Generators and relations for Soergel bimodules

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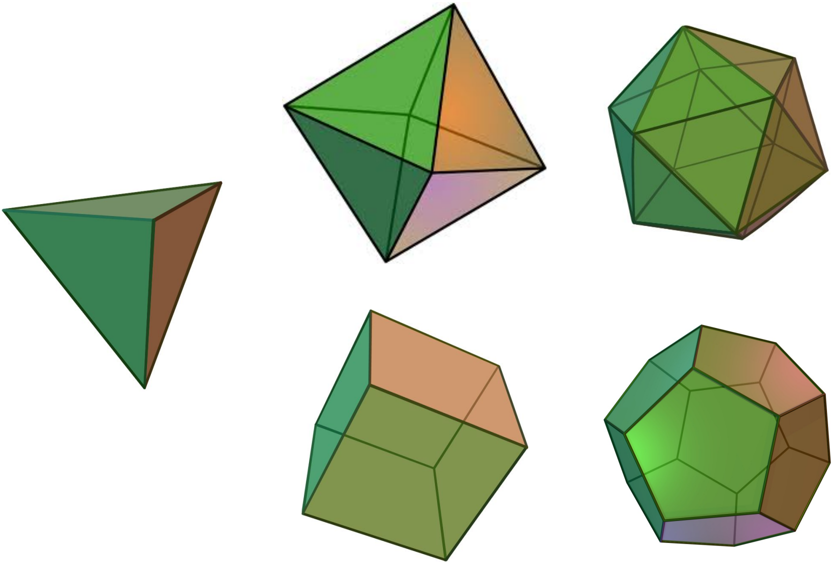
January 24, 2011

In this talk I will present **work in progress** with Ben Elias.

The goal is a better understanding of Soergel bimodules.

However we will begin by discussing more elementary matters . . .

We begin with the five platonic solids ...



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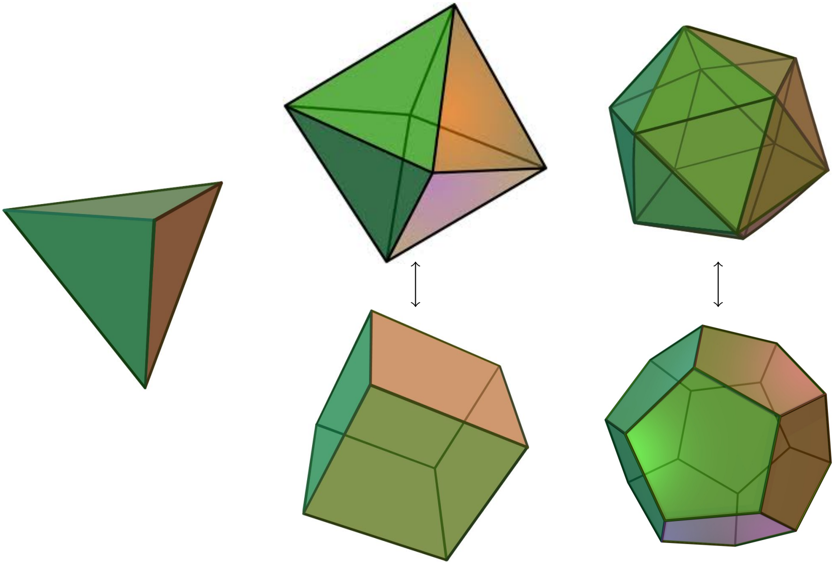

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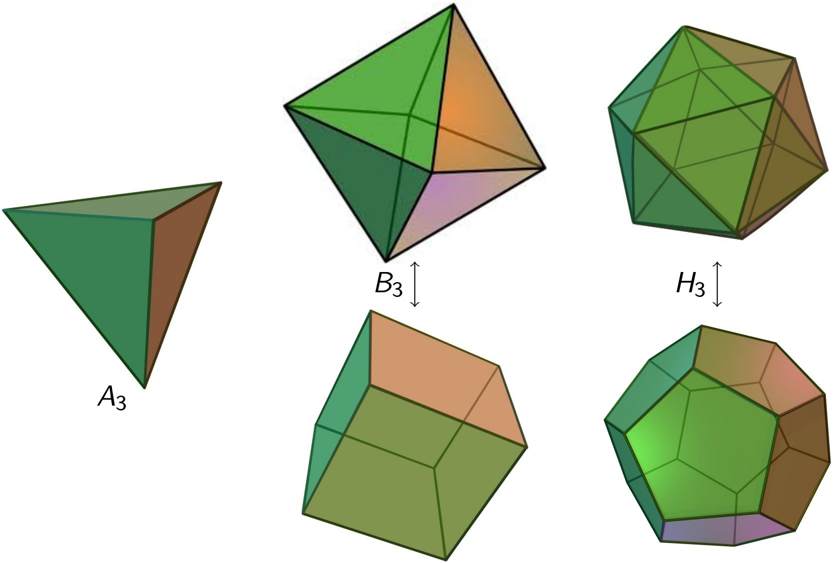

Let (W, S) be a Coxeter system:

$$W = \langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \rangle$$

For example, we could take W to be a real reflection group.

