\Delta$ is the full subcategory of ${}^I\mathcal{R}^J$ whose objects are modules $M \in {}^I\mathcal{R}^J$ such that, for all upwardly closed subsets $U \subset W_I\backslash W/W_J$ and minimal elements $p \in U$, the subquotient*

$$\Gamma_U M/\Gamma_{U\setminus\{p\}}M$$

is isomorphic to a direct sum of shifts of ${}^IR^J_p$.

Just as for objects with nabla flags there is a "hin-und-her" lemma, whose proof is similar to that for objects with nabla flags (and works because the support of $M \in {}^I\mathcal{R}^J$ is always contained in ${}^I\mathrm{Gr}^J_C$ for some finite subset $C \subset W_I\backslash W/W_J$).

Lemma 4.3.2 ("Hin-und-her lemma for delta flags"). *Let $p_1, p_2, \dots$ be an enumeration of the elements of $W_I\backslash W/W_J$ refining the Bruhat order and let $\check{C}(m) = \{p_{m+1}, p_{m+2}, \dots\}$. Then $M \in {}^I\mathcal{R}^J$ is in ${}^I\mathcal{F}^J_\nabla$ if and only if, for all m, the subquotient*

$$\Gamma_{\check{C}(m-1)}M/\Gamma_{\check{C}(m)}M$$

is isomorphic to a direct sum of shifts of ${}^IR^J_{p_m}$.

Moreover, if $M \in {}^I\mathcal{R}^J$ and $p = p_m$ then the natural map

$$\Gamma_{\geq p}M/\Gamma_{>p}M \to \Gamma_{\check{C}(m-1)}M/\Gamma_{\check{C}(m)}M$$

is an isomorphism.

For each $p \in W_I\backslash W/W_J$ we renormalise ${}^IR^J_p$ and define

$${}^I\Delta^J_p = {}^IR^J_p[-\ell(p_-)].$$

If $id \in p$ we sometimes omit p and write ${}^I\Delta^J$ for ${}^I\Delta^J_p$. If $M \in {}^I\mathcal{F}^J_\Delta$ then we may find polynomials $h_p(M) \in \mathbb{N}[v, v^{-1}]$ such that, for all $p \in W_I\backslash W/W_J$, we have an isomorphism

$$\Gamma_{\geq p}M/\Gamma_{>p}M \cong h_p(M) \cdot {}^I\Delta^J_p.$$

We define the *delta character* to be the map

$$\begin{aligned} \mathrm{ch}_\Delta : {}^I\mathcal{F}^J_\Delta &\to {}^I\mathcal{H}^J \\ M &\mapsto \sum_{p \in W_I\backslash W/W_J} v^{\ell(p_-)-\ell(p_+)} h_p(M)\, {}^IH^J_p. \end{aligned}$$

The analogue of Theorem 4.1.5 in this context is the following:

THEOREM 4.3.3. *Let $K \subset S$ with either $J \subset K$ or $J \supset K$.*

(1) If $M \in {}^I\mathcal{F}^J_\Delta$ then $B \cdot {}^J\vartheta^K \in {}^I\mathcal{F}^K_\Delta$.

(2) The following diagrams commute:

$$\begin{tikzcd} {}^I\mathcal{F}^J_\Delta \arrow[r, "-\cdot {}^J\vartheta^K"] \arrow[d, "\mathrm{ch}_\Delta"] & {}^I\mathcal{F}^K_\Delta \arrow[d, "\mathrm{ch}_\Delta"] \\ {}^I\mathcal{H}^J \arrow[r, "*_J\ {}^JH^K"] & {}^I\mathcal{H}^K \end{tikzcd} \qquad \begin{tikzcd} {}^I\mathcal{F}^J_\Delta \arrow[r, "{[1]}"] \arrow[d, "\mathrm{ch}_\Delta"] & {}^I\mathcal{F}^J_\Delta \arrow[d, "\mathrm{ch}_\Delta"] \\ {}^I\mathcal{H}^J \arrow[r, "v\cdot"] & {}^I\mathcal{H}^J \end{tikzcd}$$

We define a *duality* functor

$$\begin{aligned} D : R^I\text{-Mod-}R^J &\to R^I\text{-Mod-}R^J \\ M &\mapsto \mathrm{Hom}_{R^I}(M, R^I[2\ell(w_J)]) \end{aligned}$$

where we make DM into a bimodule using the bimodule structure on M. That is, if $f \in DM$, then

$$(r_1 f r_2)(m) = f(r_1 m r_2) \qquad \text{for all } m \in M.$$

We do not include reference to I and J in the notation for D, and hope this will not lead to confusion. The following proposition shows that the translation functors commute with duality.

PROPOSITION 4.3.4. *Let $K \subset S$ be finitary with either $J \subset K$ or $J \supset K$, and let $M \in R^I$-Mod-R^J. In R^I-Mod-R^K one has*

$$D(M \cdot {}^J\vartheta^K) \cong (DM) \cdot {}^J\vartheta^K.$$

PROOF. If $J \subset K$ then the isomorphism $D(M \cdot {}^J\vartheta^K) \cong (DM) \cdot {}^J\vartheta^K$ is a tautology. So assume that $J \supset K$. We will use standard isomorphisms discussed in Section 1 and switch between left and right modules as appropriate (note that we have already done this once in the definition of D). In R^I-Mod-R^K we have

$$\begin{aligned} D(M \cdot {}^J\vartheta^K) &= \mathrm{Hom}_{R^I}(M \otimes_{R^J} R^K, R^I[2\ell(w_K)]) \\ &\cong \mathrm{Hom}_{R^J}(R^K, \mathrm{Hom}_{R^I}(M, R^I[2\ell(w_K)]) && (1.0.8) \\ &\cong \mathrm{Hom}_{R^I}(M, R^I[2\ell(w_K)]) \otimes_{R^J} \mathrm{Hom}_{R^J}(R^K, R^J) && (1.0.9) \\ &\cong \mathrm{Hom}_{R^I}(M, R^I[2\ell(w_J)]) \otimes_{R^J} R^K && (3.2.3) \\ &= (DM) \cdot {}^J\vartheta^K \end{aligned}$$

□

Theorem 4.3.3 now follows from Theorem 4.1.5 and the following proposition, which also explains the name "duality".

PROPOSITION 4.3.5. *The restriction of D to ${}^I\mathcal{F}^J_\nabla$ defines an equivalence of ${}^I\mathcal{F}^J_\nabla$ with ${}^I\mathcal{F}^{J\,opp}_\Delta$ and we have a commutative diagram:*

$$\begin{tikzcd} {}^I\mathcal{F}^J_\nabla \arrow[rr, "D"] \arrow[dr, "\mathrm{ch}_\nabla"'] & & {}^I\mathcal{F}^{J\,opp}_\Delta \arrow[dl, "\mathrm{ch}_\Delta"] \\ & {}^I\mathcal{H}^J & \end{tikzcd}$$

Before we begin the proof we state a lemma, describing the effect of D on a nabla module.

LEMMA 4.3.6. *If $p \in W_I\backslash W/W_J$ we have*

$$D({}^I\nabla^J_p) \cong {}^I\Delta^J_p[\ell(p_+) - \ell(p_-)].$$

PROOF. Let $K = I \cap p_- J p_-^{-1}$. In R^K-Mod we have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{R^I}(R^K, R^I[2\ell(w_J)]) &\cong R^K[2(\ell(w_I) + \ell(w_J) - \ell(w_K))] && \text{(Cor. 3.2.3)} \\ &\cong R^K[2(\ell(p_+) - \ell(p_-))] && \text{(Cor. 1.2.11).} \end{aligned}$$

As a left module, ${}^IR^J_p$ is equal to R^K where R^I acts via the inclusion $R^I \hookrightarrow R^K$. Hence

$$D({}^IR^J_p) \cong {}^IR^J_p[2(\ell(p_+) - \ell(p_-))]$$

and we have

$$D({}^I\nabla^J_p) \cong D({}^IR^J_p[\ell(p_+)]) \cong {}^IR^J_p[\ell(p_+) - 2\ell(p_-)] \cong {}^I\Delta^J_p[\ell(p_+) - \ell(p_-)]$$

as claimed. $\square$

PROOF OF PROPOSITION 4.3.5. Let $M \in {}^I\mathcal{F}^J_\nabla$. We have to show that $DM \in {}^I\mathcal{F}^J_\Delta$, and that $\mathrm{ch}_\nabla(M) = \mathrm{ch}_\Delta(DM)$. Choose an enumeration $p_1, p_2, \ldots$ of the elements of $W_I\backslash W/W_J$ refining the Bruhat order and let $C(m) = \{p_1, \ldots, p_m\}$ and $\check{C}(m) = \{p_{m+1}, p_{m+2}, \ldots\}$. As $M \in {}^I\mathcal{F}^J_\nabla$ we can find polynomials $g_m \in \mathbb{N}[v, v^{-1}]$ such that, for all m, we have an exact sequence

$$\Gamma_{C(m-1)}M \hookrightarrow \Gamma_{C(m)} \twoheadrightarrow g_m \cdot {}^I\nabla^J_{p_m}.$$

Consider the "cofiltration":

$$\cdots \twoheadrightarrow M/\Gamma_{C(m-1)}M \twoheadrightarrow M/\Gamma_{C(m)}M \twoheadrightarrow \cdots \tag{4.3.1}$$

By the third isomorphism theorem we have an exact sequence

$$g_m \cdot {}^I\nabla^J_{p_m} \hookrightarrow M/\Gamma_{C(m-1)}M \twoheadrightarrow M/\Gamma_{C(m)}M.$$

We know that ${}^I\nabla^J_p$ is graded free as an R^I-module for all p. We conclude, using induction and the above exact sequence that the same is true of every module in (4.3.1). In particular, D is exact when applied to (4.3.1) and we obtain a filtration of DM

$$\cdots \hookleftarrow D(M/\Gamma_{C(m-1)}M) \hookleftarrow D(M/\Gamma_{C(m)}M) \hookleftarrow \cdots \tag{4.3.2}$$

with successive subquotients isomorphic to

$$D(g_m \cdot {}^I\nabla^J_{p_m}) \cong \overline{g_m} \cdot D({}^I\nabla^J_{p_m}) \cong v^{\ell(p_+)-\ell(p_-)}\overline{g_m} \cdot {}^I\Delta^J_{p_m} \tag{4.3.3}$$

(for the second isomorphism we use Lemma 4.3.6 above). It follows that the filtration (4.3.2) is identical to

$$\cdots \hookleftarrow \Gamma_{\check{C}(m-1)}DM \hookleftarrow \Gamma_{\check{C}(m)}DM \hookleftarrow \cdots . \tag{4.3.4}$$

Thus, by the "hin-und-her" lemma we conclude that $M \in {}^I\mathcal{F}^J_\Delta$. Using (4.3.3) and the "hin-und-her" lemma again we see that

$$\mathrm{ch}_\nabla(M) = \sum \overline{g_m}\, {}^IH^J_{p_m} = \mathrm{ch}_\Delta(DM).$$

Lastly, the restriction of D to ${}^I\mathcal{F}^J_\nabla$ gives an equivalence with ${}^I\mathcal{F}^{J\,opp}_\Delta$ because the objects in both categories are free as left R^I-modules. □

5. Singular Soergel bimodules and their classification

In this section we complete the categorication of the Hecke category in terms of Soergel bimodules. After the preliminary work completed in the previous sections, the only remaining difficulty is the classification of the indecomposable objects in ${}^I\mathcal{B}^J$. The key to the classification is provided by Theorem 5.4.1 which explicitly describes the graded dimension of $\mathrm{Hom}(M, N)$ for certain combinations of Soergel bimodules and modules with nabla and delta flags.

In Section 5.1 we define the categories of singular Soergel bimodules, as well as a certain smaller category of bimodules (the "Bott-Samelson bimodules"), for which a description of homomorphisms is straightforward (Theorem 5.2.2). In order to extend this description to all special bimodules we need to consider various localisations of Soergel bimodules, which occupies Section 5.3. In Section 5.4 we then prove the Theorem 5.4.1 and the classification follows easily. In the last section we investigate the characters of indecomposable Soergel bimodules more closely, recall Soergel's conjecture and show that it implies a formula the characters of all indecomposable special bimodules in ${}^I\mathcal{B}^J$ in terms of Kazhdan-Lusztig polynomials.

5.1. Singular Bott-Samelson and Soergel bimodules.

We finally come to the definition of Soergel bimodules.

Definition 5.1.1. *We define the categories of* Bott-Samelson bimodules, *denoted ${}^I\mathcal{B}^J_{BS}$, to be the smallest collection of full additive subcategories of R^I-Mod-R^J for all finitary subsets $I, J \subset S$ satisfying:*

1. *${}^I\mathcal{B}^I_{BS}$ contains ${}^IR^I$ for all finitary subsets $I \subset S$;*
2. *If $B \in {}^I\mathcal{B}^J_{BS}$ then so is $B[\nu]$ for all $\nu \in \mathbb{Z}$;*
3. *If $B \in {}^I\mathcal{B}^J_{BS}$ and $K \subset S$ is finitary, satisfying $J \subset K$ or $J \supset K$, then $B \cdot {}^J\vartheta^K \in {}^I\mathcal{B}^K_{BS}$;*
4. *If $B \in {}^I\mathcal{B}^J_{BS}$ then all objects isomorphic to B are in ${}^I\mathcal{B}^J_{BS}$.*

We define the categories of singular Soergel bimodules*, denoted* ${}^I\mathcal{B}^J$*, to be the smallest collection of additive subcategories of* R^I*-Mod-*R^J *for all finitary* $I, J \subset S$ *satisfying:*

(1) ${}^I\mathcal{B}^J$ *contains all objects of* ${}^I\mathcal{B}^J_{BS}$*;*

(2) ${}^I\mathcal{B}^J$ *is closed under taking direct summands.*

We write $\mathcal{B}_{BS}$ *and* $\mathcal{B}$ *instead of* ${}^\emptyset\mathcal{B}^\emptyset_{BS}$ *and* ${}^\emptyset\mathcal{B}^\emptyset$*.*

The definition of the category of singular Soergel bimodules is more technical than that used in the introduction. However, from condition 3) it is clear that ${}^I\mathcal{B}^J_{BS}$ contains all tensor products

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{n-1}}} R^{I_n}$$

where $I = I_1 \subset J_1 \supset I_2 \subset \cdots \subset J_{n-1} \supset I_n = J$ are all finitary subsets of S. It follows that the definition of ${}^I\mathcal{B}^J$ given above and in the introduction are the same.

By Theorems 4.1.5 and 4.3.3 it follows by induction that any object $M \in {}^I\mathcal{B}^J_{BS}$ lies in ${}^I\mathcal{F}^J_\nabla$ and ${}^I\mathcal{F}^J_\Delta$. As the categories ${}^I\mathcal{F}^J_\nabla$ and ${}^I\mathcal{F}^J_\Delta$ are closed under taking direct summands, the same is true of ${}^I\mathcal{B}^J$.

5.2. Homomorphisms between Bott-Samelson bimodules. In this section we use the fact that translation onto and out of the wall are adjoint (up to a shift) to establish a formula for all homomorphisms between Bott-Samelson bimodules.

We start by proving the adjunction.

Lemma 5.2.1. *Let* $I, J, K \subset S$ *be finitary with either* $J \subset K$ *or* $J \supset K$*. Let* $M \in R^I$*-Mod-*R^J *and* $N \in R^I$*-Mod-*R^K*. We have an isomorphism in* R^I*-Mod:*

$$\mathrm{Hom}(M \cdot {}^J\vartheta^K, M) \cong \mathrm{Hom}(M, N \cdot {}^K\vartheta^J)[\ell(w_K) - \ell(w_J)].$$

Proof. If $J \supset K$ we have isomorphisms of R^I-modules:

$$\begin{aligned} \mathrm{Hom}_{R^I-R^K}(M \cdot {}^J\vartheta^K, N) &\cong \mathrm{Hom}_{R^I-R^J}(M, \mathrm{Hom}_{R^K}(R^K, N)) \quad (1.0.6)\\ &\cong \mathrm{Hom}_{R^I-R^J}(M, N_{R^J})\\ &\cong \mathrm{Hom}_{R^I-R^J}(M, N \cdot {}^J\vartheta^K)[\ell(w_K) - \ell(w_J)] \end{aligned}$$

If $J \subset K$ then, setting $\nu = \ell(w_K) - \ell(w_J)$ we have isomorphisms of R^I-modules:

$$\begin{aligned} &\mathrm{Hom}_{R^I-R^K}(M \cdot {}^J\vartheta^K, N) \cong\\ &\quad\cong \mathrm{Hom}_{R^I-R^K}(M \otimes_{R^J} R^J, N)[-\nu]\\ &\quad\cong \mathrm{Hom}_{R^I-R^J}(M, \mathrm{Hom}_{R^K}(R^J, N))[-\nu] && (1.0.6)\\ &\quad\cong \mathrm{Hom}_{R^I-R^J}(M, N \otimes \mathrm{Hom}_{R^K}(R^J[\nu], R^K)) && (1.0.7)\\ &\quad\cong \mathrm{Hom}_{R^I-R^J}(M, N \otimes_{R^K} R^J)[\nu] && (\text{Cor. } 3.2.3)\\ &\quad\cong \mathrm{Hom}_{R^I-R^J}(M, N \cdot {}^K\vartheta^J)[\ell(w_K) - \ell(w_J)] && \square \end{aligned}$$

We can now establish the first version of the homomorphism formula.

THEOREM 5.2.2. *If* $M \in {}^I\mathcal{B}^J_{BS}$, $N \in {}^I\mathcal{F}^J_\nabla$ *or* $M \in {}^I\mathcal{F}^J_\Delta$, $N \in {}^I\mathcal{B}^J_{BS}$ *then* $\mathrm{Hom}(M,N)$ *is graded free as an* R^I*-module and we have an isomorphism*

$$\mathrm{Hom}(M,N)[-\ell(w_J)] \cong \overline{\langle \mathrm{ch}_\Delta(M), \mathrm{ch}_\nabla(N)\rangle} \cdot R^I$$

of graded R^I*-modules.*

PROOF. Let us first assume that $M \in {}^I\mathcal{B}^J_{BS}$ and $N \in {}^I\mathcal{F}^J_\nabla$. Using Lemma 5.2.1 we see that, as R^I-modules

$$\mathrm{Hom}(M \cdot {}^J\vartheta^K, N)[-\ell(w_K)] \cong \mathrm{Hom}(M, N \cdot {}^K\vartheta^J)[-\ell(w_J)].$$

By (2.2.4) and Theorems 4.1.5 and 4.3.3 we have

$$\begin{aligned}\langle \mathrm{ch}_\Delta(M \cdot {}^J\vartheta^K), \mathrm{ch}_\nabla(N)\rangle &= \langle \mathrm{ch}_\Delta(M) *_J {}^JH^K, \mathrm{ch}_\nabla(N)\rangle = \\ &= \langle \mathrm{ch}_\Delta(M), \mathrm{ch}_\nabla(N) *_K {}^KH^J\rangle = \langle \mathrm{ch}_\Delta(M), \mathrm{ch}_\nabla(N \cdot {}^K\vartheta^J)\rangle\end{aligned}$$

We conclude that the formula is true for $(M \cdot {}^J\vartheta^K, N)$ if and only if it is true for $(M, N \cdot {}^K\vartheta^J)$. It is also clear that it is true for (M,N) if and only it if it true for any shift of M or N. Thus, without loss of generality, we may assume that $M = {}^IR^I = {}^I\Delta^I$.

By Lemma 2.2.10 we know

$$\langle \mathrm{ch}_\Delta({}^I\Delta^I), \mathrm{ch}_\nabla(N)\rangle = \langle v^{-\ell(w_I)}\, {}^IH^I, \mathrm{ch}_\nabla(N)\rangle = \text{coefficient of } {}^IH^I \text{ in } \mathrm{ch}_\nabla N.$$

Thus, by definition of ch_∇, we have

$$\Gamma_{W_I} N \cong \overline{\langle \mathrm{ch}_\Delta({}^I\Delta^I), \mathrm{ch}_\nabla(N)\rangle} \cdot {}^I\nabla^I \cong$$

It follows that

$$\mathrm{Hom}({}^I\Delta^I, N)[-\ell(w_I)] = \Gamma_{W_I}(N)[-\ell(w_I)] = \overline{\langle \mathrm{ch}_\Delta({}^I\Delta^I), \mathrm{ch}_\nabla(N)\rangle} \cdot {}^IR^I$$

which settles the case when $M \in {}^I\mathcal{B}^J_{BS}$ and $N \in {}^I\mathcal{F}^J_\nabla$.