Recall that a standard parabolic subgroup is a subgroup of W generated by a subset $I \subset S$.

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Recall that a standard parabolic subgroup is a subgroup of W generated by a subset $I \subset S$.

Using this language, we see that Coxeter systems have

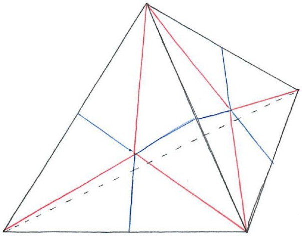
generators $\leftrightarrow$ rank 1 standard parabolic subgroups

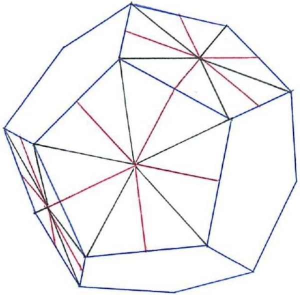
relations $\leftrightarrow$ finite rank 2 standard parabolic subgroups.

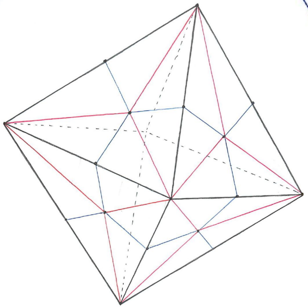
To a Coxeter system (W, S) one may associate a simplicial complex $CC(W)$ called the Coxeter complex of W.

Let $n = |S|$ denote the rank of W. Its construction is as follows:

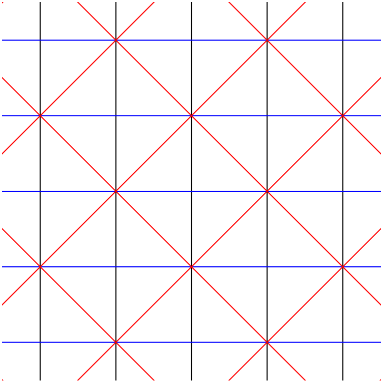
- colour the n faces of the standard $n-1$-simplex by the set S,
- take one such simplex for each element $w \in W$,
- glue the simplex corresponding to w to that corresponding to ws along the wall coloured by s.

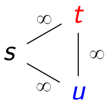






$$s \overset{4}{\text{———}} \color{red}{t} \overset{4}{\text{———}} \color{blue}{u}$$

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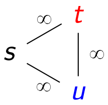
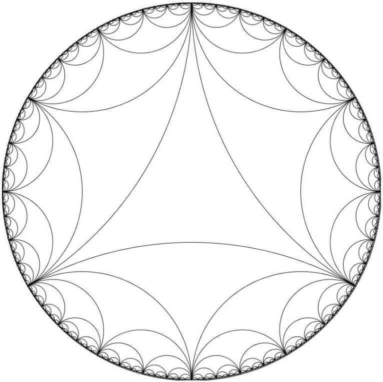
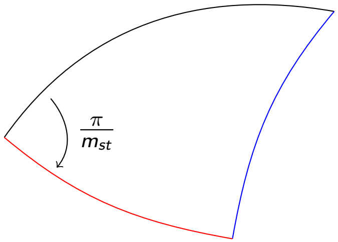


The Coxeter complex gives a tesselation of the hyperbolic plane by ideal triangles:



;

Because any triangle is either spherical, euclidean or hyperbolic, any rank 3 Coxeter group is either spherical, affine or hyperbolic.



Given a group G we can construct a (strict) monoidal category $\mathcal{G}$ as follows:

It has one object r_g for each element of G and

$$\mathrm{Hom}(r_g, r_h) = \begin{cases} \{id_{r_g}\} & \text{if } g = h, \\ \varnothing & \text{otherwise.} \end{cases}$$

The monoidal structure is given by $r_g r_h = r_{gh}$ (we always omit the $\otimes$ for the product in a monoidal category).

If G is generated by a subset S, then $\mathcal{G}$ is generated by $\{r_s \mid s \in S\}$ as a monoidal category.

However, in general it is a difficult problem to describe $\mathcal{G}$ by generators and relations.

(This is more difficult than giving a presentation of G.)

We will explain a solution to this problem for Coxeter systems.

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- we have morphisms $\epsilon : \mathbf{1} \to r_s r_s$ and $\eta : r_s r_s \to \mathbf{1}$ making (r_s, r_s) into a dual pair, and such that $\eta \circ \epsilon = id$.

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- for all $s \neq t$ with $m_{st} < \infty$ we have morphisms

  $$r_s r_t \cdots \xrightarrow{f_{st}} r_t r_s \dots$$

  (m_{st}-terms on both sides), such that $f_{st} \circ f_{ts} = id$.

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$$K_0(\mathcal{W}_{naive}) = W.$$

But in general it will not be true that $\mathrm{End}(r_s r_t \ldots) = \{id\}$.

Take $W = S_4$. Then the following are all reduced expressions for the longest element (edges indicate braid relations):

$121321 = 123121$

212321 $\quad$ 123212

$213231 = 231213$ $\quad$ $132312 = 312132$

232123 $\quad$ 321232

$323123 = 321323$

We must also require the "Zamolodchikov equation":

$$
\begin{tikzcd}
 & & r_1r_2r_1r_3r_2r_1 \arrow[r] \arrow[dl] & r_1r_2r_3r_1r_2r_1 \arrow[dr] & & \\
 & r_2r_1r_2r_3r_2r_1 \arrow[dl] & & & r_1r_2r_3r_2r_1r_2 \arrow[dr] & \\
r_2r_1r_3r_2r_3r_1 \arrow[d] & & & & & r_1r_3r_2r_3r_1r_2 \arrow[d] \\
r_2r_3r_1r_2r_1r_3 \arrow[dr] \arrow[rrrrr, phantom, "="] & & & & & r_3r_1r_2r_1r_3r_2 \arrow[dl] \\
 & r_2r_3r_2r_1r_2r_3 \arrow[dr] & & & r_3r_2r_1r_2r_3r_2 \arrow[dl] & \\
 & & r_3r_2r_3r_1r_2r_3 \arrow[r] & r_3r_2r_1r_3r_2r_3 & &
\end{tikzcd}
$$

In fact, any finite rank three parabolic subgroup leads to a Zamolodchikov like equation.

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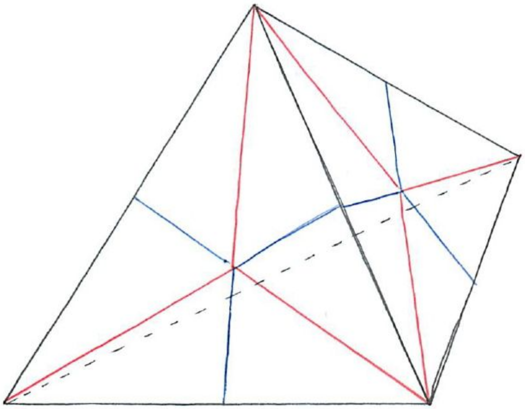
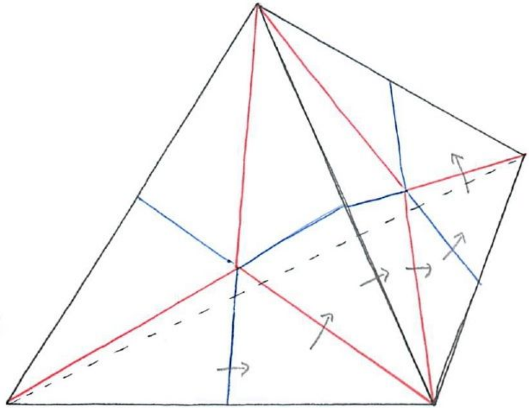
Let W denote a finite rank 3 Coxeter group, and let $CC(W)$ denote its Coxeter complex. Because W is finite, $CC(W)$ is homeomorphic to a sphere.



We may view reduced expressions for the longest element as paths from the identity alcove to its opposite.



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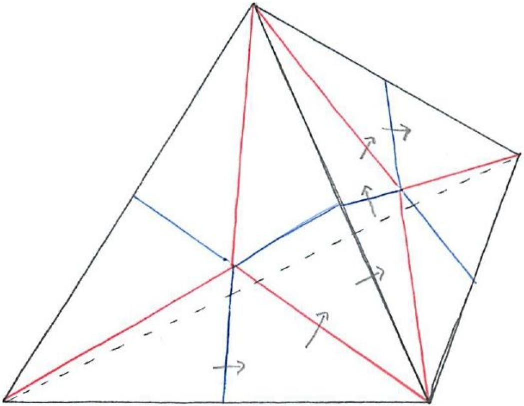

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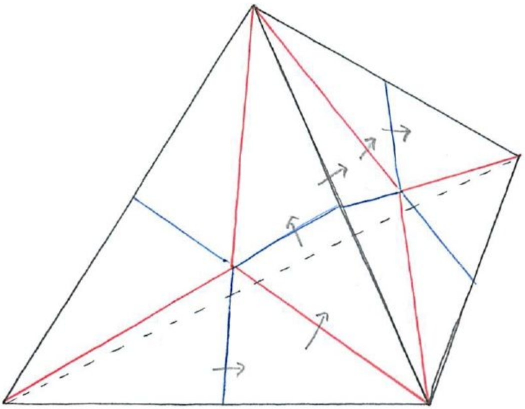

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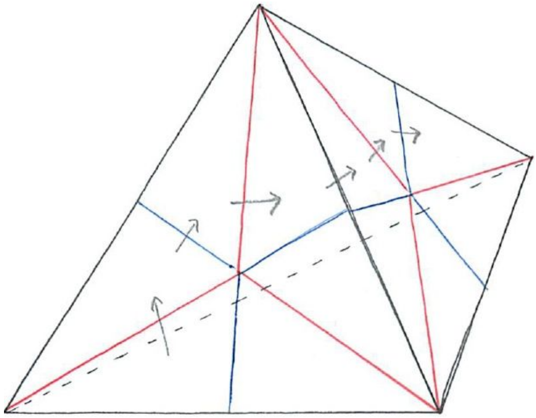