If $M \in {}^I\mathcal{F}^J_\Delta$ and $N \in {}^I\mathcal{B}^J_{BS}$ then identical arguments to those above allow us to assume that $N = {}^I\nabla^I$. We have

$$\Gamma^{W_I} M = \langle \mathrm{ch}_\Delta M,\ {}^IH^I\rangle \cdot {}^I\Delta^I$$

and hence

$$\begin{aligned}\mathrm{Hom}(M, {}^I\nabla^I)[-\ell(w_I)] &= \mathrm{Hom}(M, {}^IR^I) \\ &= \mathrm{Hom}(\Gamma^{W_I} M, {}^IR^I) \\ &\cong \overline{\langle \mathrm{ch}_\Delta M,\ {}^IH^I\rangle} \cdot \mathrm{Hom}({}^IR^I, {}^IR^I) \\ &\cong \overline{\langle \mathrm{ch}_\Delta M, \mathrm{ch}_\nabla({}^I\nabla^I)\rangle} \cdot R^I \qquad \square\end{aligned}$$

5.3. Some local results. We would like to generalise the homomophism formula of the previous section to all objects in ${}^I\mathcal{B}^J$. The crucial point is determining $\mathrm{Hom}(M, {}^I\nabla_p^J)$ and $\mathrm{Hom}({}^I\Delta_p^J, N)$ for $M, N \in {}^I\mathcal{B}^J$. For this we consider various localisations of special bimodules, which is the purpose of this section.

Given any reflection $t \in W$ let $R^{(t)}$ denote the local ring of $V^t \subset V$. In other words, in $R^{(t)}$ we invert all functions $f \in R$ which do not vanish identically on V^t.

The ring $R^{(t)}$ is no longer graded and we will denote by $R^{(t)}$-mod-R the category of $(R^{(t)}, R)$-bimodules. The lack of a grading on $R^{(t)}$ means that we do not know if objects in $R^{(t)}$-mod-R satisfy Krull-Schmidt, which explains some strange wording below.

If $M, N \in R$-Mod-R are free as left R-modules, with M finitely generated we have an isomorphism

$$\mathrm{Hom}_{R^{(t)}-R}(R^{(t)} \otimes_R M, R^{(t)} \otimes_R N) \cong R^{(t)} \otimes_R \mathrm{Hom}_{R-R}(M, N).$$

It follows that, with the same assumptions on M and N,

$$\mathrm{Ext}^1_{R^{(t)}-R}(R^{(t)} \otimes_R M, R^{(t)} \otimes_R N) \cong R^{(t)} \otimes_R \mathrm{Ext}^1_{R-R}(M, N).$$

Lemma 4.2.1 tells us that that $\mathrm{Ext}^1_{R-R}(R_x, R_y)$ is non-zero if and only if $y = rx$ for some reflection $r \in T$, in which case it is supported on $\mathrm{Gr}_x \cap \mathrm{Gr}_{rx}$. We conclude that

$$\mathrm{Ext}^1_{R^{(t)}-R}(R^{(t)} \otimes_R R_x, R^{(t)} \otimes_R R_y) = 0 \text{ unless } y = tx. \tag{5.3.1}$$

(Alternatively, one may explicitly split the extension of scalars of the generator of $\mathrm{Ext}^1(R_x, R_{rx})$ to $R^{(t)}$-mod-R using a Demazure operator, if $r \neq t$.)

Suppose that $M \in R$-Mod-R has a filtration with successive subquotients isomorphic to a direct sum of shifts of R_x, and that no (shift of) R_x occurs in two different subquotients. By inducting over the filtration of M and using (5.3.1), we see that $R^{(t)} \otimes_R M$ has a decomposition in which each summand is either isomorphic to $R^{(t)} \otimes_R R_x$ or is an extension between $R^{(t)} \otimes_R R_x$ and $R^{(t)} \otimes_R R_{tx}$.

The next two results makes this decomposition more precise for special classes of modules.

LEMMA 5.3.1. *Let $I, J \subset S$ be finitary and $p \in W_I \backslash W / W_J$ be a double coset. In $R^{(t)}$-mod-R we have an isomorphism*

$$R^{(t)} \otimes_R R(p) \cong \begin{cases} \bigoplus_{x \in p} R^{(t)} \otimes_R R_x & \text{if } tp \neq p \\ \bigoplus_{x \in p; x < tx} R^{(t)} \otimes_R R_{x,tx} & \text{if } tp = p. \end{cases}$$

PROOF. Note that, by Proposition 1.2.12, either $tp = p$ or $tp \cap p = \emptyset$. The lemma then follows by applying $R^{(t)} \otimes_R -$ to the exact sequence in Proposition 2.3.4. $\square$

PROPOSITION 5.3.2. *If $B \in {}^I\mathcal{B}^J$ then $R^{(t)} \otimes_{R^I} B \otimes_{R^J} R \in R^{(t)}$-mod-$R$ is isomorphic to a direct summand in a direct sum of modules of the form $R^{(t)} \otimes_R R_x$ and $R^{(t)} \otimes_R R_{x,tx}$ with $x < tx$.*

PROOF. IF the statement is true for B, then it is true for any direct summand of B, and hence we may assume that $B \in {}^I\mathcal{B}^J_{BS}$. If $B = {}^IR^I$ then $R \otimes_{R^I} {}^IR^I \otimes_{R^I} R \cong R(W_I)$ (Theorem 2.4.1) and the necessary decomposition is provided by Lemma 5.3.1. By the inductive definition of ${}^I\mathcal{B}^J_{BS}$ it is enough to show that, if the lemma is true for $B \in {}^I\mathcal{B}^J$, then it is also true for $B \cdot {}^J\vartheta^K \in {}^I\mathcal{B}^K$ with $J \subset K$ or $J \supset K$. The case $J \supset K$ is trivial, and so we are left with the case $J \subset K$.

The module $B \otimes_{R^K} R$ is a direct summand in $B \otimes_{R^J} R \otimes_{R^K} R$ and, by assumption, $R^{(t)} \otimes_R B \otimes_{R^J} R$ is a direct summand in a direct sum of the modules $R^{(t)} \otimes_R R_x$ and $R^{(t)} \otimes_R R_{x,tx}$ with $x < tx$. Hence it is enough to show that the statement of the lemma is true for $R^{(t)} \otimes_R R_x \otimes_{R^K} R$ and $R^{(t)} \otimes_R R_{x,tx} \otimes_{R^K} R$.

In the first case $R_x \otimes_{R^K} R \cong R(xW_K)$ (Theorem 2.4.1 again) and the decomposition follows again from Lemma 5.3.1 together with the fact that $tx > x$.

In the second case there are two possibilities. If $tx = xt'$ for a reflection $t' \in W_K$ then $R_{x,tx}$ splits upon restriction to R^K (Lemma 4.2.3) and we may apply Lemma 5.3.1 again.

If $tx \neq xt'$ for any reflection $t' \in W_K$ then the sets xW_K and txW_K are disjoint. By applying $- \otimes_{R^K} R$ to the exact sequence $R_x[-2] \hookrightarrow R_{x,tx} \twoheadrightarrow R_{tx}$ and using the identification $R_x \otimes_{R^K} R \cong R(W_K)$ we see that $R_{x,tx} \otimes_{R^K} R$ has a filtration with subquotients (a shift of) R_w with $w \in xW_K$ or txW_K. It follows that we have an isomorphism

$$R^{(t)} \otimes_R R_{x,tx} \otimes_{R^K} R \cong \bigoplus_{y \in W_K} E_{xy,txy}$$

where $E_{xy,txy}$ is a (possibly trivial) extension of $R^{(t)} \otimes_R R_{xy}$ and $R^{(t)} \otimes_R R_{txy}$.

To identify $E_{xy,txy}$ we tensor the surjection $R(W_K) \twoheadrightarrow R_y$ with the exact sequence $R_x[-2] \hookrightarrow R_{x,tx} \twoheadrightarrow R_{tx}$ to obtain a diagram

$$\begin{tikzcd} R_x \otimes_{R^K} R[-2] \arrow[r, hook] \arrow[d, two heads] & R_{x,tx} \otimes_{R^K} R \arrow[r, two heads] \arrow[d, two heads] & R_{tx} \otimes_{R^K} R \arrow[d, two heads] \\ R_{xy}[-2] \arrow[r, hook] & R_{xy,txy} \arrow[r, two heads] & R_{txy}. \end{tikzcd}$$

After tensoring with $R^{(t)}$ the left and right surjections split by Lemma 5.3.1. It follows that $E_{xy,txy}$ is isomorphic to $R^{(t)} \otimes_R R_{xy,txy}$ for all $y \in W_K$ and the lemma follows. $\square$

We now come to the goal of this section, which is to relate $\mathrm{Hom}({}^I\Delta^J_p, B)$ and $\mathrm{Hom}(B, {}^I\nabla^J_p)$ for a singular Soergel bimodule $B \in {}^I\mathcal{B}^J$ to the nabla and delta filtrations on B. This provides the essential (and trickiest) step in generalising the homomorphism formula for Bott-Samelson bimodules to all Soergel bimodules.

The arguments used to establish this relation are complicated and so we first sketch the basic idea. Let us consider a nabla filtration on a Bott-Samelson bimodule B. By Theorem 5.2.2 we know the rank of $\mathrm{Hom}({}^I\Delta_p^J, B)$ in terms of $\Gamma_p^{\leq} B$ and a simple calculation confirms that $\mathrm{Hom}({}^I\Delta_p^J, B)$ and $\Gamma_p^{\leq} B[-\ell(p_-)]$ have the same graded rank as left R^I-modules.

Given a morphism $\alpha : {}^I\Delta_p^J \to B$ one may consider the image of a non-zero element of lowest degree in $\Gamma_p^{\leq} B$ and one obtains in this way an injection

$$\mathrm{Hom}({}^I\Delta_p^J, B) \to \Gamma_p^{\leq} B[\ell(p_-)].$$

One might hope that this maps into a submodule isomorphic to $\Gamma_p^{\leq} B[-\ell(p_-)]$, which would explain the above equality of ranks.

In order to show that this is the case we choose a decomposition

$$\Gamma_p^{\leq} B \cong P \cdot {}^I R_p^J$$

and recall that ${}^I R_p^J$ has the structure of a graded algebra compatible with the bimodule structure. In particular, elements in ${}^I R_p^J$ define endomorphisms of $\Gamma_p^{\leq} B$ (which in general do not come from acting by an element in $R^I \otimes R^J$). Given an element $m \in {}^I R_p^J$, we will abuse notation and denote by $m\Gamma_p^{\leq} B$ the image of this endomorphism.