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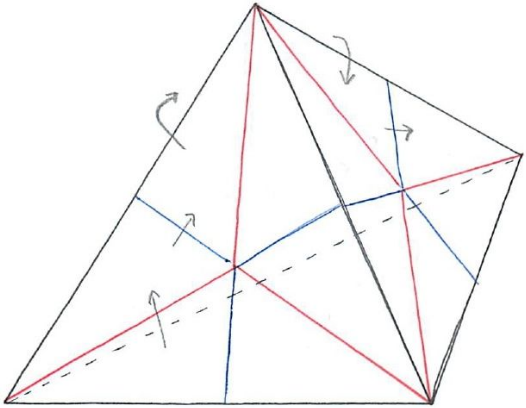
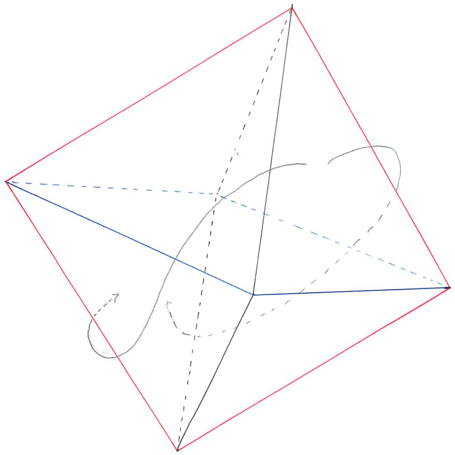
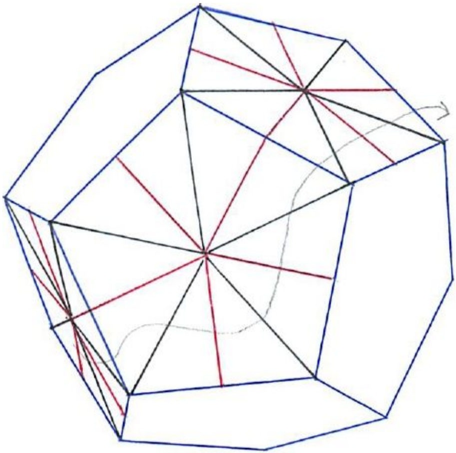


If we $W = \langle s, t, u \rangle$ and s, t and u all commute, then $CC(W)$ is an octahedron, and we retrieve the hexagon axiom:



$$
\begin{tikzcd}
 & stu \arrow[dl] \arrow[dr] & \\
tsu \arrow[d] & & sut \arrow[d] \\
tus \arrow[dr] \arrow[rr, phantom, "="] & & ust \arrow[dl] \\
 & uts &
\end{tikzcd}
$$

If $W = H_3$ ...



... then the Zamolodchikov relation is ...

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It turns out that the Zamolodchikov relations for finite rank 3 parabolic subgroups are all that one needs to add.

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To explain this precisely it is convenient to use a diagrammatic language.

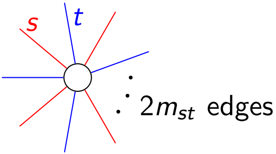
Consider the monoidal category with objects sequences of dots, coloured by the elements of S.

For example, if $S = \{s, t, u\}$ then

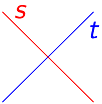
represents what we used to call $r_t r_u r_t r_s r_t$.

Morphisms are isotopy classes of string diagrams coloured by S.

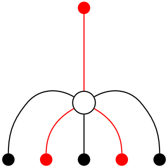
There are cup and cap diagrams, as well as a $2m_{st}$-valent vertex for each pair s and t of simple reflections with $s \neq t$:



If s and t commute we draw this simply as a crossing:



For example, the following represents a morphism from $r_s r_t r_s r_t r_s$ to r_t (where $m_{st} = 3$):

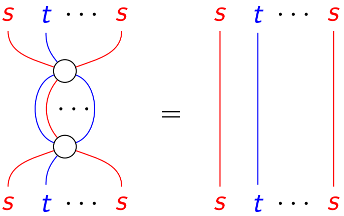


The relations we impose below will make this an isomorphism.

These morphisms are subject to the relations:

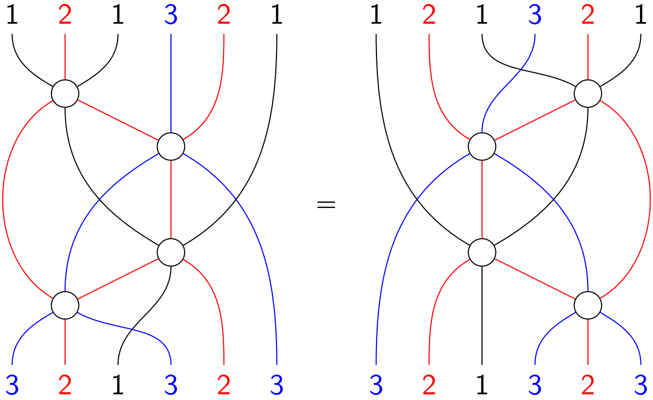
- circles disappear:

- we have:

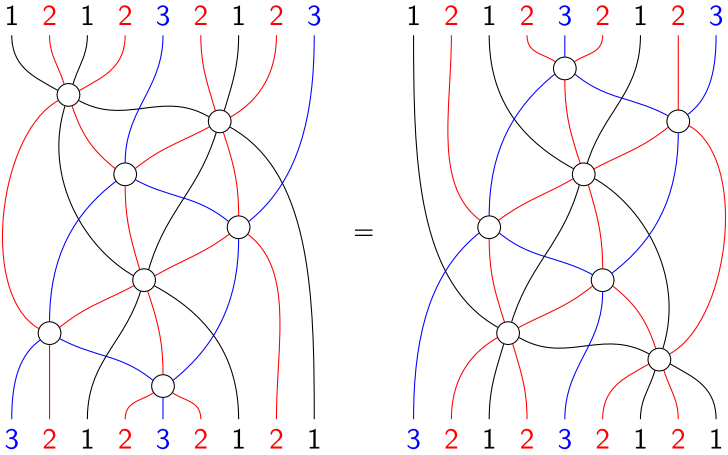


- Zamolodchikov relaions for every finite rank 3 standard parabolic subgroup.

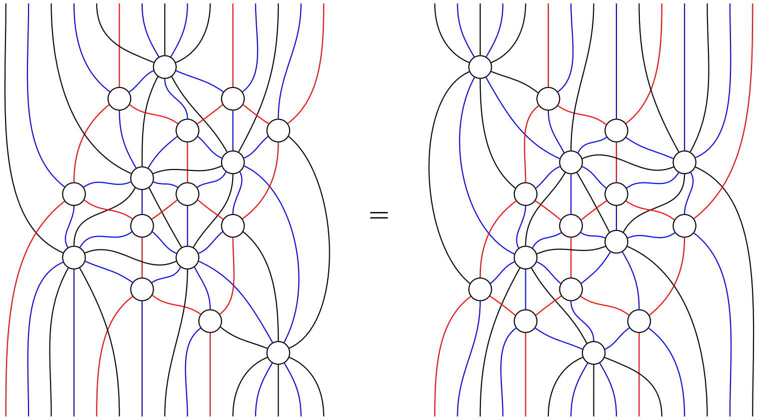
The A_3 Zamalodchikov is:



The B_3 Zamolodchikov:



The H_3 Zamolodchikov:



These relations can be read off by flattening the Coxeter complex.

(Just as the Coxeter relations can be read off the Coxeter complexes for finite rank 2 parabolic subgroups).

Theorem

The above category is monoidally equivalent to $\mathcal{W}$, the monoidal category associated to W.

The proof of this theorem relies on the following result, which is proved in Ronan, *Buildings*.

Let $\underline{w}$ be a word in S, and let $\Gamma(\underline{w})$ be the graph with vertices words which are braid equivalent to $\underline{w}$, and edges given by braid relations.

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Theorem

The only non-trivial loops in $\Gamma(\underline{w})$ *come from Zamolodchikov loops for finite rank 3 standard parabolic subgroups.*

The above has an "almost" interpretation as follows:

Let W denote a rank 3 Coxeter group, and let $CC(W)$ denote its Coxeter complex. We have a bijection:

$$\left\{ \begin{array}{c} \text{expressions} \\ \text{for } id \in W \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{paths starting and ending} \\ \text{at the identity in } CC(W) \end{array} \right\}$$

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Hence it seems natural to expect:

$$\mathrm{End}(\mathbf{1}) = \pi_1(\Omega_{id} CC(W)) = \pi_2(CC(W))$$

and the above result follows because:

$$\pi_2(CC(W)) = \begin{cases} \mathbb{Z} & \text{if } W \text{ is finite} \\ 0 & \text{if } W \text{ is affine or hyperbolic} \end{cases}$$

As a consequence of this theorem, we can deduce a theorem about actions of Coxeter groups on categories.