We define an element $m_p \in {}^I R_p^J$ of degree $2\ell(p_-)$ and argue (using localisation) that the above injection lands in

$$m_p \Gamma_p^{\leq} B[\ell(p_-)] \cong \Gamma_p^{\leq} B[-\ell(p_-)].$$

Thus the two modules $\Gamma_p^{\leq} B[-\ell(p_-)]$ and $\mathrm{Hom}({}^I\Delta_p^J, B)$ are isomorphic.

Remark 5.3.3. *If W is a finite one may make the arguments in this section simpler by considering certain elements (similar to our $\phi_x \in R(p)$) constructed using Demazure operators. This is discussed in* [**So6**], *Bemerkung 6.7.*

We begin by defining the special elements $m_p \in {}^I R_p^J$. Recall that, by definition, the modules ${}^I R_p^J$ are the invariants in R under W_K, where $K = I \cap p_- J p_-^{-1}$.

Lemma 5.3.4. *The element*

$$m_p = \prod_{\substack{t \in T \\ tp_- < p_-}} h_t \in R.$$

lies in ${}^I R_p^J$.

Proof. Because $xh_s = h_{xsx^{-1}}$ if $x \in W$ (2.1.2) it is enough to show that if $s \in I \cap p_- J p_-^{-1}$ and $t \in T$ with $tp_- < p_-$, then $(sts)p_- < p_-$. Choose $r \in J$ such that $sp_- = p_- r$. We have either $(sts)sp_- = stp_- \leq$

sp_- or $stp_- \geq sp_-$. However the latter is impossible as $tp_- \notin p$. Similarly, either $stp_-r \leq sp_-r = p_-$ or $stp_-r \geq sp_-r$ and the latter is again impossible. It follows that $(sts)p_- \leq p_-$ as claimed. $\square$

We now come to the main goal of this section.

THEOREM 5.3.5. *Let $I, J \subset S$ be finitary, $B \in {}^I\mathcal{B}^J$ and $p \in W_I \backslash W/W_J$. We have isomorphisms*

(1) $\mathrm{Hom}({}^IR^J_p, B) \cong \mathrm{Hom}({}^IR^J_p, \Gamma^{\leq}_{p} B)[-2\ell(p_-)]$,

(2) $\mathrm{Hom}(B, {}^IR^J_p) \cong \mathrm{Hom}(\Gamma^{\geq}_{p} B, {}^IR^J_p)[-2\ell(p_-)]$.

The proof depends on a lemma which we establish by considering various localisations of B. Given a subset $A \subset W$ we extend the notion to sections supported in Gr_A to modules $M \in R^{(t)}$-mod-R as follows. Writing I_A for the ideal of functions vanishing on Gr_A, we define $\Gamma_A M$ to be the submodule of elements annihilated by $\langle I_A \rangle$, the ideal generated by I_A in $R^{(t)} \otimes R$.

LEMMA 5.3.6. *For any pair of morphisms*

$$M \to B \to {}^IR^J_p$$

with $M \in {}^I\mathcal{F}^J_\Delta$ such that $\Gamma_{\geq p} M = M$, the composition lands in $m_p {}^IR^J_p$.

PROOF. As in Lemma 2.4.2 let us regard ${}^IR^J_p$ as the subalgebra of $W_I \times W_J$-invariants in $R(p)$. Using Theorem 2.4.1 we obtain, for all $t \in T$, a commutative diagram (where the vertical inclusions are inclusions of abelian groups):

$$\begin{array}{ccccc}
m \in M & \longrightarrow & B & \longrightarrow & {}^IR^J_p \\
\cap & & \cap & & \cap \\
R \otimes_{R^I} M \otimes_{R^J} R & \longrightarrow & R \otimes_{R^I} B \otimes_{R^J} R & \longrightarrow & R(p) \ni (f_x) \\
\cap & & \cap & & \cap \\
R^{(t)} \otimes_{R^I} M \otimes_{R^J} R & \to & R^{(t)} \otimes_{R^I} B \otimes_{R^J} R & \to & R^{(t)} \otimes_R R(p)
\end{array}$$

Denote by $f = (f_x)$ the image of $m \in M$ in $R(p)$ as shown. By $W_I \times W_J$-invariance, it is enough to show that f_{p_-} is divisible by m_p.

To this end, let $t \in T$ satisfy $tp_- < p_-$. Considering elements supported on Gr_{p_-} and Gr_{tp_-} and using Lemma 5.3.1 and Proposition 5.3.2 we see that the bottom row admits a morphism to a composition of the form

$$R^{(t)} \otimes_R R_{p_-} \to \bigoplus R^{(t)} \otimes_R R_{tp_-, p_-} \to R^{(t)} \otimes_R R_{p_-}.$$

The composition of any two such maps must land in $h_t R^{(t)} \otimes_R R_{p_-}$. It follows that

$$f_{p_-} \in R \cap \bigcap_{\substack{t \in T \\ tp_- < p_-}} h_t R^{(t)} \otimes_R R = m_p R$$

and the lemma follows. $\square$

Proof of Theorem 5.3.5. First note that if the theorem is true for a module B, then it is true to any direct summand of B. Thus we may assume without loss of generality that $B \in {}^I\mathcal{B}^J_{BS}$.

We begin with 1). Let $\alpha : {}^IR^J_p \to B$ be a morphism. As $\operatorname{supp} {}^IR^J_p = {}^I\mathrm{Gr}^J_p$ the image of α is contained in $\Gamma_{\le p} B$ and, by composing with the quotient map we obtain a map ${}^IR^J_p \to \Gamma^{\le}_{p} B$. This yields a morphism

$$\Phi : \mathrm{Hom}({}^IR^J_p, B) \to \mathrm{Hom}({}^IR^J_p, \Gamma^{\le}_{p} B).$$

As B has a nabla flag, any element of B has support consisting of a union of ${}^I\mathrm{Gr}^J_q$ for $q \in W_I\backslash W/W_J$ by Lemma 2.5.3. It follows that Φ is injective.

Let us now fix an isomorphism

$$\Gamma^{\le}_{p} B \cong P \cdot {}^IR^J_p.$$

By Lemma 5.3.6 above, given any $\alpha \in \mathrm{Hom}({}^IR^J_p, B)$ the image of $\Phi(\alpha)$ is contained in $P \cdot m_p {}^IR^J_p \cong \Gamma^{\le}_{p} B[-2\ell(p_-)]$. Thus we obtain an injection

$$\mathrm{Hom}({}^IR^J_p, B) \to \mathrm{Hom}({}^IR^J_p, \Gamma^{\le}_{p} B)[-2\ell(p_-)]. \tag{5.3.2}$$

We compare ranks in order to show that this is an isomorphism.

Let us write $g \in \mathbb{N}[v, v^{-1}]$ for the coefficient of ${}^IH^J_p$ in $\mathrm{ch}_\nabla(N)$ written in the standard basis. By Theorem 5.2.2, we have, as left R^I-modules,

$$\begin{aligned}
\mathrm{Hom}({}^IR^J_p, B)[\ell(p_-) - \ell(w_J)] &\cong \mathrm{Hom}({}^I\Delta^J_p, B)[-\ell(w_J)] \\
&\cong \overline{\langle v^{\ell(p_-)-\ell(p_+)}\, {}^IH^J_p, \mathrm{ch}_\nabla(B)\rangle} \cdot R^I \\
&\cong \overline{g}\frac{\pi(p)}{\pi(J)} \cdot R^I.
\end{aligned}$$

One the other hand,

$$\begin{aligned}
&\mathrm{Hom}({}^IR^J_p, \Gamma^{\le}_{p} B)[-\ell(p_-) - \ell(w_J)] \cong \\
&\qquad \cong \overline{g} \cdot {}^I\nabla^J_p[-\ell(p_-) - \ell(w_J)] && \text{(Cor. 2.4.4)} \\
&\qquad = \overline{g} \cdot {}^IR^J_p[\ell(p_+) - \ell(p_-) - \ell(w_J)] \\
&\qquad = \overline{g} \cdot {}^IR^J_p[\ell(w_I) - \ell(w_{I,p,J})] && (1.2.1) \\
&\qquad = \overline{g}\frac{\pi(I)}{\pi(I,p,J)} \cdot R^I && \text{(Cor. 2.1.4)} \\
&\qquad = \overline{g}\frac{\pi(p)}{\pi(J)} \cdot R^I. && (1.2.3)
\end{aligned}$$

Thus (5.3.2) is an isomorphism and 1) follows.

We now turn to 2) which, of course, is similar. Let $\alpha : B \to {}^IR^J_p$ be a morphism. For support reasons, α annihilates $\Gamma_{>p} B$ and hence factorises to yield a map $\Gamma_p^{\geq} B \to {}^I R_p^J$. We obtain in this way an injection

$$\Phi : \mathrm{Hom}(B, {}^I R_p^J) \to \mathrm{Hom}(\Gamma_p^{\geq} B, {}^I R_p^J).$$

Let us fix an isomorphism

$$\Gamma_p^{\geq} B \cong P \cdot {}^I R_p^J$$

for some $P \in \mathbb{N}[v, v^{-1}]$. By the above lemma if $\alpha \in \mathrm{Hom}(B, {}^I R_p^J)$ then the image of $\Phi(\alpha)$ is contained in $P \cdot m_p {}^I R_p^J$ and thus we obtain an injection

$$\mathrm{Hom}(B, {}^I R_p^J) \to \mathrm{Hom}(\Gamma_p^{\geq} B, {}^I R_p^J)[-2\ell(p_-)].$$

Again we compare ranks. Choose $h \in \mathbb{N}[v, v^{-1}]$ such that $\Gamma_p^{\geq} B \cong h \cdot {}^I \Delta_p^J$. By Theorem 5.2.2 we have isomorphisms of left R^I-modules:

$$\begin{aligned} \mathrm{Hom}(B, {}^I R_p^J)[\ell(p_+) - \ell(w_J)] &\cong \mathrm{Hom}(B, {}^I \nabla_p^J)[-\ell(w_J)] \\ &\cong \overline{h} \frac{\pi(p)}{\pi(J)} \cdot R^I. \end{aligned}$$

On the other hand

$$\begin{aligned} \mathrm{Hom}(\Gamma_p^{\geq} B, {}^I R_p^J)&[-2\ell(p_-) + \ell(p_+) - \ell(w_J)] \cong \\ &\cong \mathrm{Hom}(h \cdot {}^I \Delta_p^J, {}^I R_p^J)[-2\ell(p_-) + \ell(p_+) - \ell(w_J)] \\ &\cong \overline{h} \cdot {}^I R_p^J[\ell(p_+) - \ell(p_-) - \ell(w_J)] \qquad \text{(Cor. 2.4.4)} \\ &\cong \overline{h} \frac{\pi(p)}{\pi(J)} \cdot R^I \end{aligned}$$

which completes the proof of 2). $\square$

5.4. The general homomorphism formula and classification. We can now prove the natural generalisation of Theorem 5.2.2 to all Soergel bimodules. For the duration of this section fix $I, J \subset S$ finitary.