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Recall that if g is a group, then a (strict) action of G on a category is the data of

- equivalences F_g for all $g \in G$,
- an isomorphism $u : id \to F_{id}$,
- isomorphisms $F_g F_h \xrightarrow{\sim} F_{gh}$ for all $g, h \in G$

such that the morphisms $F_{id} F_g \to F_g$ and $F_g F_{id} \to F_g$ are deduced from u, and such that the following diagram commutes:

$$\begin{array}{ccc} F_g F_h F_i & \rightarrow & F_{gh} F_i \\ \downarrow & & \downarrow \\ F_g F_{hi} & \longrightarrow & F_{ghi} \end{array}$$

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Equivalently, an action of a group on a category is just a tensor functor

$$\mathcal{G} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}).$$

Theorem

To give an action of W on a category it is enough to give

- *equivalences F_s for all $s \in S$,*
- *morphisms $\epsilon : \mathbf{1} \to F_s F_s$ and $\eta : F_s F_s \to \mathbf{1}$*
- *isomorphisms $f_{st} : F_s F_t \cdots \to F_t F_s \dots$ (m_{st} terms) for all $s, t \in S$ with $s \neq t$*

such that

- *the morphisms ϵ and η make (F_s, F_s) into a dual pair,*
- *$\eta \circ \epsilon = id$,*
- *$f_{ts} \circ f_{st} = id$,*
- *f_{st} and f_{ts} are "mates under adjunction",*
- *For all finite standard rank 3 parabolic subgroups, the Zamolodchikov equation holds.*

Other applications:

- a similar theorem should be true for actions of braid groups on categories.

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- conjectural generalisation:

$$\text{generators} \leftrightarrow \text{rank 1 standard parabolic subgroups}$$
$$\text{relations} \leftrightarrow \text{finite rank 2 standard parabolic subgroups}$$

Other applications:

- a similar theorem should be true for actions of braid groups on categories.
- conjectural generalisation:

$$\begin{aligned} \text{generators} &\leftrightarrow \text{rank 1 standard parabolic subgroups} \\ \text{0-relations} &\leftrightarrow \text{finite rank 2 standard parabolic subgroups} \\ \text{1-relations} &\leftrightarrow \text{finite rank 3 standard parabolic subgroups} \\ &\vdots \end{aligned}$$

We now turn to Soergel bimodules ...

Let $\mathcal{H}$ denote the Hecke algebra of (W, S).

It is a $\mathbb{Z}[v^{\pm 1}]$-algebra generated by $\{H_s \mid s \in S\}$ subject to the relations

$$H_s^2 = (v^{-1} - v)H_s + 1 \quad \text{and} \quad H_s H_t H_s \cdots = H_t H_s H_t \dots .$$

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For later use, note the all-important specialisation homomorphism

$$\mathcal{H} \xrightarrow{v \mapsto 1} \mathbb{Z}W.$$

Let $\{\underline{H}_w \mid w \in W\}$ denote the *Kazhdan-Lusztig basis* of $\mathcal{H}$.

For example $\underline{H}_s = H_s + v$ and one has

$$\underline{H}_s^2 = (v + v^{-1})\underline{H}_s.$$

If st has order 3, then

$$\underline{H}_s\underline{H}_t\underline{H}_s + \underline{H}_t = \underline{H}_t\underline{H}_s\underline{H}_t + \underline{H}_s.$$

Soergel bimodules provide a categorification of the Hecke algebra.

Let V be a reflection faithful representation of W. For example, if W is a real reflection group, we can take V to be its natural representation.

Let R denote the regular functions on V, graded so that $\deg V^* = 2$. Then W acts on R. For any simple reflection $s \in S$ let R^s denote the invariants in R under s.

Given a graded module $M = \bigoplus M^i$ let $M[n]^i = M^{n+i}$.

Let

$$\mathcal{B} = \left\{ \begin{array}{c} \text{additive monoidal category of graded } R\text{-bimodules} \\ \text{generated by shifts of } b_s := R \otimes_{R^s} R[1]. \end{array} \right\}$$

and let $\overline{\mathcal{B}}$ denote the idempotent completion of $\mathcal{B}$. The category $\overline{\mathcal{B}}$ is the category of *Soergel bimodules*.

Theorem (Soergel)

The split Grothendieck group of $\overline{\mathcal{B}}$ is isomorphic to the Hecke algebra:

$$K_0(\overline{\mathcal{B}}) \cong \mathcal{H}$$

Under this isomorphism, $[b_s]$ corresponds to $\underline{H}_s$.

Let K denote the field of fractions of R. For any $w \in W$ consider the K-bimodule K_w which is isomorphic to K as a left K-module, and have right K-action twisted by w. In formulas:

$$f \cdot 1 = f \qquad 1 \cdot f = w(f).$$

Clearly

$$K_x \otimes_K K_y \cong K_{xy}.$$

It follows that the monoidal category of K-bimodules generated by the bimodules K_x for $x \in W$ gives a categorification of W.

We denote this category by $\mathcal{W}_K$.

Given any R-bimodule M, we can extend scalars to obtain a K-bimodule M_K. Hence we obtain a tensor functor

$$\overline{\mathcal{B}} \to K - \text{bimod}.$$

We have

$$(b_s)_K \cong K_{id} \oplus K_s.$$

It follows that, in fact, extension of scalars gives a functor

$$\overline{\mathcal{B}} \to \mathcal{W}_K$$

which categorifies the specialisation homomorphism

$$\mathcal{H} \to \mathbb{Z}W.$$

In his ICM paper, Rouquier raises the possibility of presenting the monoidal category of Soergel bimodules via generators and relations.

He points out that such a presentation should have several consequences in representation theory. For example, it should be possible to use such a presentation to give new proofs of the Kazhdan-Lusztig conjecture, as well as analogues for affine Weyl groups.

According to Rouquier:

> *The representation-theoretic and the geometrical categories should be viewed as two realizations of the same "2-representation" of* $\overline{\mathcal{B}}$.

In 2008, Nicolas Libedinsky found such a presentation for right-angled Coxeter groups (all $m_{st} \in \{2, \infty\}$).

Soon afterwards Ben Elias and Mikhail Khovanov found a presentation for Soergel bimodules in type A.

Let us consider the monoidal subcategory $\mathcal{B}_s$ generated by b_s. We have morphisms

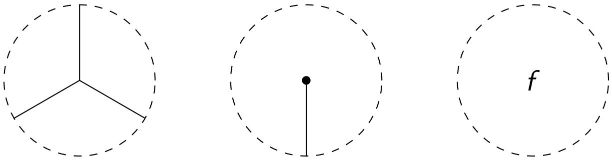
$$\begin{aligned} i_{s*} &: b_s \to \mathbf{1} : f \otimes g \mapsto fg \quad \text{degree } 1, \\ i_s^* &: \mathbf{1} \to b_s : 1 \mapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s \quad \text{degree } 1, \\ t_{s*} &: b_s \to b_s b_s : 1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 \quad \text{degree -1}, \\ t_s^* &: b_s b_s \to b_s : 1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1 \quad \text{degree -1}. \end{aligned}$$

It turns out that these morphisms generate all morphisms in $\mathcal{B}_s$.

(Here, and in what follows α_s denotes an equation for reflecting hyperplane of s.)

The above morphisms satisfy many relations, but Elias-Khovanov noticed that, when interpreted diagrammatically, many of these relations become “natural”.

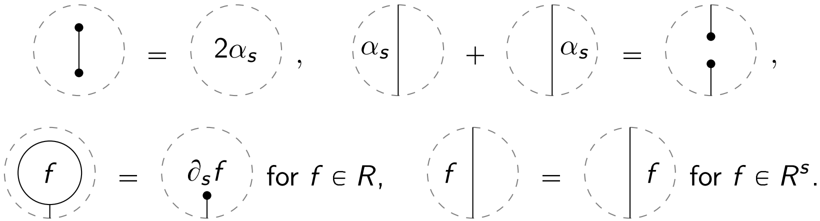
Consider the monoidal category $\mathcal{B}_s^{diag}$ generated by a single object, with morphisms linear combinations of isotopy classes of string diagrams composed from the morphisms



where $f \in R$ is a polynomial.

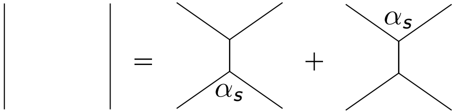
Subject to the relations:

- the generating object is a Frobenius object,
- we have



Then $\mathcal{B}_s^{diag}$ is equivalent to $\mathcal{B}_s$ (Libedinsky and Elias-Khovanov).

For example, these relations imply



which is the idempotent decomposition $b_s^2 = b_s[1] \oplus b_s[-1]$.

Theorem (Libedinsky)

For arbitrary (W, S) $\mathcal{B}$ is generated by the following morphisms:

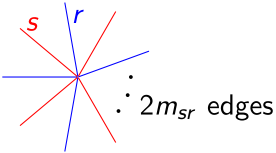
- *the morphisms i_{s*}, i_s^*, t_{s*} and t_s^* for all $s \in S$,*
- *the unique up to scalar degree zero morphism morphism*

$$f_{sr} : b_s b_r \cdots \to b_r b_s \ldots \quad (m_{sr}\text{-terms})$$

for all pairs s, r of simple reflections with $m_{sr} < \infty$.

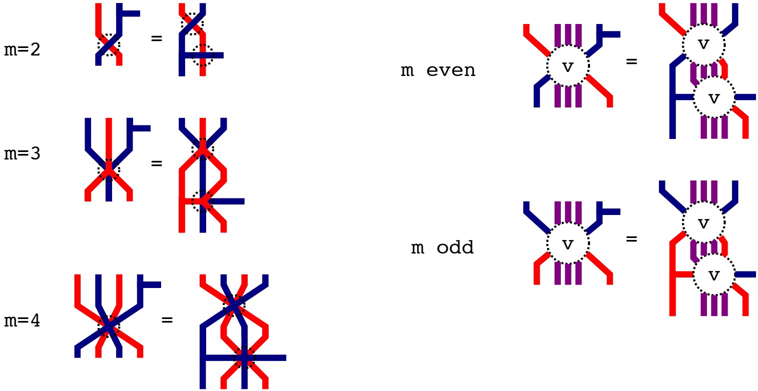
Suppose now that W is of rank 2. If $m_{sr} = \infty$ then (as was shown by Libedinsky) there are no new relations.

In the diagrammatic language, the morphism f_{sr} is represented by an $2m_{sr}$-valent vertex



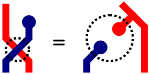
This year, all rank 2 relations were calculated by Elias.