THEOREM 5.4.1. *If $M \in {}^I\mathcal{B}^J$, $N \in {}^I\mathcal{F}_\nabla^J$ or $M \in {}^I\mathcal{F}_\Delta^J$, $N \in {}^I\mathcal{B}^J$ then* $\mathrm{Hom}(M, N)$ *is graded free as an R^I-module and we have an isomorphism*

$$\mathrm{Hom}(M, N)[-\ell(w_J)] \cong \overline{\langle \mathrm{ch}_\Delta(M), \mathrm{ch}_\nabla(N) \rangle} \cdot R^I$$

of graded R^I-modules.

PROOF. We handle first the case $M \in {}^I\mathcal{F}_\Delta^J$ and $N \in {}^I\mathcal{B}^J$. We will prove the theorem via induction on the length of a delta flag of M. The base case where $M \cong {}^I\Delta_p^J$ for some $p \in W_I \backslash W / W_J$ follows by essentially the same calculations as those in the proof of Theorem

5.3.5. Namely, if we write g for the coefficient of ${}^I H_p^J$ in $\mathrm{ch}_\nabla(N)$, we have

$$\begin{aligned}
\mathrm{Hom}({}^I\Delta_p^J, N) &\cong \Gamma_p^{\leq} N[-\ell(p_-)] \\
&\cong \overline{g} \cdot {}^I R_p^J[\ell(p_+) - \ell(p_-)] \qquad \text{(Theorem 5.3.5)} \\
&\cong \overline{g}\frac{\pi(I)}{\pi(I,p,J)} \cdot R^I[\ell(w_J)] \\
&\cong \overline{g}\frac{\pi(p)}{\pi(J)} \cdot R^I[\ell(w_J)] \\
&\cong \overline{\langle \mathrm{ch}_\Delta({}^I\Delta_p^J), \mathrm{ch}_\nabla(N)\rangle} \cdot R^I[\ell(w_J].
\end{aligned}$$

For the general case we may choose $p \in W_I\backslash W/W_J$ minimal with $\Gamma^p M \neq 0$ and obtain an exact sequence

$$\Gamma_{\neq p} M \hookrightarrow M \twoheadrightarrow \Gamma^p M. \tag{5.4.1}$$

By the minimality of p, both $\Gamma_{\neq p}M$ and $\Gamma^p M$ are in ${}^I\mathcal{F}_\Delta^J$ and

$$\mathrm{ch}_\Delta M = \mathrm{ch}_\Delta(\Gamma_{\neq p}M) + \mathrm{ch}_\Delta(\Gamma^p M).$$

As $N \in {}^I\mathcal{B}^J$ there exists some $\widetilde{N} \in {}^I\mathcal{B}_{BS}^J$ in which N occurs as a direct summand. The homomorphism formula for Bott-Samelson modules (5.2.2) tells us that $\mathrm{Hom}(-, \widetilde{N})$ is exact when applied to (5.4.1). Hence the same is true for $\mathrm{Hom}(-, N)$ and we conclude by induction that we have isomorphisms of graded R^I-modules:

$$\begin{aligned}
\mathrm{Hom}(M, N) &\cong \mathrm{Hom}(\Gamma_{\neq p}M, N) \oplus \mathrm{Hom}(\Gamma^p M, N) \\
&\cong \langle\overline{\mathrm{ch}_\Delta(M), \mathrm{ch}_\nabla(N)} \cdot R^I[\ell(w_J)].
\end{aligned}$$

The case when $M \in {}^I\mathcal{B}^J$ and $N \in {}^I\mathcal{F}_\nabla^J$ is handled similarly. If N is isomorphic to ${}^I\nabla_p^J$ for some $p \in W_I\backslash W/W_J$, then similar calculations to those in Theorem 5.3.5 verify the theorem in this case. For general N we choose p minimal with $\Gamma_p N \neq 0$ and obtain an exact sequence

$$\Gamma_p N \hookrightarrow N \twoheadrightarrow N/\Gamma_p N.$$

Applying $\mathrm{Hom}(M, -)$ this stays exact for the same reasons as above, and the isomophism in the theorem follows by induction. $\square$

We now come to the classification.

THEOREM 5.4.2. *For every $p \in W_I\backslash W/W_J$ there is, up to isomorphism, a unique indecomposable module ${}^I B_p^J \in {}^I\mathcal{B}^J$ satisfying*

(1) $\mathrm{supp}\, {}^I B_p^J \subset {}^I\mathrm{Gr}_{\leq p}^J$*;*
(2) $\Gamma^p({}^I B_p^J) \cong {}^I\nabla_p^J$*.*

The bimodule ${}^I B_p^J$ is self-dual and any indecomposable object in ${}^I\mathcal{B}^J$ is isomorphic to ${}^I B_p^J[\nu]$ for some $p \in W_I\backslash W/W_J$ and $\nu \in \mathbb{Z}$

In keeping with our notational convention, if $I = J = \emptyset$ we will write B_w instead of ${}^I B_w^J$.

Proof. Choose $p \in W_I \backslash W / W_J$ and let $(I, p_i, J_i)_{0 \le i \le n}$ be a right reduced translation sequence with end-point (I, p, J) (see Section 1.3). Consider the module

$$\widetilde{B} = {}^I\nabla^I \cdot {}^{J_0}\vartheta^{J_1} \cdot {}^{J_1}\vartheta^{J_2} \cdot \dots \cdot {}^{J_{n-1}}\vartheta^{J_n} \in {}^I\mathcal{B}^J.$$

By Theorem 4.1.5 and Proposition 2.2.7 we have

$$\mathrm{ch}_\nabla \widetilde{B} = {}^{J_0}H^{J_1} *_{J_1} {}^{J_1}H^{J_2} *_{J_2} \cdots *_{J_{n-1}} {}^{J_{n-1}}H^{J_n} = {}^IH^J_p + \sum_{q<p} \lambda_q \, {}^IH^J_q.$$

Hence $\widetilde{B}$ satisfies conditions 1) and 2) in the theorem. Let ${}^IB^J_p$ te the unique indecomposable summand of $\widetilde{B}$ with non-zero support on ${}^I\mathrm{Gr}^J_p$. Clearly ${}^IB^J_p$ also satisfies conditions 1) and 2).

Note that $\widetilde{B}$ is self-dual (because ${}^I\nabla^I$ is and the translation functors commute with duality by Proposition 4.3.4). As ${}^IB^J_p$ is the only direct summand of $\widetilde{B}$ with support containing ${}^I\mathrm{Gr}^J_p$, it follows that ${}^IB^J_p$ is also self-dual.

Let M and N be objects in ${}^I\mathcal{B}^J$ and assume that p is maximal for both modules with $\Gamma^p M \neq 0$ and $\Gamma^p N \neq 0$. Using Theorem 5.4.1 we see that $\mathrm{Hom}(M, -)$ is exact when applied to the sequence

$$\Gamma_{\neq p} N \hookrightarrow N \twoheadrightarrow \Gamma^p N.$$

In other words we have a surjection

$$\mathrm{Hom}(M, N) \twoheadrightarrow \mathrm{Hom}(M, \Gamma^p N) = \mathrm{Hom}(\Gamma^p M, \Gamma^p N).$$

By symmetry, we also have a surjection

$$\mathrm{Hom}(N, M) \twoheadrightarrow \mathrm{Hom}(\Gamma^p N, \Gamma^p M).$$

These surjections tell us that we can lift homomorphisms between $\Gamma^p M$ and $\Gamma^p N$ to M and N.

Now assume that M and N are indecomposable. After shifting M and N if necessary we may find $\alpha : \Gamma^p M \to \Gamma^p N$ and $\beta : \Gamma^p N \to \Gamma^p M$ of degree zero, such that $\beta \circ \alpha$ is the identity on a fixed direct summand ${}^I\nabla^J_p$ in $\Gamma^p M$ and zero elsewhere. By the above arguments we may find lifts $\tilde{\alpha} : M \to N$ and $\tilde{\beta} : N \to M$ of α and β of degree zero. As M is indecomposable and $\tilde{b} \circ \tilde{\alpha}$ is not nilpotent it must be an isomorphism. Thus $\Gamma^p M \cong {}^I\nabla^J_p$ and M is isomorphic to a direct summand of N. However N is indecomposable by assumption and thus M and N are isomorphic.

We conclude that, for any fixed $p \in W_I \backslash W / W_J$, there is at most one isomorphism class (up to shifts) of indecomposable bimodules $B \in {}^I\mathcal{B}^J$ such that p is maximal with $\Gamma^p B \neq 0$. The theorem then follows as we already know that ${}^IB^J_p$ satisfies these conditions. $\square$

The classification allows us to prove that indecomposable Soergel bimodules stay indecomposable when translated out of the wall:

Proposition 5.4.3. *Let $K \subset I$ and $L \subset J$ be finitary subsets of S and*

$$\mathrm{qu} : W_K \backslash W / W_L \to W_I \backslash W / W_J$$

be the quotient map. Choose $p \in W_I \backslash W / W_J$ and let q be the unique maximal element in $\mathrm{qu}^{-1}(p)$.