It turns out that one only needs two relations. The first is "2-colour associativity":

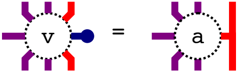


The second relation expresses the composition of the map f_{sr} with a "dot" (either i_s^* or i_{s*}).

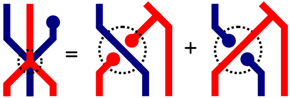
m=2



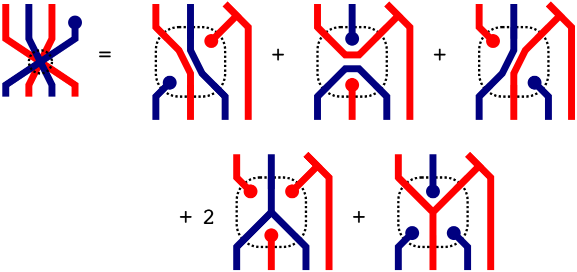
m arbitrary



m=3



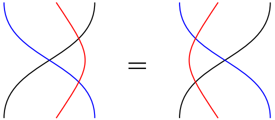
m=4



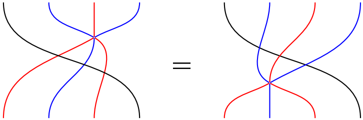
Theorem (Elias-Khovanov)

If W is of type A, then $\mathcal{B}$ is presented by the above relations, together with the following rank 3 relations:

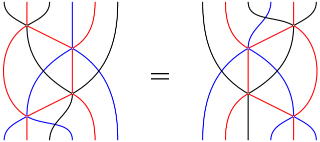
if s, t and u commute,



if s commutes with $t - u$,



if $s - t - u$,



Now, let $\mathcal{B}^{diag}$ denote the monoidal category with

- objects sequences of dots coloured by S,
- morphisms planar diagrams as above modulo:
  - all rank 1 and 2 relations as above,
  - Zamalodchikov relations for all finite rank 3 parabolic subgroups.

Then $\mathcal{B}^{diag}$ is an R-linear category. Let $\mathcal{B}^{diag}_{tf}$ denote quotient of $\mathcal{B}^{diag}$ by its torsion ideal, and $\overline{\mathcal{B}^{diag}_{tf}}$ its idempotent completion.

Theorem (Elias-W)

Suppose that W does not contain a standard parabolic subgroup isomorphic to H_3. Then we have equivalences

$$\mathcal{B}_{tf}^{diag} \cong \mathcal{B}, \quad \overline{\mathcal{B}_{tf}^{diag}} \cong \overline{\mathcal{B}}.$$

The proof is very straightforward, and uses the fact that Soergel bimodules become easy when tensored with K.

First one constructs a monoidal functor

$$r : \mathcal{B}^{diag} \to \mathcal{B}$$

(by verifying relations amongst Soergel bimodules).

Because homomorphism spaces between Soergel bimodules are free R-modules, this induces to a functor

$$r : \mathcal{B}^{diag}_{tf} \to \mathcal{B}$$

which is easily seen to be essentially surjective.

It remains to show that R_{tf} is fully-faithful. By Libedinsky's theorem it is a surjection on homomorphism spaces:

$$\mathrm{Hom}_{diag}(M, N) \quad \twoheadrightarrow \quad \mathrm{Hom}_{\mathcal{B}}(r(M), r(N))$$

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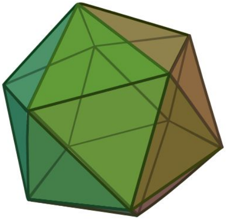
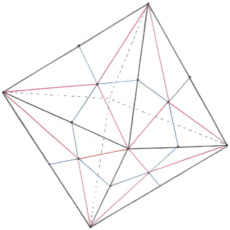
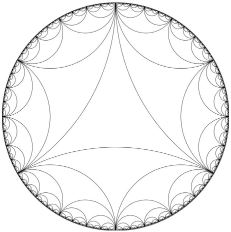
- just as one should have a similar version of our theorem for actions of Coxeter groups on ∞-categories, there should be an ∞-version of Soergel bimodules. (What's the center?!)
- the categories are defined over $\mathbb{Z}[\frac{1}{2}]$ for Weyl groups, and over certain rings of integers general. For example, (if we can check the Zamalodchikov) then H_3 and H_4 are defined over $\mathbb{Z}[\frac{1}{2}, \sqrt{5}]$.



Happy birthday Francois Digne and Jean Michel!





Literature

These slides are available at:

`http://people.maths.ox.ac.uk/~williamsong/GR4SB.pdf`

For more details, see:

Deligne, *Action du groupe de tresses sur une catégorie*, Invent. Math. 128, 159-175 (1997).

Libedinsky, *Presentation of right-angled Soergel categories by generators and relations*, `arXiv:0810.2395v1` (appeared in JPAA).

Elias, Khovanov, *Diagrammatics for Soergel categories*, `arXiv:0902.4700`.

Elias, W, *Generators and relations for Soergel bimodules*, in preparation.