(1) In R^K-Mod-R^L we have an isomorphism

$$R^K \otimes_{R^I} {}^I B_p^J \otimes_{R^J} R^L \cong {}^K B_q^L.$$

(2) In R^I-Mod-R^J we have an isomorphism

$${}_{R^I}({}^K B_q^L)_{R^J} \cong \frac{\widetilde{\pi}(I)\widetilde{\pi}(J)}{\widetilde{\pi}(K)\widetilde{\pi}(L)} \cdot {}^I B_p^J.$$

Proof. For the course of the proof let use define

$$P = \frac{\widetilde{\pi}(I)\widetilde{\pi}(J)}{\widetilde{\pi}(K)\widetilde{\pi}(L)}.$$

The composition of inducing to R^K-Mod-R^L and restricting to R^I-Mod-R^J always produces a factor of P. To get started, note that $\Gamma^p({}^I B_p^J) \cong {}^I \nabla_p^J$ and hence (using Proposition 4.1.6)

$$\begin{aligned} \Gamma_{\mathrm{qu}^{-1}(\{\le p\})}(R^K \otimes_{R^I} {}^I B_p^J \otimes_{R^J} R^L)/\Gamma_{\mathrm{qu}^{-1}(\{< p\})}(R^K \otimes_{R^I} {}^I B_p^J \otimes_{R^J} R^L) \cong \\ \cong R^K \otimes_{R^I} {}^I \nabla_p^J \otimes_{R^J} R^L \end{aligned}$$

The latter is isomorphic to a shift of $R(p)^{W_K \times W_L}$ by Theorem 2.4.1 and hence is indecomposable. By the classification, we may write

$$R^K \otimes_{R^I} {}^I B_p^J \otimes_{R^J} R^L \cong {}^K B_q^L \oplus M \tag{5.4.2}$$

for some $M \in {}^K \mathcal{B}^L$ whose support is contained in ${}^K \mathrm{Gr}^L_{\mathrm{qu}^{-1}(\{< q\})}$. It follows that

$$\Gamma_{\mathrm{qu}^{-1}(\{\le p\})}({}^K B_q^L)/\Gamma_{\mathrm{qu}^{-1}(\{< p\})}({}^K B_q^L) \cong R^K \otimes_{R^I} {}^I \nabla_p^J \otimes_{R^J} R^L$$

This tells us (again by Proposition 4.1.6) that

$$\begin{aligned} \Gamma_{\le p}({}_{R^I}({}^K B_q^L)_{R^J})/\Gamma_{< p}({}_{R^I}({}^K B_q^L)_{R^J}) &\cong {}_{R^I}(R^K \otimes_{R^I} {}^I \nabla_p^J \otimes_{R^J} R^L)_{R^J} \\ &\cong P \cdot {}^I \nabla_p^J \end{aligned}$$

Therefore we may write

$${}_{R^I}({}^K B_q^L)_{R^J} \cong P \cdot {}^I B_p^J \oplus N$$

for some $N \in {}^I \mathcal{B}^J$. Restricting (5.4.2) to R^I-Mod-R^J yields

$$P \cdot {}^I B_p^J \cong {}_{R^I}({}^K B_q^L)_{R^J} \oplus {}_{R^I} M_{R^J} \cong P \cdot {}^I B_p^J \oplus {}_{R^I} M_{R^J} \oplus N$$

whence $M = N = 0$. Both claims then follow. $\square$

5.5. Characters and Soergel's conjecture. In this section we turn our attention to the characters of Soergel bimodules. We will see in the following theorem that the nabla character of a singular Soergel bimodule is determined by its delta character (and vice versa). Therefore we simplify notation and define

$$\mathrm{ch}(B) = \mathrm{ch}_\Delta(B)$$

for all Soergel bimodules B.

THEOREM 5.5.1. *Let I, J and K be finitary subsets of S.*

(1) For all $B \in {}^I\mathcal{B}^J$ we have $\mathrm{ch}_\nabla(B) = \overline{\mathrm{ch}_\Delta(B)}$.

(2) We have a commutative diagram

$$\begin{tikzcd} {}^I\mathcal{B}^J \times {}^J\mathcal{B}^K \arrow[r, "- \otimes_{R^J} -"] \arrow[d, "\mathrm{ch} \times \mathrm{ch}"] & {}^I\mathcal{B}^K \arrow[d, "\mathrm{ch}"] \\ {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \arrow[r, "- *_J -"] & {}^I\mathcal{H}^K \end{tikzcd} .$$

(3) The set $\{\mathrm{ch}({}^IB^J_p) \mid p \in W_I\backslash W/W_J\}$ builds a self-dual basis for ${}^I\mathcal{H}^J$.

PROOF. We begin with 1). As $\mathrm{ch}_\nabla({}^IR^I) = \overline{\mathrm{ch}_\Delta({}^IR^I)}$ we may use Theorems 4.1.5 and 4.3.3 to conclude that the statement is true for all Bott-Samelson bimodules. We now use induction over the Bruhat order on $W_I\backslash W/W_J$ to show that $\mathrm{ch}_\nabla({}^IB^J_p) = \overline{\mathrm{ch}_\Delta({}^IB^J_p)}$ for all $p \in W_I\backslash W/W_J$, which implies the claim. If p contains the identity, then ${}^IB^J_p$ is Bott-Samelson and so the claim is true. For general $p \in W_I\backslash W/W_J$ we may (as in the proof of Theorem 5.4.2) find a Bott-Samelson module N such that $N \cong {}^IB^J_p \oplus \widetilde{N}$ and the support of $\widetilde{N}$ is contained in ${}^I\mathrm{Gr}^J_{<p}$. We have

$$\mathrm{ch}_\nabla({}^IB^J_p) + \mathrm{ch}_\nabla(\widetilde{N}) = \mathrm{ch}_\nabla(N) = \overline{\mathrm{ch}_\Delta(N)} = \overline{\mathrm{ch}_\Delta({}^IB^J_p)} + \overline{\mathrm{ch}_\Delta(\widetilde{N})}.$$

By induction $\mathrm{ch}_\nabla(\widetilde{N}) = \overline{\mathrm{ch}_\Delta(\widetilde{N})}$ and the claim follows.

Statement 2) follows by a very similar argument. It is clear from Theorem 4.3.3 that the statement is true for Bott-Samelson bimodules. Let us fix $M \in {}^I\mathcal{B}^J$. It is enough to show that $\mathrm{ch}(M \otimes_{R^J} {}^JB^K_p) = \mathrm{ch}(M) *_J \mathrm{ch}({}^JB^K_p)$ for all $p \in W_J\backslash W/W_K$. Again we induct over the Bruhat order on $W_J\backslash W/W_K$. If p is minimal then ${}^JB^K_p$ is Bott-Samelson and the claim follows by Theorem 4.3.3. If $p \in W_J\backslash W/W_K$ is arbitrary then we may find, as above, a Bott-Samelson bimodule $N \in {}^J\mathcal{B}^K_{BS}$ which decomposes as $N \cong {}^JB^K_p \oplus \widetilde{N}$ with the support of $\widetilde{N}$ contained in ${}^I\mathrm{Gr}^J_{<p}$. We have

$$\begin{aligned}\mathrm{ch}(M \otimes_{R^J} {}^JB^K_p) + \mathrm{ch}(M \otimes_{R^J} \widetilde{N}) &= \mathrm{ch}(M \otimes_{R^J} N) = \\ &= \mathrm{ch}(M) *_J \mathrm{ch}(N) = \mathrm{ch}(M) *_J \mathrm{ch}({}^JB^K_p) + \mathrm{ch}(M) *_J \mathrm{ch}(\widetilde{N}).\end{aligned}$$

By induction $\mathrm{ch}(M \otimes_{R^J} \widetilde{N}) = \mathrm{ch}(M) *_J \mathrm{ch}(\widetilde{N})$ and the claim follows.

We now turn to 3). By Theorem 5.4.2, we have

$$\mathrm{ch}({}^I B^J_p) = {}^I H^J_p + \sum_{q<p} \lambda_q \, {}^I H^J_q$$

for some $\lambda_q \in \mathbb{N}[v, v^{-1}]$. It follows that the set $\{\mathrm{ch}({}^I B^J_p)\}$ gives a basis for ${}^I\mathcal{H}^J$. The self-duality of $\mathrm{ch}({}^I B^J_p)$ follows from the self-duality of ${}^I B^J_p$ and Proposition 4.3.5:

$$\mathrm{ch}({}^I B^J_p) = \mathrm{ch}_\Delta(D^I B^J_p) = \mathrm{ch}_\nabla({}^I B^J_p) = \overline{\mathrm{ch}({}^I B^J_p)}. \qquad \square$$

Given the theorem it is desirable to understand this basis $\{\mathrm{ch}({}^I B^J_p)\}$ for $p \in W_I \backslash W / W_J$ more explicitly. We will finish this chapter by recalling Soergel's conjecture on the characters of the indecomposable bimodules in $\mathcal{B}$ (recall that we write $\mathcal{B}$ instead of ${}^\emptyset\mathcal{B}^\emptyset$).

In [**So6**] Soergel considers the full subcategory of R-Mod-R consisting of all objects isomorphic to direct sums, summands and shifts of objects of the form

$$R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R \tag{5.5.1}$$

where $s, t, \ldots, u \in S$ are simple reflections. A priori, this category may not contain all objects of $\mathcal{B}$. However using the same arguments as in the proof of Theorem 5.4.2 one can show that one obtains all indecomposable objects in $\mathcal{B}$ as direct summands of bimodules of the form (5.5.1) for reduced expressions $st \ldots u$. Thus Soergel's category is precisely $\mathcal{B}$.

The following is Vermutung 1.13 in [**So6**].

Conjecture 5.5.2. *(Soergel) For all $w \in W$ we have* $\mathrm{ch}(B_w) = \underline{H}_w$.

If Soergel's conjecture is true then, by Proposition 5.4.3,

$$\mathrm{ch}(R \otimes_{R^I} {}^I B^J_p \otimes_{R^J} R) = \mathrm{ch}(B_{p_+}) = \underline{H}_{p_+}.$$

By Theorem 5.5.1, $\mathrm{ch}(R \otimes_{R^I} {}^I B^J_p \otimes_{R^J} R)$ is equal to $\mathrm{ch}({}^I B^J_p)$ regarded as an element of $\mathcal{H}$. Hence

$$\mathrm{ch}({}^I B^J_p) = {}^I \underline{H}^J_p.$$

CHAPTER 4

Soergel bimodules in low rank

In this chapter we determine the characters of some indecomposable Soergel bimodules for finite, low rank Coxeter groups. We concentrate on the non-singular situation (i.e. $I = J = \emptyset$) however (as we have seen in Section 5.5 in the last chapter), verifying Soergel's conjecture in this case determines the characters of all singular Soergel bimodules.

If the ground field is $\mathbb{C}$ all the characters we discuss below (with the exception of type H_3 and H_4) are known by geometric methods (see [**So5**]). However, as we mention in the introduction, it is also possible to define Soergel bimodules in positive characteristic and finding an approach to the characters in this situation seems to be a difficult problem. Here we present a simple, combinatorial method by which many characters can be determined if one knows the W-graph of the corresponding Coxeter system.

The idea is that the basis $\{\mathrm{ch}(B_x) \mid x \in W\}$ of $\mathcal{H}$ given by the characters of indecomposable Soergel bimodules is self-dual and has positivity properties shared by the Kazhdan-Lusztig basis. By expanding products of the form $\underline{H}_s * \mathrm{ch}(B_w)$ and $\mathrm{ch}(B_w) * \underline{H}_s$ and using the fact that the result must consist of a positive combination of bimodule characters, one may often conclude that $\mathrm{ch}(B_x) = \underline{H}_x$ inductively.

In order to carry this out it is essential to know how $\underline{H}_s$ acts on the Kazhdan-Lusztig basis from the left and right. This precisely the information provided by the W-graph of the Coxeter system (W, S).

The structure of this chapter is as follows. In Section 1 we recall the definition of the W-graph. In Section 2 we define a subset $\sigma(W) \subset W$ based on the W-graph and show that $\mathrm{ch}(B_x) = \underline{H}_x$ if $x \in \sigma(W)$. In the last Section 3 we discuss the results of computer calculations determining the subset $\sigma(W) \subset W$ for all finite Coxeter groups of rank less than or equal to 6.

1. The W-graph

Let (W, S) be a Coxeter system. In Section 2 of Chapter 2 we defined the Hecke algebra $\mathcal{H}$, its standard basis $\{H_w \mid w \in W\}$ and its Kazhdan-Lusztig basis $\{\underline{H}_w \mid w \in W\}$. If $x \leq w$ we defined $\mu(x, w)$ to be the coefficient of v in the Kazhdan-Lusztig polynomial $h_{x,w}$ and stated the multiplication formula:

$$\underline{H}_s \underline{H}_w = \begin{cases} (v+v^{-1})\underline{H}_w & \text{if } sw < w \\ \underline{H}_{sw} + \sum_{x<w; sx<x} \mu(x,w)\underline{H}_x & \text{if } sw > w. \end{cases} \tag{1.0.2}$$

We want to explain how one may simplify this formula slightly. Given $w \in W$ we define the *left* and *right descent set* to be

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \text{ and } \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Let us now extend $\mu(x,y)$ so that μ is symmetric and $\mu(x,y) = 0$ if x and y are not comparable in the Bruhat order. If $w \leq x$, $sw > w$ and $sx < x$ then $h_{w,x} = vh_{sw,x}$ and hence $\mu(w,x) \neq 0$ only when $w = sx$. We may therefore rewrite (1.0.2) as follows (see [**KL1**]):

$$\underline{H}_s \underline{H}_w = \begin{cases} (v+v^{-1})\underline{H}_{sw} & \text{if } s \in \mathcal{L}(w) \\ \sum_{x\in W; s\in\mathcal{L}(x)} \mu(x,w)\underline{H}_x & \text{if } s \notin \mathcal{L}(w). \end{cases} \tag{1.0.3}$$

Similarly one has

$$\underline{H}_w \underline{H}_s = \begin{cases} (v+v^{-1})\underline{H}_{sw} & \text{if } s \in \mathcal{R}(w) \\ \sum_{x\in W; s\in\mathcal{R}(x)} \mu(x,w)\underline{H}_x & \text{if } s \notin \mathcal{R}(w). \end{cases} \tag{1.0.4}$$

It follows that all the information about the action of $\underline{H}_s$ on the left and right on the Kazhdan-Lusztig basis may be encoded in a labelled graph, known as the *W-graph*. The vertices correspond to the elements of W and are labelled with the left and right descent sets. There is an edge between x and $y \in W$ if $\mu(x,y) \neq 0$, in which case it is labelled by $\mu(x,y)$. The important point for us is that, in order to know the action of $\underline{H}_s$ on the Kazhdan-Lusztig basis, it is only necessary to know the W-graph and not all Kazhdan-Lusztig polynomials. This significantly simplifies the necessary computer calculations in Section 3.

2. Separated Elements

Let V be a reflection faithful representation of a Coxeter system (W,S), let $\mathcal{B}$ denote the category of (non-singular) Soergel bimodules and choose representatives $\{B_w \mid w \in W\}$ for each isomorphism class of indecomposable Soergel bimodule (normalised as in Theorem 5.4.2). We would like to show that their characters are given by the Kazhdan-Lusztig basis. Using Theorem 5.5.1 one sees that the set $\{\text{ch}(B_w) \mid w \in W\}$ yields a self-dual basis of $\mathcal{H}$ with certain positivity properties which are shared by the Kazhdan-Lusztig basis. Sometimes this allows one to conclude that $\text{ch}(B_w) = B_w$. This is the motivation behind the set $\sigma(W)$ to be defined below.

Example 2.0.3. *As motivation, let us consider some examples:*

1. *Let $x \in W$ and suppose that $sx < x$, $\underline{H}_s\underline{H}_{sx} = \underline{H}_x$ and* $\text{ch}(B_{sx}) = \underline{H}_{sx}$. *We know that B_x is a direct summand of $B_s \otimes_R B_{sx}$ with self-dual character. Hence* $\text{ch}(B_{sx}) = \underline{H}_{sx}$ *by the uniqueness of the Kazhdan-Lusztig basis.*

(2) *Fix $x \in W$ and suppose that* $\mathrm{ch}(B_{sx}) = \underline{H}_{sx}$ *for all* $s \in \mathcal{L}(x)$. *Suppose further that the only Kazhdan-Lusztig basis element that appears with non-zero coefficient in all expressions* $\underline{H}_s \underline{H}_{sx}$ *with* $s \in \mathcal{L}(x)$ *is* $\underline{H}_x$. *Then, using the fact that* B_x *occurs as a direct summand in* $B_s \otimes_R B_{sx}$ *for all* $s \in \mathcal{L}(x)$, *it follows that* $\mathrm{ch}(B_x) = \underline{H}_x$.

We start with some definitions. Given an element $h = \sum a_x \underline{H}_x \in \mathcal{H}$ we define the *Kazhdan-Lusztig support* to be the set

$$\mathrm{supp}_{KL}(h) = \{x \mid a_x \neq 0\}.$$

We say that h is *KL-supported in degree 0* if all $a_x \in \mathbb{Z}$.

Given $h, h' \in \mathcal{H}$ we may write the difference $h' - h$ in the standard basis as

$$h' - h = \sum a_x H_x.$$

If all $a_x \in \mathbb{N}[v, v^{-1}]$ we write $h \leq h'$. Note that if M is a direct summand of $N \in \mathcal{B}$ then $\mathrm{ch}(M) \leq \mathrm{ch}(N)$.

LEMMA 2.0.4. *Suppose M is a direct summand of a Soergel bimodule N whose character is self-dual and KL-supported in degree 0. Then the character of M is also self-dual, KL-supported in degree zero and*

$$\mathrm{supp}_{KL}(\mathrm{ch}(\mathcal{F})) \subset \mathrm{supp}_{KL}(\mathrm{ch}(\mathcal{G})).$$

PROOF. We may write

$$\mathrm{ch}(M) = \sum a_x \underline{H}_x \quad \text{for some } a_x \in \mathbb{Z}[v, v^{-1}].$$

Now M occurs as a direct summand of the self-dual N whose character is KL-supported in degree zero. Thus:

$$\mathrm{ch}(N) \in \bigoplus_{x \in W} \mathbb{Z}[v] \underline{H}_x = \bigoplus_{x \in W} \mathbb{Z}[v] H_x$$

$$\mathrm{ch}(M) \leq \mathrm{ch}(N) \text{ and } \overline{\mathrm{ch}(M)} \leq \mathrm{ch}(N)$$

Hence $a_x \in \mathbb{Z}$ for all $x \in W$ and the last claim follows by considering the coefficients of v^0 in $\mathrm{ch}(N) = \sum b_x H_x$. $\square$

Given a subset $X \subset W$ define

$${}^sX = \{x \in X \mid sx > x\} \text{ and } X^s = \{x \in X \mid xs > x\}.$$

We now define a function $f_W : Y \to \mathcal{P}(W)$ from some subset $Y \subset W$ to the power set of W. This function and its domain are defined inductively as follows:

(1) $f_W(id) = \{id\}$.
(2) Suppose we have defined f_W on all $y < x$. Then it is possible to define f_W on x if there exists $s \in \mathcal{L}(x)$ or $t \in \mathcal{R}(x)$ such that either

$${}^sf_W(sx) = f_w(sx) \text{ or } f_W(xt)^t = f_w(xt).$$

In this case we define $f_W(x)$ to be the set:

$$\bigcap_{s\in\mathcal{L}(x)}\left(\bigcup_{w\in{}^s f_W(sx)}\operatorname{supp}_{KL}(\underline{H}_s\underline{H}_w)\right)\cap\bigcap_{t\in\mathcal{R}(x)}\left(\bigcup_{w\in f_W(xt)^t}\operatorname{supp}_{KL}(\underline{H}_w\underline{H}_t)\right)$$

REMARK 2.0.5. *The condition in the definition for f_W to be defined at $x \in W$ may seem a strange. It is one way to force* $\mathrm{ch}(B_x)$ *to be supported in degree zero, which is crucial to our argument below. In all examples that we have considered f_W is defined on all of W. However we see no reason why this should be true in general.*

DEFINITION 2.0.6. *If f_W is defined on $x \in W$ and $f_W(x) = \{x\}$ we say that x is* separated*. The set of all separated elements will be denoted by $\sigma(W)$.*

EXAMPLE 2.0.7. *Let W be a dihedral group*

$$D_n = \langle s, t \mid s^2 = t^2 = (st)^n = id\rangle.$$

If $(st)^m$ (resp. $(st)^m s$) is not the longest element then

$$\begin{aligned} f_W((st)^m) &= \{(st)^m, (st)^{m-1}, \dots, st\} \\ f_W((st)^m s) &= \{(st)^m s, (st)^{m-1}s, \dots, s\} \end{aligned}$$

and similarly for $(ts)^m$ and $(ts)^m t$. For the longest element w_0 one has

$$f_W(w_0) = \{w_0\}.$$

It follows that the separated elements are $\{id, s, t, st, ts, w_0\}$. In particular, A_2 and $A_1 \times A_1$ are the only rank two Coxeter groups in which $\sigma(W) = W$.

The following proposition shows the usefulness of the set $\sigma(W)$:

PROPOSITION 2.0.8. *If f_W is defined on $x \in W$ then* $\mathrm{ch}(B_x)$ *is KL-supported in degree 0 and*

$$\operatorname{supp}_{KL}(\mathrm{ch}(B_x)) \subset f_W(x).$$

In particular, if $x \in \sigma(W)$ is separated we have $\mathrm{ch}(B_x) = \underline{H}_x$.

PROOF. Clearly $\mathrm{ch}(B_{id}) = \underline{H}_{id}$ and so we may assume by induction that $\operatorname{supp}(\mathrm{ch}(B_w)) \subset f_W(w)$ for all $w < x$, where $x \in W$ is some element on which f_W is defined. Without loss of generality we may assume, by the indutive definition of f_W above, that there exists some $s \in \mathcal{L}(x)$ so that ${}^s f_W(sx) = f_W(sx)$. Hence, by (1.0.2), the character of $B_s \otimes_R B_{sx} = \underline{H}_s * \mathrm{ch}(B_{sx})$ is KL-supported in degree zero.

We may now apply Lemma 2.0.4 to conclude that $\mathrm{ch}(B_x)$ is KL-supported in degree zero. As B_x is a direct summand of all $B_t \otimes B_{tx}$ for $t \in \mathcal{L}(x)$ we conclude (again using Lemma 2.0.4) that

$$\operatorname{supp}_{KL}(\mathrm{ch}(\mathcal{F}_x)) \subset \bigcup_{w\in{}^t f_W(tx)}\operatorname{supp}_{KL}(\underline{H}_t\underline{H}_w).$$

(We may ignore those $w \notin {}^t f_W(tx)$ because we know that $\mathrm{ch}(B_{tx})$ is supported in degree zero, hence so is $\mathrm{ch}(B_w)$, and so they make no contribution.) Similarly, if $t \in \mathcal{R}(x)$ then

$$\mathrm{supp}_{KL}(\mathrm{ch}(B_x)) \subset \bigcup_{w \in f_W(xt)^t} \mathrm{supp}_{KL}(\underline{H}_w \underline{H}_t).$$

Taking the intersection over these conditions yields $f_W(x)$ and the proposition. □

3. Results of Computer Calculations

In this section we give some examples of the sets $\sigma(W) \subset W$ for finite, low rank Coxeter groups. As is clear from the definition of f_W and the multiplication formulas in Section 1, the only information needed to calculate $\sigma(W)$ and f_W is the Weyl group W together with its W-graph. However no general description of the W-graph is known (for descriptions of some subgraphs see [**LS**] and [**Ke**] and for a description of the computational aspects of the problem see [**dC2**]).

Thus, in order to calculate f_W and $\sigma(W)$ we have to restrict ourselves to examples. This involves two steps:

(1) calculation of the W-graph of (W, S);
(2) calculation of the function f_W using the W-graph.

Step 1) is computational quite difficult. Luckily there exists the program *Coxeter* written by Fokko du Cloux [**dC1**], which calculates the W-graph very efficiently.[1] Step 2) is then relatively straightforward. A crude implementation in Magma (whose routines for handling Coxeter groups proved very useful) as well as the W-graphs obtained from *Coxeter* are available at:

`home.mathematik.uni-freiburg.de/geordie/torsion/`

This site also contains a complete description of the sets $\sigma(W)$ and f_W for all examples discussed below.

By definition $f_W(w) = \{w\}$ if and only if $w \in \sigma(W)$. If $f_W(w) \neq \{w\}$ then $f_W(w)$ is a subset of W containing w as a maximal element. Thus, in order to know f_W is is enough to know the sets $f_W(w)$ for all $w \notin \sigma(W)$. We will refer to these sets as the *critical sets* and call the maximal element w the *index* of set $f_W(w)$. For convenience we will list the index first.

Thus for example, a listing

$$\{w, x\}$$

means that w is the index, $f_W(w) = \{w, x\}$ and either

$$\mathrm{ch}(B_w) = \underline{H}_w \text{ or } \mathrm{ch}(B_w) = \underline{H}_w + \lambda \underline{H}_x \text{ for some } \lambda \in \mathbb{N}.$$

[1] Although the task of calculating the W-graph is computationally orders of magnitude more difficult than the calculation of the function f_W, for any given Coxeter group our program was always slower than Fokko's!

Because of invariance properties of the W-graph with respect to diagram automorphisms and inversion the sets $\sigma(W)$ and $\{f_W(w)|w \notin \sigma(W)\}$ are invariant under these operations. Hence, when listing critical pairs we will choose a representative of each orbit under inversion and any diagram automorphisms.

3.1. $A_n, n \leq 6$**.** Here $\sigma(W) = W$. Thus, in any characteristic in which one may define and classify Soergel bimodules one has $\mathrm{ch}(B_w) = \underline{H}_w$ for all $w \in W$.

3.2. A_7**.** Here 9 of the 40 320 elements in W do not lie in $\sigma(W)$. We display the elements in string and diagram form. Recall that the string form of a permutation $w \in Sym_n$ is the sequence $w(1)w(2)\dots w(n)$. The critical sets (up to inversion and the diagram autmorphism $s_1 \mapsto s_7$, $s_2 \mapsto s_6$ etc.) are shown in Figure 1. Interestingly, the indices of

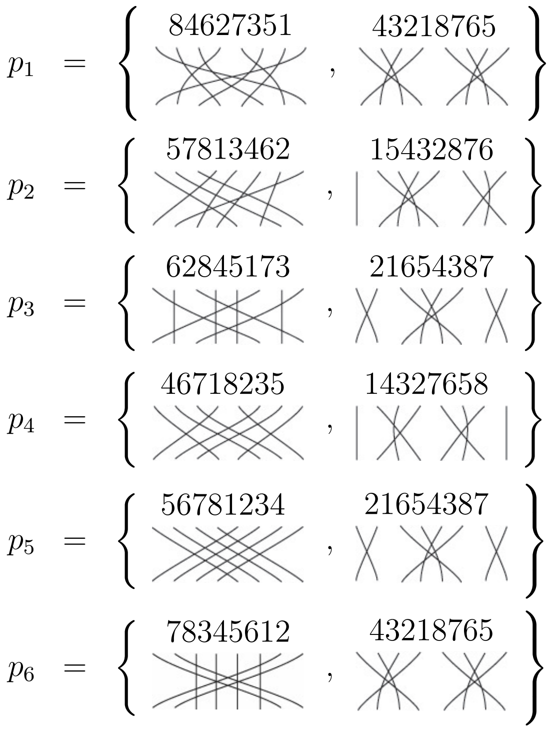
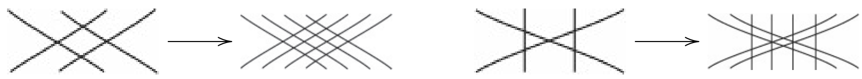


FIGURE 1. Critical Sets in A_7

p_4 and p_5 have already appeared in Kazhdan-Lusztig combinatorics. They are *hexagon permutations* as defined by Billey and Warrington [**BW**]. In [**Bra**] Braden has investigated the intersection cohomology complex in $SL_8(\mathbb{C})/B$ corresponding to the index of p_5 and reports that the intersection cohomology complex over $\mathbb{Z}$ has 2-torsion at the T-fixed point corresponding to the permutation 15372648.

We believe (very tentatively) to have an argument that shows that $\mathrm{ch}(B_w) = \underline{H}_w$ for w the index in p_i for $i = 1, 2, 3, 4$. Hence p_5 and p_6 are the only remaining cases in A_7. Interesting, these permutations are obtained by "doubling" the indices of the two singular Schubert varieties in $SL_4(\mathbb{C})/B$:



3.3. B_3 **and** B_4**.** We describe the function f_W for B_4. The ordering of the generators is as follows:

$$s \;\text{——}\; t \;\text{——}\; u \;=\!=\!=\; v$$

There are seven elements of W which do not lie in $\sigma(W)$. The critical sets are:

$$\begin{aligned}
p_1 &= \{vuv, v\} \\
p_2 &= \{uvu, u\} \\
p_3 &= \{vutvuv, uvuv\} \\
p_4 &= \{vutsvutvuv, utvutvuv\} \\
p_5 &= \{sutvutsvu, suvuv\} \\
p_6 &= \{stsuvuts, stsv\} \\
p_7 &= \{stsutvutsvut, stsuts\}
\end{aligned}$$

Note that p_1, p_2 and p_3 all lie in the parabolic subgroup isomorphic to B_3 and describe f_W on the 3 elements of W of type B_3 which don't lie in $\sigma(W)$ in this case. In Example 2.0.7 we have already seen the existence of the sets p_1 and p_2.

3.4. B_5 **and** B_6**.** In B_5, 21 of the 3840 elements of W do not lie in $\sigma(W)$. In B_6, 228 of the 46080 elements do not lie in $\sigma(W)$. In both cases this is less than 1% of all elements.

3.5. D_4**.** We label our generators s, t, u and v of W as follows:

$$\begin{array}{ccc}
 & & u \\
 & & / \\
s - t & & \\
 & & \backslash \\
 & & v
\end{array}$$

Here 4 of the 192 elements of D_4 are not in $\sigma(W)$. Representatives for the critical sets are:

$$\begin{aligned}
p_1 &= \{sutvtsu, suv\} \\
p_2 &= \{stsuvts, sts\}
\end{aligned}$$

The critical set p_1 is stable under the automorphism $s \mapsto u \mapsto v \mapsto s$ and the orbit of p_2 gives the other three sets. Braden has discovered 2-torsion in intersection cohomology of the Schubert variety corresponding to the index in p_1 at the point suv, which is a nice coincidence with our results.

3.6. D_5 **and** D_6**.** In D_5, 15 of the 1920 elements do not lie in $\sigma(W)$. In D_6, 107 of the 23040 elements in D_6 do not lie in $\sigma(W)$. In both cases these correspond to less than 1% of all elements.

3.7. E_6**.** Here 691 of the 51840 elements of W (roughly 1%) do not lie in $\sigma(W)$.

3.8. F_4**.** In F_4, 44 of the 1152 do not lie in $\sigma(W)$. This consists of almost 4% of all elements.

3.9. G_2**.** In this case we have already calculated $\sigma(W)$ in Example 2.0.7. Here we obtain nothing new. If $W = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$ then $\sigma(W) = \{1, s, t, st, ts, w_0\}$. In this case direct arguments may to used to verify that, in fact, $\mathrm{ch}(B_w) = \underline{H}_w$ for all $w \in W$ (see [**So6**]).

3.10. H_3 **and** H_4**.** In H_3, 8 of the 120 elements do not lie in $\sigma(W)$. In H_4, 1021 of the 14400 elements do not lie in $\sigma(W)$. This high percentage seems to be due to the large dihedral subgroups.

3.11. Further Calculations. The order of the group seems to be the greatest obstacle to further computer calculations. It would be interesting to know how many elements in A_8 do not lie in $\sigma(W)$ however this computation is out of reach at the moment (*Coxeter* can calculate the W-graph in a few hours, and it is 88MB). It would also be interesting to extend these calculations to the fundamental box of low rank affine Weyl groups. By recent results of Fiebig [**Fie4**], this situation has strong connections to the Lusztig conjecture.

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